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A two-dimensional study of finite amplitude sound waves in a trumpet using the discontinuous Galerkin method

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A model for nonlinear sound wave propagation for the trumpet is proposed. Experiments have been carried out to measure the sound pressure waveforms of the B_4^b and B_4^b notes played *forte*. We use these pressure measurements at the mouthpiece as an input for the proposed model. The compressible Euler equations are used to incorporate nonlinear wave propagation and compressibility effects. The equations of motion are solved using the discontinuous Galerkin method (DGM) and the suitability of this method is assessed. The third spatial dimension is neglected and the consequences for such an assumption are examined. The numerical experiments demonstrate the validity of this approach. We obtain a good match between experimental and numerical data after the dimensionality of the problem is taken into account.

Keywords: Nonlinear acoustic wave propagation; DGM; trumpet; sound pressure measurements; wave steepening; shock waves.

1. Introduction

The amplitude of most audible sound waves is only a small fraction of atmospheric pressure. In such cases, a small amplitude linearization of the gas dynamics equations can be made resulting in a linear model of sound propagation ¹. However, for brass musical instruments these simplifying assumptions start to break down once loud or high frequency notes are played ¹. For such notes, the pressure variations can be a significant fraction of atmospheric pressure in the narrow tubing of the instrument ². This can potentially allow the nonlinear behavior from the high amplitude propagating waves to distort the waveforms ^{1,3}.

If the pressure pulse leaving the mouthpiece of an instrument has large enough amplitude, the crest will travel noticeably faster than the trough. This will cause the waveform

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to steepen and excite the higher frequency components 4 . This influences the timbre of the sound giving it a more 'brassy' effect ⁵. If the cylindrical bore of the instrument is long enough and the nonlinear effects are strong enough, it is possible for the wave to steepen into a shock wave exaggerating the brassy sound ¹.

Work addressing wave production for the trumpet began in 1971 by Bachus *et al.*⁶, and for the trombone in 1982 by Elliot *et al.*⁷. In these papers, the motion of the lips coupled to the air column was examined. They concluded that the nonlinear motion of the lips did not contribute to the harmonic generation. They also thought that the amplitude of the sound waves did not influence the wave propagation behavior. Bachus *et al.*⁶ state that the transfer function between the mouthpiece pressure and radiated sound was basically linear.

Since they assumed that the wave propagation could be considered as linear, Webster's horn equation was considered to be an adequate description of sound wave propagation in a horn ^{8,9,10}. This expression was published in a paper by Webster in 1919 ¹¹. However, credit should also be given to Bernoulli, Euler and Lagrange. They all derived this equation and examined the solution before Webster published his work ¹². The equation is a one-dimensional (1D) differential equation that is an approximation for linear sound waves under the assumption that the acoustic pressure is constant in the tube and that the duct does not flare too quickly ¹³.

In 1980 however, this work was contradicted by Beauchamp. He suggested that linear models would not be sufficient to understand the wave motion in brass instruments ¹⁴. In his experiments with trombones, he measured the radiated harmonics and compared them to the corresponding harmonics of the initial wave at the mouthpiece. He found that at different playing volumes the difference between the harmonics was not constant, i.e., the transfer function was not linear. The amplitude of the initial wave influenced the sound pressure levels of the radiated harmonics.

Hirschberg *et al.* in 1995 ² observed that shock waves were present outside the trombone bell for certain notes played at certain volumes. It is often assumed that the nonlinear effects in the trumpet are also strong enough to produce shock waves. By using schlieren imaging, Pandya *et al.* ¹⁵ reported that they observed shock waves outside the trumpet bell. However, there is some uncertainty since the trumpet is much shorter than the trombone. The typical length of the trombone is approximately 2.76 m versus 1.48 m for the trumpet. Some results in the literature suggest that the shock distance is about the length of the trumpet for a B_3^b played *forte* ^{15,16}.

A number of papers proposed acoustic models to describe or mathematically explain nonlinear wave propagation within brass instruments. For example, Thompson *et al.* ¹⁶ proposed a linear and a nonlinear frequency domain model. The nonlinear model matches the experimental and numerical results very well for quietly played notes. For loudly played notes, the model deviates from the experiment for all harmonic components below 6000 Hz except for the two lowest harmonics. Burgers' equation or its variations have been used to explain the presence of experimentally observed shocks, e.g., ^{5,17}. Overall though, most work has been done on the trombone rather than the trumpet.

It was the interest of the authors to investigate the complex acoustic phenomena in the

trumpet experimentally and to develop an accurate numerical time domain model. Since there is little published experimental data available (in terms of recorded frequencies and their corresponding sound pressure levels), we took pressure measurements in a lab and published the results in the appendix for others to use for future research. This data will also be used in our model as an input at the mouthpiece.

To solve the equations of motion, we have decided to use the discontinuous Galerkin method (DGM) because of its many advantages. Firstly, the DGM has sufficiently small numerical dispersion and dissipation errors. This implies that the propagation of high harmonics can be simulated accurately. Secondly, the DGM can easily handle the complex geometry of the trumpet. Further discussions of the numerical scheme can be found in ¹⁸ and ¹⁹.

Since majority of the mathematical descriptions of wave steepening in brass instruments are 1D, the extension to two dimensions seemed natural. The use of lower dimensional models is usually justified by the symmetry of the trumpet which is accurate within the cylindrical bore but not the flare. In addition, we do not want to assume that the bends can be neglected without sufficient numerical evidence. We want to carefully model the cylindrical bore and a bend before assuming that an axisymmetric two-dimensional (2D) model is a good approximation. Therefore, we base our model on the full 2D Euler system describing the motion of compressible inviscid fluid rather than Burgers' equations. This allows us to better take into account the spreading of the waves at the bell.

However, the spreading of the waves in three and two dimensions is also different: the amplitude decays inversely proportional to the distance or a square root of it, respectively, i.e., we still expect there to be some amplitude discrepancy in our results. We will analyze this amplitude difference between 2D and 3D for the trumpet. For now however, losses will not be included in our model. Though in reality there are losses in the system due to wall boundary layers, wall vibration and the viscosity of the medium ^{20,21}.

The end goal of this research is to describe nonlinear wave propagation in the trumpet and this includes the generation of shock waves if they are in fact produced. However, this is a large task and this paper will focus on the first step of this investigation: to determine if neglecting the bends and third dimension in the full Euler system is reasonable once the dimension factor is taken into consideration. Once this is established, we can than investigate the production of shock waves in a future paper.

