NONCOMMUTATIVE CHOQUET THEORY

KENNETH R. DAVIDSON AND MATTHEW KENNEDY

Abstract. We introduce a new and extensive theory of noncommutative convexity along with a corresponding theory of noncommutative functions. We establish noncommutative analogues of the fundamental results from classical convexity theory, and apply these ideas to develop a noncommutative Choquet theory that generalizes much of classical Choquet theory.

The central objects of interest in noncommutative convexity are noncommutative convex sets. The category of compact noncommutative sets is dual to the category of operator systems, and there is a robust notion of extreme point for a noncommutative convex set that is dual to Arveson's notion of boundary representation for an operator system.

We identify the C*-algebra of continuous noncommutative functions on a compact noncommutative convex set as the maximal C*-algebra of the operator system of continuous noncommutative affine functions on the set. In the noncommutative setting, unital completely positive maps on this C*-algebra play the role of representing measures in the classical setting.

The role of noncommutative convex functions is crucial to our theory, and this is a new notion in the theory of nc functions. The nc convex functions determine an order on the set of unital completely positive maps that is analogous to the classical Choquet order on probability measures. We characterize this order in terms of the extensions and dilations of the maps, providing a powerful new perspective on the structure of completely positive maps on operator systems.

Finally, we establish a noncommutative generalization of the Choquet-Bishop-de Leeuw theorem asserting that every point in a compact noncommutative convex set has a representing map that is supported on the extreme boundary. In the separable case, we obtain a corresponding integral representation theorem.

2010 *Mathematics Subject Classification.* Primary 46A55, 46L07, 47A20; Secondary 46L52, 47L25.

Key words and phrases. noncommutative convexity, noncommutative Choquet theory, noncommutative functions, operator systems, completely positive maps.

First author supported by NSERC Grant Number 2018-03973.

Second author supported by NSERC Grant Number 50503-10787.

Contents

1. Introduction

Classical Choquet theory is now a fundamental part of infinite-dimensional analysis. The integral representation theorem of Choquet-Bishop-de Leeuw, which has found numerous applications throughout mathematics, is undoubtedly the most well known result in the theory. It asserts that every point in a compact convex set can be represented by a probability measure supported on the extreme points of the set. However, this result is just one piece of classical Choquet theory, which is now a very powerful framework for the analysis of convex sets.

Many objects in mathematics, especially in the theory of operator algebras, exhibit "higher order" convex structure. Various attempts have been made to capture this structure within an abstract framework, most notably in Wittstock's [51] theory of matrix convexity. However, each of these frameworks suffers from the same serious issue: the non-existence of a suitable notion of extreme point.

In this paper we introduce a new theory of noncommutative convexity that we believe finally resolves this issue. The central objects of interest in the theory are noncommutative convex sets, for which there is a robust notion of extreme point. Working within this framework, we establish analogues of the fundamental results from classical convexity theory, along with a corresponding theory of noncommutative functions. We then apply these ideas to develop a corresponding noncommutative Choquet theory that generalizes much of classical Choquet theory. For example, we obtain a noncommutative generalization of the Choquet-Bishop-de Leeuw integral representation theorem for points in compact noncommutative convex sets.

An nc (noncommutative) convex set over an operator space E is a graded set $K = \coprod_n K_n$, with each K_n consisting of $n \times n$ matrices over

E. The graded components K_n are related by requiring that K be closed under direct sums and compressions by isometries. The union is taken over all cardinal numbers $n \leq \kappa$, where κ is a fixed infinite cardinal number depending on E. The fact that n is permitted to be infinite here is an essential part of the theory, since even if K is completely determined by its finite dimensional part, the finite part $\coprod_{n \in \mathbb{N}} K_n$ of K may not contain any extreme points at all.

For example, if A is a separable unital C*-algebra, then the nc state space K of A is a (compact) nc convex set over A^* defined by $K = \coprod_{n \leq \aleph_0} K_n$ with $K_n = \{\varphi : A \to \mathcal{B}(H_n)$ unital and completely positive$\}$, where H_n is a fixed Hilbert space of dimension n and $\mathcal{B}(H_n)$ denotes the C*-algebra of bounded operators on H_n. The extreme points ∂K of K are precisely the irreducible representations of A, and they completely determine K in the sense that every point in K is a limit of nc convex combinations of points in ∂K. Yet if A is simple and infinite dimensional, e.g. if A is the Cuntz algebra $\mathcal{O}_2$, then it has no finite dimensional representations, so in this case ∂K has empty intersection with the finite part of K.