2. Acoustic Laboratory Experiments

2.1. Experimental Set Up

Pressure measurements were taken on a B^b Barcelona BTR-200LQ trumpet shown in Figure 1. Three microphones were attached at three locations along the trumpet. One quarter-inch microphone was mounted to the shank, i.e., the cylindrical part of the trumpet mouthpiece about 4.5 cm from the beginning of the instrument. The other quarter-inch microphone was attached to the tubing of the trumpet at the first bend which is approximately 42 cm along the length of the trumpet. The last, a half-inch microphone was placed on the central axis

of the trumpet approximately 16 cm - 17 cm outside the bell. In the rest of the paper, we will refer to these locations as *mouthpiece*, *bend*, and *bell*.



Fig. 1: Placement of microphones on the Barcelona BTR-200LQ trumpet.

To mount the microphones at the mouthpiece and bend, small holes were cut into the cylindrical part of the mouthpiece and trumpet bore. Quarter inch o-ring compression fittings were then soldered to the trumpet bore over these holes. In an attempt not to alter the acoustic properties of the trumpet, the microphones were placed so that the diaphragms made the boundary of the tube as smooth as possible. The microphones were connected to three inputs of an Agilent four channel digital oscilloscope and the pressure data from all three microphones was collected simultaneously while the instrument was held in the normal playing position. The data was captured once the note of choice was steady and only a couple of periods of the raw data was saved into the local computer.

The Agilent digital oscilloscope quantizes the input signals with 8-bit converters, so we might expect the microphone data to have a maximum of 256 levels. However, due to the large oversampling of the internal converters, many samples are added together to produce output samples with much better resolution. We found that our data displayed over 2000 levels, so it was captured with 11-bit precision, having a signal-to-noise ratio of about 66 dB.

In order to make the path of the airflow more direct we avoided using the values of the B^b trumpet. For this reason, we chose to play C_3 and C_4 notes, which correspond to a concert B_3^b and B_4^b , respectively. Both of these notes were played *forte* (f), which means loud.

2.2. Experimental Results

One period of the recorded pressure measurements for the B_3^b and B_4^b notes played f is depicted in the left plots of Figures 2 and 3, respectively. The plots show deviation of the sound wave pressure from atmospheric pressure, which is 101325 Pa. Since the pressure at the bell is much lower than inside the instrument, its waveform looks like a straight line in

the left plots of Figures 2-3. A close up can be seen on the right plots. Since the data is periodic, the starting point for plotting of all three waveshapes was chosen arbitrarily, i.e., if a different point were chosen the phase shifts would differ.



Fig. 2: Left: Measured pressure waveform at the mouthpiece, bend and bell of a B_3^b . Right: Close up of the measurements at the bell.



Fig. 3: Left: Measured pressure waveform at the mouthpiece, bend and bell of a B_4^b . Right: Close up of the measurements at the bell.

We will use the experimental data at the mouthpiece as the boundary condition on the pressure. The data at the bell will be used to judge accuracy of the proposed model. To this end, we applied the FFT to the discrete experimental data to obtain the spectral components of the wave. When using an N-point discrete Fourier transform (DFT), we must multiply the spectrum thus obtained by $\frac{\sqrt{2}}{N}$, in order for the peak of a spectral line to represent the rms value of that component. These are reported in Figure 4. As customary, the results are presented as sound pressure level (SPL) which is measured in decibels (dB).

Conversion from Pascals (Pa) to dB is given by

$$L_p = 20 \log_{10} \left(\frac{P}{2 \times 10^{-5}} \right).$$
 (1)

The total SPL is equal to 166.9 dB for the B_3^b and 167.2 dB for the B_4^b in the shank of the mouthpiece. Dynamic levels are arbitrary in the sense that there is no specific decibel level that defines *forte*; but typically, these SPLs fall into the *forte* range reported in the literature. As expected, our obtained results are also similar in character to the pressure measurements presented in ^{2,4,6,7,16}. In particular, the waveform shapes and SPLs for analogous notes in these papers resemble our plots shown in Figures 2, 3 and 4.



Fig. 4: Frequency spectra at the mouthpiece, bend and bell. Left: B_3^b . Right: B_4^b .

Applying Fourier synthesis to the results in Figure 4 gives us a continuous expression for the pressure. We write the trumpet notes as a sum of sine/cosine waves with frequencies $f, 2f, 3f, \ldots$, i.e., harmonics, each with a corresponding amplitude, A, and phase shift, ϕ . The sinusoidal function of a harmonic can be rewritten as a phase-shifted cosine function that is a superposition of the real and imaginary parts with amplitudes α and β ²²

$$\alpha_i \cos(\omega_i t) + \beta_i \sin(\omega_i t) = A_i \cos\left(2\pi f_i t + \phi_i\right),\tag{2}$$

where *i* is the harmonic number, *t* is time, f_i is *i*th frequency, $\omega_i = 2\pi f_i$ is the angular frequency. The amplitude is defined as

$$A_i = \sqrt{(\alpha_i^2 + \beta_i^2)},\tag{3}$$

and the corresponding phase angle ϕ_i is given by

$$\phi_i = \arctan\left(\frac{\beta_i}{\alpha_i}\right). \tag{4}$$

Therefore, one period of the entire pressure waveform of a desired note is expressed as

$$p = A_0 + \sum_{i}^{N/2} 2A_i \cos\left(2\pi f_i t + \phi_i\right),$$
(5)

where A_0 is the term corresponding to the direct current, and N is the number of points in our data, and $f_{N/2}$ is the Nyquist frequency. The amplitude is multiplied by a factor of two since the A_i are double-sided.

3. Numerical Set Up

3.1. Number of Harmonics

Next, we need to determine how many harmonics in (5) are required in order to reconstruct and accurately describe a pressure waveform for a given note. We obtained slightly more than 200 harmonics from our FFT analysis. Not all of these frequencies are meaningful. Once the SPLs drop by approximately 50 dB, these frequencies represent noise in our measurements. Our oscilloscope had a dynamic range of 66 dB, so it does not contribute to the noise in our measurements. The number of retained harmonics is important as it influences the size of the mesh: we normally require a minimum number of mesh cells per wavelength in order to resolve it properly. This number depends on the order of the numerical scheme.

In early investigations, it was typical for only six to eight harmonics to be considered in describing a note, e.g., ^{2,3,6}. We found, however, that more harmonics are necessary for an accurate representation of a note at the mouthpiece and an even higher number for measurements taken outside the bell. This is in line with recent results in the literature ²³. The required number of harmonics depend on the acoustical system and available computational resources. In order to find the proper number, we truncate the FFT reconstruction (5) at $N_f = 5, 10, 15, 20, 25$, and 30 number of harmonics and measure the resulting error (Table 1). The relative error was measured in the L^2 norm

$$error(\%) = \frac{\|p_{FFT} - p_{Trun}\|_2}{\|p_{FFT}\|_2} * 100\%, \tag{6}$$

where p_{FFT} is the fully reconstructed waveform and p_{Trun} is the reconstruction truncated after a given number of harmonics. We observe that for the mouthpiece at least fifteen harmonics are necessary to describe B_3^b and at least ten are needed for B_4^b . Adding extra frequencies does not significantly reduce the error. The difference in the number of harmonics for B_3^b and B_4^b can be explained by looking at the waveforms in Figures 2-3 where the B_4^b

waveform is visually smoother. This is further supported by the frequency spectra in Figure 4 where the B_4^b spectrum tapers off at lower frequencies.

Table 1: Error associated with the number of harmonics used to reconstruct the measured pressure waveform of a B_3^b and B_4^b note at the mouthpiece.

Number of Harmonics	Error for B_3^b	Error for B_4^b
5	10.46~%	2.69~%
10	2.16~%	0.53~%
15	0.52~%	0.32~%
20	0.44~%	0.29~%
25	0.32~%	0.26~%
30	0.24~%	0.24~%

We also calculate the number of harmonics that are needed to be retained in the truncated FFT reconstruction in order to represent the pressure waveform measured outside the bell. This is necessary for a meaningful comparison of the experimental and computational data. Table 2 reports the errors for both notes. We observe again that a greater number of harmonics are needed for B_3^b . Furthermore, we note that the number of meaningful harmonics in the recorded data is higher at the bell than at the mouthpiece for the same accuracy. Approximately thirty harmonics are needed for the waveforms at the bell. Thus, we use $N_f = 30$ in the boundary conditions (13). We again refer to Figures 2-3 where the signal at the bell is steeper than at the mouthpiece. This phenomenon is due to nonlinear behavior of the pressure wave, which steepens the wave as it moves through the trumpet.

Table 2: Error associated with the number of harmonics used to reconstruct the measured pressure waveform of a B_3^b and B_4^b note outside the bell.

Number of Harmonics	Error for B_3^b	Error for B_4^b
5	75.48~%	37.67~%
10	46.86~%	8.53~%
15	20.87~%	3.08~%
20	8.78~%	1.68~%
25	4.69~%	0.77~%
30	2.94~%	0.35~%
35	2.00~%	0.27~%

3.2. Euler Equations

It is tempting to use Webster's horn equation, an ordinary differential equation in the axial coordinates, to describe the wave propagation in the trumpet. However, this 1D equation

is inadequate for our purposes. The method neglects the nonlinear coupling of the forward and backward moving waves. Furthermore, the pressure field is assumed to be uniform throughout the width of the duct and the flare expansion of the bell has to be gradual. If the rate of change of the flare is too large, this method fails even for low frequency waves because we cannot decouple the modes ¹³. To describe nonlinear wave propagation in the trumpet for loud, high frequency notes, it is essential to take the pressure variations and mode coupling into consideration without placing any restrictions on the flare of the bell.

We base our model on the 2D compressible Euler equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0, \tag{7a}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} + \frac{\partial(\rho u v)}{\partial y} = 0,$$
(7b)

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho u v)}{\partial x} + \frac{\partial(\rho v^2 + p)}{\partial y} = 0,$$
(7c)

$$\frac{\partial E}{\partial t} + \frac{\partial (u(E+p))}{\partial x} + \frac{\partial (v(E+p))}{\partial y} = 0,$$
(7d)

where ρ is the density of air, $(\rho u, \rho v)$ are the momenta in the x and y direction respectively, p is the internal pressure, and E is the total energy. The equation of state connects E to the other variables:

$$E = \frac{p}{\gamma - 1} + \frac{\rho}{2}(u^2 + v^2).$$
(8)

For air, the specific heat ratio γ is approximately 1.4. The system (7) describes the motion of inviscid, compressible gas. Thus, our model neglects viscosity of the air and wall losses in the trumpet. In addition, to reduce the 3D problem to two dimensions we simply took the projected shape of the trumpet. We will discuss how these assumptions have influenced our results later. In order to solve equations (7-8), we need to specify initial and boundary conditions.

3.3. Initial and Boundary Conditions

We assume that initially the flow is at rest, i.e., the velocity in the horizontal and vertical direction is zero. The rest of the variables are scaled from physical values to values more convenient for computation. In particular, the speed of sound c_0 , which is approximately 343 m/s in air, and atmospheric pressure p_0 , which is 101325 Pa (i.e., one atmosphere), are scaled to be equal to 1. Assuming that the flow is isentropic, i.e., $c_0 = \sqrt{\frac{\gamma p_0}{\rho_0}}$, the initial density should then be taken to be 1.4. In summary

Initial State:
$$\begin{cases} p_0 = 1.0, \\ \rho_0 = 1.4, \\ u_0 = 0.0, \\ v_0 = 0.0. \end{cases}$$
(9)

The reflecting boundary conditions were prescribed on the inner and outer walls of the trumpet (Figures 6, 7, 8) excluding the mouthpiece end which is discussed below. At the straight wall, we use the solid wall boundary conditions. We specify a ghost state at each boundary quadrature point where the density, pressure and tangential velocity are taken to be the same as the corresponding value inside the cell. The normal velocity is reflected with respect to the wall, i.e., taken with a change of sign. Then, the ghost and inner states are passed to the Riemann solver. The resulting Riemann state corresponds to no flow through the solid wall boundary condition. For more detail see 24 .

We found that special care needs to be taken in imposing boundary conditions on the curved part of the bell. Computations are very sensitive to accuracy of geometric description. We refine the mesh and use the curvature boundary conditions proposed in ²⁵ to simulate the flow around the curved geometry of the bell on the straight-sided mesh shown in Figures 6, 7 and 8. Curved boundary conditions take into account the curvature of the true geometry of the trumpet while computing on a simpler straight-sided mesh. This allows us to speed up computations and at the same time, avoid spurious entropy production at the corners of the straight-sided triangles. Another approach would be to use high-order curve-sided triangles and the standard solid wall treatment described above. In the numerical section, both types of reflective boundary conditions (curved and straight) have been considered in certain simulations.