This marks the key point of divergence from the theory of matrix convexity which, on the surface, resembles the theory of noncommutative convexity, but does not allow points corresponding to infinite matrices. As the previous example demonstrates, it is for precisely this reason that there is no suitable notion of extreme point in the matrix convex setting. We will see that this results in major differences between the theory of noncommutative convexity and the theory of matrix convexity.

The fundamental idea underlying classical Choquet theory is the dual equivalence between the category of compact convex sets and the category of function systems. The functor implementing this duality maps a compact convex set C to the corresponding function system $\mathrm{A}(C)$ of continuous affine functions on C, while the inverse functor maps a function system to its state space. This result is Kadison's [29] representation theorem.

An analogous result holds in the noncommutative setting. The category of compact nc convex sets is dually equivalent to the category of operator systems, which are closed unital self-adjoint subspaces of C*-algebras. The functor implementing this duality maps a compact nc convex set K to the corresponding operator system $\mathrm{A}(K)$ of continuous nc affine functions on K. The inverse functor maps an operator system S to its noncommutative state space $K = \coprod K_n$, where as above, $K_n = \{\varphi : S \to \mathcal{B}(H_n)$ unital and completely positive$\}$. In particular, S is completely order isomorphic to the operator system $\mathrm{A}(K)$ of continuous nc affine functions on K, providing a noncommuative analogue of Kadison's representation theorem. A similar result was obtained in the matrix convex setting by Webster and Winkler [48, Proposition 3.5].

For a compact convex set C, the C*-algebra $\mathrm{C}(C)$ of continuous functions on C is generated by the function system $\mathrm{A}(C)$. A probability measure μ on C is said to represent a point $x \in C$ and x is said to be the barycenter of μ if the restriction $\mu|_{\mathrm{A}(C)}$ satisfies $\mu|_{\mathrm{A}(C)} = x$. Since the point mass δ_x represents x, every point in C has at least one representing measure. The points $x \in C$ for which δ_x is the unique representing measure are precisely the extreme points of C. This interplay between the function system $\mathrm{A}(C)$ and the C*-algebra $\mathrm{C}(C)$ plays an essential role in classical Choquet theory.

Something similar is true in the noncommutative setting, and it is here that major differences begin to appear between the theory of noncommutative convexity and the theory of matrix convexity.

For a compact nc convex set K, we introduce a notion of nc function on K. The space $\mathrm{C}(K)$ of continuous nc functions on K is a C*-algebra that is generated by the space $\mathrm{A}(K)$ of continuous nc affine functions on K. By applying Takesaki and Bichteler's noncommutative Gelfand theorem [5, 44], we identify $\mathrm{C}(K)$ with the maximal C*-algebra $\mathrm{C}^*_{\max}(\mathrm{A}(K))$ of $\mathrm{A}(K)$ introduced by Kirchberg and Wasserman [31].

Motivated by the classical setting, we say that a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{B}(H_n)$ represents a point $x \in K_n$ and that x is the barycenter of μ if the restriction $\mu|_{\mathrm{A}(K)}$ satisfies $\mu|_{\mathrm{A}(K)} = x$. The corresponding point evaluation $\delta_x : \mathrm{C}(K) \to \mathcal{B}(H_n)$ represents x, so every point in K has at least one representing map. As in the classical setting, the points in $x \in K$ for which δ_x is both irreducible and the unique representing map for x are precisely the extreme points of K.

In fact, this characterization of the extreme points of a compact nc convex set K implies that they are dual to the boundary representations of the operator system $\mathrm{A}(K)$ in the sense of Arveson [2]. Hence viewed from the perspective of noncommutative convexity, Arveson's conjecture about the existence of boundary representations is equivalent to the existence of extreme points in compact nc convex sets. This conjecture was resolved only recently, by Arveson himself [3] in the separable case, and by the authors [12] in complete generality. As an application of the ideas in this paper, we obtain a new proof of this result that is conceptually much different.

We establish a noncommutative Krein-Milman theorem asserting that a compact nc convex set is the closed nc convex hull of its extreme points, as well as an analogue of Milman's partial converse to the Krein-Milman theorem. In the matrix convex setting, Webster and

Winkler [48] obtained variants of these results for "matrix extreme points." However, we will see that even in the special case that a matrix convex set is generated by points that are extreme in the sense of noncommutative convexity, there are generally many more matrix extreme points, meaning that our results are much stronger.