The far boundary of the computational domain around the trumpet has pass-through boundary conditions. The condition is imposed by specifying the unperturbed flow state as the ghost state on the outside of the domain. Due to the negligibly small velocity and low amplitude of the pressure wave outside the trumpet (.05% of atmospheric pressure), there is very little numerical reflection at the outer wall boundary of our mesh. To verify this, we considered a box that was four times larger than the one shown in the left of Figure 6 and found no difference in the computed pressure.

The data obtained from the experiments is used as the boundary condition on the pressure. We do not attempt to model the flow behavior inside the mouthpiece as it is quite complex. Our measurements were taken on the straight part of the mouthpiece where it connects with the bore. Since the waveform in the bore behaves like a planar 1D wave, we impose that boundary condition on the left vertical wall where the mouthpiece would be. This corresponds to the zero value of the horizontal variable and the interval [-0.58,0.58] in the vertical direction in the geometric description in Figure 5. Since the measurements were recorded when the note was being played for some time, they represent the boundary conditions when the quasi steady state is reached. Thus, we can expect for the simulation

that the first time the wave transverses the trumpet, the output will not be accurate. However, if simulations are performed correctly, then the reflected waves should fit well with the prescribed values at the wall as they form the standing wave pattern.

The boundary conditions for the dimensionless pressure are given by

$$\hat{p}(t) = \hat{A}_0 + \sum_{i=1}^{N_f} 2\hat{A}_i \cos\left(2\pi \hat{f}_i t + \hat{\phi}_i\right),$$
(10)

where \hat{A}_i , \hat{f}_i , and $\hat{\phi}_i$ are scaled versions of data in (5) and N_f is the number of used frequencies. The values for this variables are given in Appendix B.

The boundary conditions for the rest of the variables are set in the following way. The density is computed using the isentropic flow law

$$\hat{\rho}(t) = C\hat{p}(t)^{\frac{1}{\gamma}},\tag{11}$$

where C is the proportionality constant. We require C to hold for the initial conditions (9) which gives $C = \gamma$. Finally, we need to prescribe the initial velocity. We connect velocity to pressure using the 1D expression derived from linear acoustic theory

$$\hat{u}(t) = \frac{\hat{p}(t) - p_o}{\rho_o c}.$$
(12)

This results in an input of the pressure wave of the form given in (10), moving from left to right. Since there is little return from the reflections at the bell (especially for the frequency components we are interested in), the plane wave velocity expression is a good approximation. We note that the linearization is applied at the mouthpiece boundary, i.e., on the line corresponding to the beginning of our trumpet. The velocity of the air particles inside of the instrument is governed by the Euler equations and, hence, is nonlinear. The linearization is justified since the speed of the air particles inside the instrument is low relative to the speed of sound. Velocity measurements for trombones reported in ²⁶ give the maximum speed about 17 m/s, i.e., about 5% of the speed of sound, in the throat of the mouthpiece. We were unable to find in literature measurements taken inside the instrument. We discuss later the limitations of the boundary conditions.

Therefore, the boundary conditions at the mouthpiece of the computational trumpet, i.e., at about 4.5 cm from the beginning of the physical instrument, are given by

Boundary Conditions:
$$\begin{cases} \hat{p} = \hat{A}_{0} + \sum_{i=1}^{N_{f}} 2\hat{A}_{i} \cos\left(2\pi \hat{f}_{i}t + \hat{\phi}_{i}\right), \\ \hat{\rho} = \gamma \hat{p}^{\frac{1}{\gamma}}, \\ \hat{\mu} = \frac{\hat{p} - p_{o}}{\rho_{o}c}, \\ \hat{v} = 0.0, \\ \hat{E} = \frac{\hat{p}}{\gamma - 1} + \frac{\hat{\rho}}{2}(\hat{u}^{2} + \hat{v}^{2}). \end{cases}$$
(13)

3.4. Trumpet Mesh

To create a 2D trumpet mesh, we first wanted to determine if neglecting the bends would be a reasonable approximation. In this paper, we verify that the trumpet bends do not greatly influence wave propagation inside the instrument. Nonetheless, considering a bend in the mesh does produce slightly better numerical results. This hypothesis is tested in the numerical section and is shown to be accurate within the error margins of our model. Thus, our mesh attempts to accurately reflect the length of a trumpet, the bore and slowly increasing diameter of the flare of the bell.

Figure 5 depicts the two different geometries of the trumpet (when the bend is neglected) that were used as an input by a mesh generator. Valves on the trumpet are not used in playing B_3^b and B_4^b so the length of the instrument is taken to be 1.48 m. The diameter of the tube is constant except near the end where it slowly increases and then flares rapidly at the bell. To accurately depict the geometry, measurements were taken as the diameter of the bore expanded near the bell. These points were passed to the mesh generating software GMSH.



Fig. 5: Geometries for the trumpet meshes.

The initial measurements were taken at the university's machine shop. However, measurements were only taken every few centimeters throughout the flare. This mesh, which we will call *mesh 1*, was constructed and is illustrated in Figure 6. Mesh 1 was used for simulations in the numerical section; however, we were concerned with the lack of smoothness as the flare expanded. Therefore, we created another mesh, called *mesh 2*, and it can be seen in Figure 7. To obtain a better set of points along the flare, we took a picture of the trumpet and downloaded the *grabit* software from MathWorks Inc.. This allowed us to trace out the shape of the bell with more points making the geometry more precise, and smoother. Mesh

2 was also used for simulations presented in the numerical section. By using both meshes, we were able to compare how the shape of the flare influenced our results. A third mesh, *mesh* 3 in Figure 8 will be used in the numerical section to justify that the bends can be neglected. The *grabit* software was also used to create the bend shape.

For all meshes, the points on the straight part of the bore were connected by lines; the curve representing the bell was interpolated using cubic splines. The meshes were constructed so that one unit on the axes represents 1 cm. We enclosed the trumpet in a box with dimensions 2.20 m by 2.92 m. We found that increasing the box's size does not influence the accuracy of the computations. The total number of cells is 8467, 8038, and 9563 in meshes 1, 2 and 3, respectively. There are about four to five triangles along the width of the bore. In the length direction, we have approximately 350 triangles per wavelength. At the bell, the mesh is refined by a factor of two and a half, which would correspond to 875 triangles per wavelength. The mesh is coarser near the boundary of the box. According to 23 , this is more than enough triangles per wavelength regardless of the discretization order.



Fig. 6: Mesh 1: Two-dimensional trumpet mesh constructed by taking measurements in machine shop.



Fig. 7: Mesh 2: Two-dimensional trumpet mesh constructed by tracing out the shape of the bell in Matlab.

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Fig. 8: Mesh 3: Two-dimensional trumpet mesh constructed by tracing out the shape of the bell and bend in Matlab.

4. Numerical Experiments

4.1. Discontinuous Galerkin method

We use the discontinuous Galerkin method in the formulation originally proposed by Cockburn and Shu²⁴. Here we provide a brief synopsis of the numerical scheme.

In order to describe the method, we write a general conservation law

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{u}) = \mathbf{0}, \qquad \mathbf{x} \in \Omega, \qquad t > 0, \tag{14a}$$

$$\mathbf{u} = \mathbf{u}^0, \qquad t = 0. \tag{14b}$$

We divide the problem domain Ω into a collection of nonoverlapping elements

$$\Omega = \bigcup_{j=1}^{N_h} \Omega_j.$$
(15)

Then, we construct a Galerkin problem on element Ω_j by multiplying (14a) by a test function $\mathbf{v} \in (H^1(\Omega_j))^m$, where *m* is the number of equations in the system (14), integrating the result on Ω_j , and using the Divergence Theorem to obtain

$$\int_{\Omega_j} \mathbf{v} \frac{\partial \mathbf{u}}{\partial t} \, ds + \int_{\partial \Omega_j} \mathbf{v} \mathbf{F}(\mathbf{u}) \cdot \vec{n} \, d\tau - \int_{\Omega_j} \mathbf{grad} \mathbf{v} \cdot \mathbf{F}(\mathbf{u}) \, ds = \mathbf{0}, \qquad \forall \mathbf{v} \in (H^1(\Omega_j))^m, \quad (16)$$

where \vec{n} is the normal vector to $\partial \Omega_j$.

The solution **u** is approximated by a vector function $\mathbf{U}_j = (U_{j,1}, U_{j,2}, \dots, U_{j,m})^T$, where

$$U_{j,k} = \sum_{i=1}^{N_p} c_{j,k,i} \varphi_i, \quad k = 1, 2, \dots, m,$$
(17)

in a finite-dimensional subspace of the solution space. The basis $\{\varphi_i\}_{i=1}^{N_p}$ is chosen to be orthonormal in $L^2(\Omega_j)$ ¹⁸. This will produce a multiple of the identity for the mass matrix on Ω_j when the testing function **v** is chosen to be equal to the basis functions consecutively starting with φ_1 .

Due to the discontinuous nature of the numerical solution, the normal flux $\mathbf{F}_n = \mathbf{F}(\mathbf{u}) \cdot \vec{n}$, is not defined on $\partial \Omega_j$. The usual strategy is to define it in terms of a numerical flux $\mathbf{F}_n(\mathbf{U}_j, \mathbf{U}_k)$ that depends on the solution \mathbf{U}_j on Ω_j and \mathbf{U}_k on the neighboring element Ω_k sharing the portion of the boundary $\partial \Omega_{jk}$ common to both elements. In our numerical experiments, we have used the Roe numerical flux ¹⁸. Finally, the L^2 volume and surface inner products in (16) are computed using 2p and 2p + 1 order accurate Gauss quadratures ¹⁸, respectively, where p is the order of the orthonormal basis. The resulting system of ODEs is

$$\frac{d\mathbf{c}}{dt} = \mathbf{f}(\mathbf{c}),\tag{18}$$

where \mathbf{c} is the vector of unknowns and \mathbf{f} is a nonlinear vector function resulting from the boundary and volume integrals in (16).

Numerical experiments presented in Section 4.3 were computed using a linear basis and verified by running the same tests with a quadratic basis. Due to the very fine meshes used, the results were extremely similar.

4.2. Estimated Difference of Trumpet Output in Two and Three Dimensions

In that portion of the trumpet where the bore diameter is constant, the acoustic signal is essentially 1D. We can approximately compare the 2D simulations with 3D measurements if we make a few reasonable assumptions about the acoustic signals leaving the bell. Let us compare a 3D trumpet of bore with radius R with a 2D trumpet of the same projected shape. We choose our 2D trumpet with a height W such that the area of its rectangular bore is the same as that of the 3D trumpet, thus $2RW = \pi R^2$. Using the measured acoustic pressure for our simulations means that the same total power is traveling down the actual trumpet and the simulation.

Now consider that most of the acoustic power will leave each trumpet bell, since much of the power is in the higher harmonics. We assume that this will be similar in 2D and 3D, i.e., we are using a conservation of energy argument. In addition, we assume that the axial acoustic pressure² measured in similar places outside the bell is proportional to the

total power leaving the instrument. There will be some differences in 2D and 3D but we will assume that they are not significant relative to the other considerations.

The acoustic waves leaving the bell in both 2D and 3D will have curved wavefronts, and we can equate the pressure² times the wavefront area so that the power leaving each bell is the same. It is necessary to estimate (see Appendix A) these curved wavefront areas, the one in 2D being a curved strip of width h bridging the edges of the bell, and the one in 3D being a spherical cap bounded by the edge of the bell. If A_2 and A_3 represent these areas, and p_2 and p_3 represent the simulated and measured axial pressures on these areas, then we would set $p_2^2A_2 = p_3^2A_3$.

There is one more detail needed to compare the simulated and measured pressures. They have been assessed not at the bell but some distance from it, due to the original placement of the microphone well outside the bell. In comparing this position to the one on the curved wavefront bridging the bell, we must account for the different variation of amplitude with distance in 2D and 3D. In 2D, the amplitude is closely proportional to $\frac{1}{\sqrt{r}}$, whereas in 3D the amplitude varies as $\frac{1}{r}$. We need to know where the center of curvature is located for the wavefronts. All these considerations have been included in the numerical factor of 14 dB difference between 2D and 3D at a distance 16 cm from the bell. While the whole process seems a bit rudimentary, it nonetheless allows a comparison between 2D and 3D.

4.3. Computational Results

To better interpret our simulation results, we first examine the acoustic behavior of the 2D bell. We send down the trumpet a pulse given by

$$p_1 = \begin{cases} 1.0 + (0.01 - 0.01 \cos(1500 t)), & \text{if } t < \frac{2\pi}{1500} \\ 1.0, & \text{else} \end{cases}$$

which corresponds to a pressure pulse with an amplitude of approximately 2000 Pa. Since the amplitude is only 2% of an atmosphere, we can view the pulse as linear. We do not observe wave steepening. In the top plot of Figure 9, we show the pressure simulated at a point located at the mid-length of the cylindrical bore. The first peak located at approximately t = 0.0025 corresponds to the initial pulse moving from the mouthpiece towards the bell. The second inverted peak seen at about t = 0.007 corresponds to the signal traveling back to the mouthpiece after it has been reflected at the bell.

The reflected transfer data of Figure 9 (bottom), is calculated from the frequency content of the reflected pulse divided by that of the incident pulse. This curve represents the power reflected by the bell meshes. The calculated transmission data, which is the compliment of the reflected transfer data, represent the power transmitted from the bell; the transmission transfer data are the output of the bell meshes. The curves representing the output of the bell meshes have a reduction in the sound pressure level (SPL). This is mainly because the wave spreads as it exits the bell. However, we can see that the shape of the calculated and simulated transmission data are similar, especially for frequency components greater than

200 Hz. The spreading of the wave is slightly different for the low frequencies and high frequencies.



Fig. 9: Simulated pressure pulse through mesh 1 and mesh 2. Top: Initial and reflected wave. Bottom: Corresponding transfer functions of the bell.

The reflection data of Figure 9 shows that more than 50% of the incident energy at

the mouthpiece is transmitted out of the bell for frequencies above about 600 Hz for mesh 1, and 400 Hz for mesh 2. This is where the reflection plot crosses at -3 dB (in 3D, we expect that where the plot crosses will increase in frequency). This means, in theory, that frequency components less than approximately 600 Hz and 400 Hz, for mesh 1 and mesh 2 respectively, are mostly reflected before or within the bell. We hypothesize that the 200 Hz difference is due to the variation of the bell curvature. For mesh 1, the bell is not as smooth and along the length of the instrument, the flare widens further down the tube compared to mesh 2. For mesh 1, this will cause more of the frequency components to stay within the confines of the duct. This would explain why the pulse for mesh 1 in Figure 9 (bottom) has a lower peak relative to mesh 2. As the frequency increases, the wave travels further into the bell. For frequency components greater than roughly 1000 Hz, the waves are not being reflected at all and we see that the transmission data matches for both meshes in Figure 9 (bottom). This is to be expected since both mesh 1 and 2 have the same geometry by the time the bell is fully flared. We will use these findings in the discussion of our trumpet simulations.

We begin numerical experiments with notes played on a trumpet by verifying the correctness of the proposed boundary conditions (13). In classical acoustic theory, the player provides an oscillating air particle velocity, and the instrument reacts with an appropriate acoustic pressure. In our nonlinear simulation, we must specify both pressure and velocity on the mouthpiece end boundary, which is unusual from an acoustic point of view. We plot the computed frequency spectrum of the pressure sampled at the very beginning of the simulations close to the mouthpiece boundary in Figure 10. We refer to this data as the simulation data *before reflection*. We observe a good match between the experiment and the input simulated waveforms, especially for the B_4^b where the frequency components are almost perfectly aligned. For the B_3^b , the frequency components below 4000 Hz are almost identical for the experiment and simulated waveforms. For frequencies above 4000 Hz, we observe a slight variation between one to four decibels. This indicates the validity of the proposed boundary setup.

We again sample the pressure at the mouthpiece after the wave has traveled to the bell, been reflected and returned to the mouthpiece (Figure 11). We refer to this data as the simulation data after reflection. We observe that the frequency spectrum for the B_4^b on the right plot at the mouthpiece does not change drastically. The B_3^b pressure (left plot) shows more variations; specifically for the fundamental frequency, and higher harmonic components.

The results of solving (7) with initial and boundary conditions (9), (13) on mesh 1, mesh 2 and mesh 3 are shown in Figures 12, 13, and 14. Each note was simulated with 30 harmonics ($N_f = 30$ in (10)). We performed two sets of numerical experiments for each note. In the first simulation set, denoted by S_1 , only one period of the mouthpiece time pressure waveform (shown in Figures 2 and 3) will be generated at the mouthpiece boundary. For the second simulation set, denoted by S_2 , we applied the same mouthpiece waveform continuously. In Figures 12 and 13, we refer to S_1 and S_2 as the *single* and *multiple* period simulations, respectively.



Fig. 10: Computed pressure sampled at the mouthpiece at the beginning of simulations. Experimental data is plotted for comparison. Left: B_3^b . Right: B_4^b .



Fig. 11: Computed pressure sampled at the mouthpiece after the incident wave traversed the length of the trumpet twice. Experimental data is plotted for comparison. Left: B_3^b . Right: B_4^b .

The S_1 and S_2 solutions should be similar for most of the harmonics since the bell transmits most of the higher frequency components (see Figure 9). This should also be true if the boundary conditions are imposed properly. However, we anticipate that S_2 may produce slightly different amplitude values. This is because the additional reflections at the bell will influence the standing wave pattern. Figure 11 shows that the solution with multiple periods reaches a new numerical steady state which is somewhat different from the experimental steady state. In fact, what we are computing for S_2 is the output corresponding to the input shown in Figure 11.

The presented results for S_1 correspond to the wave directly propagating out of the bell. For the single period simulations, the output of the trumpet is longer than one period of the note being simulated. In order to obtain the proper frequencies that are the multiples of the fundamental, we must force the data to have the correct period. This means that



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Fig. 12: Numerical result for B_3^b .

we must 'fold' the longer response into one that has the proper duration. For S_2 , we plot the solution once a new steady state is reached. We found that this occurred at the second period of the pressure wave. We ran simulations until 12 periods were emitted from the bell. We noticed little change after the second period.

Figures 12 and 13 show the frequency spectrum of the pressure sampled 16 cm outside of the bell for B_3^b and B_4^b , respectively. As expected, we observe only a small difference between the single and multiple period simulations, with the multiple period simulations having a somewhat higher amplitude, especially for the B_3^b . We hypothesize this difference is due to the variation of the fundamental frequency shown in Figure 11. However, a comparison of the numerical (Figures 12 and 13) and experimental (Figure 4) data reveals that the numerical amplitude is approximately 19 dB higher. To compare the shape of the numerical and experimental spectra, we shifted the experimental data by the decibel difference.

With this adjustment we notice that the shape of the shifted experimental data is very similar to the solution curves. The spectra are in good agreement for the first 30 harmonics, i.e., up to 7200 Hz, with the exception of the lowest frequency components. We observe that the first three harmonics of the B_3^b are influenced by the reflections; whereas only the fundamental frequency for the B_4^b is affected. All of these frequency components are below 900 Hz, which are largely reflected for our trumpet (Figure 9). However, such deviations are to be expected for lower frequencies. In 3D, the opening at the bell changes more rapidly with



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Fig. 13: Numerical result for B_4^b .

axial distance than in 2D. This will mean that more of the lower harmonic components will be reflected because of the larger radiation impedance in 2D 9 . The frequency components that are transmitted from the bell are modeled accurately.

The adjustment (decibel shift) can be explained in the following way. The 2D - 3D dimensionality difference is approximately 14 dB. The additional 4 dB (for B_3^b) and 5 dB (for B_4^b) in the SPL may be a result of neglecting energy losses. Kausel *et al.* ²⁷ provides an extensive review of the literature stating that trumpet wall vibrations, and viscous and thermal boundary layers may reduce the radiated SPL by approximately 3 dB - 6 dB, sometimes even exceeding 6 dB. Our losses seem to be in this range.

We combine all S_1 results in Figure 14 to evaluate the influence of the computational geometry of the trumpet on the numerical results. Since there is little variation between the single and multiple period simulations, the corresponding results for S_2 are not reported. Although we see little difference between the results on mesh 2 and mesh 3, mesh 3 is in better agreement with the experimental data. This result may be easier to observe in Figures 12 and 13. However, the bend does not greatly influence the wave propagation. An axisymmetric model would be a good approximation and simplification for future work. In addition, we notice that when the bell geometry is smoother, i.e., on mesh 2 and mesh 3,



Fig. 14: Numerical results. Left: B_3^b . Right: B_4^b

the numerical results are slightly better than on mesh 1.



Fig. 15: Numerical result for B_3^b outside bell simulated with curved and straight reflective boundary conditions. Left: Mesh 1 results. Right: Mesh 2 results.

Finally, we assess the influence of the imposed boundary conditions on the curved portion of the trumpet in Figure 15. The S_1 solution is plotted for both notes when the standard reflecting boundary conditions and curvature boundary condition in ²⁵ are used. The meshes have similar cell sizes but significantly different geometry. A more crude geometry seems to increase the SPLs of the harmonic components. On the left plot of Figure 15, we can see that this is true for all the harmonic components below 4000 Hz. For mesh 2 (right plot), we observe that the straight boundary conditions are especially detrimental for frequency components greater than 4000 Hz since small wavelengths are affected more by corners in the straight-sided meshes. $A \ two-dimensional \ study \ of finite \ amplitude \ sound \ waves \ in \ a \ trumpet \ using \ the \ discontinuous \ Galerkin \ method \ 23$

5. Conclusions

Overall, our results are encouraging. Figures 12, 13, and 14 show that the energy represented in the measured mouthpiece pressure (Figure 4) comes out of the bell producing similar spectra. Although the 2D - 3D amplitude discrepancy can mostly be explained (see Appendix A), 2D simulations are not adequate to simulate nonlinear wave propagation within the trumpet. Neglecting the spreading of the wave in the third dimension is too significant to dismiss. However, considering an axisymmetric 2D model could provide better results.

In future work, we intend to determine a more accurate steady-state velocity for the boundary condition at the mouthpiece. In principle, if we know the acoustic input impedance of the trumpet mouthpiece, we could calculate the steady-state velocity that would accompany the measured steady-state pressure. However, since the bell transmits most of the energy above 800 Hz - 1000 Hz, we do not think the correction is too serious. The details of this will be explored in future a paper.

More importantly, we plan to examine our model in three dimensions. From our current results, we can say with confidence that the model and numerical method chosen is an appropriate one. Therefore, we will continue using the discontinuous Galerkin method to examine our 3D model. Until these simulations are complete, we will refrain from making any comments on the generation of shock waves in the trumpet.

Appendix

A. Derivation of Dimension Factor

Consider the 2D trumpet bell depicted in Figure A.1. The diameter of the bell is denoted by s (approximately 13 cm) and the radius of the bore is denoted by R (approximately 0.7 cm). We will simply assume that the pressure will be the same across the arc denoted by L. In other words, we are assuming that the axial pressure in both 2D and 3D is a good measure of the total energy leaving the bell (i.e., we are assuming that the total energy leaving the bell is conserved). The radius of the arc will be denoted by r and was determined by approximating the exiting wavefront by a circle C, with radius 9.168 cm. Although this circle slightly overshoots the wavefront, it is the closest approximation. We denote the angle $\theta_o \approx 45.95^o$ as the angle between the arc and the horizontal line r which only considers the wavefront approximated by C. The angle θ_0 can be found using

$$\sin\theta_0 = \frac{s}{2r},$$

which approximately gives $\theta_0 = 0.793252$ radians or 45.95 degrees.

We want to estimate the difference in the area of the curved wavefront propagating out of the bell for the 2D and 3D problems. In three dimensions, we examine the area of the cap, denoted by A_3 . This can be found by integrating a slice of the sector, i.e.,



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Fig. A.1: Trumpet bell in two dimensions.

$$A_3 = \int_0^{\theta_0} 2\pi r^2 \sin(\theta) d\theta = 2\pi r^2 [1 - \cos(\theta_0)].$$
 (A.1)

Notice that as $\theta_0 \to \pi$, $A_3 \to 4\pi r^2$ as one would expect.

Let's now consider the analogous problem in two dimensions. The area of the cross section of the circular bore (denoted by A_b) is $A_b = \pi R^2$. For the 2D model the corresponding cross section is a rectangle with the area A_s given by $A_s = W2R$, where $W = \frac{A_b}{2R} = \frac{R\pi}{2}$. The arc length of the 2D sector drawn in the diagram, denoted by 2L, when θ is in radians is

$$2L = 2\int_0^{\theta_0} rd\theta = 2r\theta_0. \tag{A.2}$$

Therefore, the area of the bell in two dimensions, denoted by A_2 , can be written as

$$A_2 = 2LW = 2Wr\theta_0. \tag{A.3}$$

We will now consider the ratio $\frac{A_2}{A_3}$ and, using the measurements for r, R, and θ_o , we obtain

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$$\frac{A_2}{A_3} = \frac{2Wr\theta_0}{2\pi r^2 [1 - \cos(\theta_0)]}$$

$$= \frac{R\theta_0}{2r[1 - \cos(\theta_0)]}$$

$$\approx 0.1066.$$
(A.4)

A wavefront exiting the bell of a trumpet will do so with a certain amount of energy, that should be conserved between the dimensions. Thus, we have

$$A_3 p_3^2 = A_2 p_2^2$$

$$\begin{pmatrix} p_2 \\ p_3 \end{pmatrix} = \sqrt{\frac{A_3}{A_2}}$$

$$\frac{p_2}{p_3} \approx 3.13.$$
(A.5)

This gives the pressure ratio of $\frac{p_2}{p_3} \approx 3.13$. Therefore, neglecting the third dimension in our simulations could produce an amplitude that is 3.1 times larger than it should be for the exiting wavefront, i.e., measured at 5 cm outside the bell. This corresponds to a SPL of approximately 10 dB.

Next, we estimate what the difference in SPL will be when the wave is further outside the bell. In two dimensions, the amplitude of the pressure waves (denoted by p_2) will drop off as a factor of $\frac{1}{\sqrt{r}}$ and in three dimensions pressure (denoted by p_3) will drop off as $\frac{1}{r}$. Therefore, the relationship between pressure at position a (denoted by p_a) and pressure at position b (denoted by p_b) for two dimensions will be

$$p_{2,b}\sqrt{b} = p_{2,a}\sqrt{a},\tag{A.6}$$

and in three dimensions

$$p_{3,b}b = p_{3,a}a.$$
 (A.7)

Therefore, $p_{2,b} = (\frac{\sqrt{a}}{\sqrt{b}})p_{2,a}$ and $p_{3,b} = (\frac{a}{b})p_{3,a}$. Considering the ratio between p_2 and p_3 gives

$$\frac{p_{2,b}}{p_{3,b}} = \alpha \left(\frac{p_{2,a}}{p_{3,a}}\right) \tag{A.8}$$

where $\alpha = \frac{\sqrt{ba}}{a}$. In our case, α is approximately 1.5374 and $\frac{p_2}{p_3} \approx 4.77$. Therefore, we conclude that the amplitude measured outside the bell will be approximately 14 dB too high.

B. Frequency Components

Index i	Frequency F_i [Hz]	Coefficient A_i	Phase ϕ_i [rad]
1	242.1307506053	0.0241039701	-2.3998599843
2	484.2615012107	0.0173638758	-1.6517886419
3	726.3922518160	0.0084349905	-0.8176568965
4	968.5230024213	0.0047298543	-0.2133008263
5	1210.6537530266	0.0033935648	-0.0655308956
6	1452.7845036320	0.0025695118	0.5243913475
7	1694.9152542373	0.0016134162	0.8604561928
8	1937.0460048426	0.0008113565	0.9217503119
9	2179.1767554479	0.0005827414	0.6229672421
10	2421.3075060533	0.0005015204	-0.2200386659
11	2663.4382566586	0.0004743206	-0.4341971836
12	2905.5690072639	0.0002840805	-1.2187641059
13	3147.6997578692	0.0001776235	-0.9824755922
14	3389.8305084746	0.0002692897	-1.5134465545
15	3631.9612590799	0.0001657290	-2.1666134433
16	3874.0920096852	0.0000612890	-2.2469981904
17	4116.2227602905	0.0000410981	-3.0718515385
18	4358.3535108959	0.0000098495	3.1217791911
19	4600.4842615012	0.0000295860	2.1787935295
20	4842.6150121065	0.0000276109	2.2441751234
21	5084.7457627119	0.0000718223	1.8159917771
22	5326.8765133172	0.0000377127	2.1726743847
23	5569.0072639225	0.0000392250	1.6726336078
24	5811.1380145278	0.0000208027	2.1763484676
25	6053.2687651332	0.0000281628	0.6244992167
26	6295.3995157385	0.0000211836	0.3845313608
27	6537.5302663438	0.0000463784	1.6615760855
28	6779.6610169491	0.0000238014	1.1883766564
29	7021.7917675545	0.0000338559	2.3188758038
30	7263.9225181598	0.0000119861	-2.2344618963

Table 3: Data used in the boundary conditions for B_3^b note according to (5).

Index i	Frequency F_i [Hz]	Coefficient A_i	Phase ϕ_i [rad]
1	484.2615012107	0.0308221480	-1.8557759854
2	968.5230024213	0.0070994743	-0.8967978456
3	1452.7845036320	0.0049450532	-0.3589912885
4	1937.0460048426	0.0008004100	0.3405715860
5	2421.3075060533	0.0012647681	-1.8409398815
6	2905.5690072639	0.0007447962	-2.5283707634
7	3389.8305084746	0.0002186134	-1.8779479147
8	3874.0920096852	0.0000888137	0.9044171863
9	4358.3535108959	0.0000593364	2.4589184751
10	4842.6150121065	0.0000272908	-0.9338145732
11	5326.8765133172	0.0000310905	0.9409443235
12	5811.1380145278	0.0000063116	3.1009038661
13	6295.3995157385	0.0000372662	0.2893748454
14	6779.6610169491	0.0000704671	0.7520546044
15	7263.9225181598	0.0000457795	1.5555365779
16	7748.1840193704	0.0000271710	2.2707815906
17	8232.4455205811	0.0000277986	2.0016742736
18	8716.7070217917	0.0000386123	2.0452294706
19	9200.9685230024	0.0000242123	2.2205670522
20	9685.2300242131	0.0000060413	1.6653082554
21	10169.4915254237	0.0000192693	1.4715414339
22	10653.7530266343	0.0000290413	1.9483626136
23	11138.0145278450	0.0000183053	2.1801855089
24	11622.2760290556	0.0000143986	1.6520225430
25	12106.5375302663	0.0000189479	2.1506604657
26	12590.7990314769	0.0000088018	2.4548264290
27	13075.0605326876	0.0000063889	1.6491937036
28	13559.3220338982	0.0000199512	1.5693955572
29	14043.5835351089	0.0000214452	2.0327473422
30	14527.8450363195	0.0000148274	2.1657610034

Table 4: Data used in the boundary conditions for B_4^b note according to (5).

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