A key technical tool in classical Choquet theory is the notion of convex envelope of a continuous nc function. For a compact convex set C and a real-valued continuous function $f \in \mathrm{C}(C)$, the convex envelope $\bar{f}$ of f is defined by $\bar{f} = \sup\{a \in \mathrm{A}(C)_{sa} : a \leq f\}$. It is the best approximation of f from below by a real-valued lower semicontinuous convex function. In particular, $\bar{f} = f$ if and only if f is convex.

In the noncommutative setting, we introduce a notion of convex nc function along with a corresponding notion of convex envelope of a continuous nc function. As in the classical setting, the convex envelope is a key technical tool. For a compact nc convex set K, the convex envelope $\bar{f}$ of an nc function $f \in \mathrm{C}(K)$ is the the best approximation from below by a lower semicontinuous convex nc function. However, since $\mathrm{C}(K)$ is generally not a lattice, $\bar{f}$ is necessarily a multivalued function, and this introduces some technical difficulties. It is a non-trivial theorem that continuous convex nc functions can be approximated from below by the continuous affine nc functions that they dominate.

For example, if $I \subseteq \mathbb{R}$ is a compact interval and $K = \coprod_{n \leq \aleph_0} K_n$ is the compact nc convex set defined by letting K_n denote the set of self-adjoint operators in $\mathrm{B}(H_n)$ with spectrum in I, then the convex nc functions on K can be identified with the operator convex functions on I. We obtain a noncommutative analogue of Jensen's inequality that specializes in this case to the Hansen-Pedersen-Jensen inequality [27].

For a compact convex set C, the classical Choquet order on the space of probability measures on C is a generalization of the even more classical majorization order considered by e.g. Hardy, Littlewood and Pòlya. For probability measures μ and ν on C, ν is said to dominate μ in the Choquet order if $\int_C f\, d\mu \leq \int_C f\, d\nu$ for every convex function $f \in \mathrm{C}(C)$. A probability measure is maximal in the Choquet order precisely when it is supported on the extreme boundary ∂C in an appropriate sense.

For a compact nc convex set K, we introduce two orders on the unital completely positive maps on $\mathrm{C}(K)$. The nc Choquet order is analogous to the classical Choquet order. It is determined by comparing the values of the maps on the set of convex nc functions in $\mathrm{C}(K)$. As in the classical case, a map is maximal in the nc Choquet order precisely when it is supported on the extreme boundary ∂K in an appropriate sense.

The nc dilation order, determined by comparing the set of dilations of the maps, has no classical counterpart. However, using the theory of convex envelopes of convex nc functions, we show that it coincides with the nc Choquet order. This result has a number of interesting consequences. For example, we obtain an intrinsic characterization of unital completely positive maps on operator systems that have a unique completely positive extension to the C*-algebra generated by the operator system. A version of this order in the commutative setting was used in [13].

The culmination of this paper is a noncommutative analogue of the integral representation theorem of Choquet-Bishop-de Leeuw [8, 10] We show that if K is a compact nc convex set, then every point $x \in K$ has a representing map μ that is supported on the extreme boundary of K in an appropriate sense. As in the classical setting, if $\mathrm{A}(K)$ is non-separable, then the extreme boundary ∂K of K may not be a Borel set. In this case we show that if f is nc function contained in the Baire-Pedersen enveloping C*-algebra of $\mathrm{C}(K)$ that vanishes on the extreme points of K, then $\mu(f) = 0$. In the separable case, we obtain an integral representation theorem expressing a unital completely positive map on $\mathrm{C}(K)$ as an integral against a unital completely positive map-valued probability measure supported on the extreme boundary of K.

Acknowledgements

The authors thank Ben Passer, Gregory Patchell and Eli Shamovich for their feedback during the writing of this paper. They also thank Bojan Magajna and the working group at IMPAN run by Tatiana Shulman and Adam Skalski for a number of helpful comments and corrections.

2. Noncommutative convex sets

2.1. Operator spaces, cardinality, dimension and topology.

We will work with operator spaces and operator systems throughout this paper. In this section, we briefly review some of the relevant technical details and introduce some notation and conventions. For detailed references on operator spaces we direct the reader to the books of Effros and Ruan [19] and Pisier [41]. In particular, the details on infinite matrices over operator systems are contained in [19, Section 10.1]. For a detailed reference on operator systems, we direct the reader to the book of Paulsen [37].

Let E be an operator space. For nonzero cardinal numbers m and n, we let $\mathcal{M}_{m,n}(E)$ denote the operator space consisting of $m \times n$ matrices over E with uniformly bounded finite submatrices. If $m = n$, then we let $\mathcal{M}_n(E) = \mathcal{M}_{n,n}(E)$. If $E = \mathbb{C}$, then we let $\mathcal{M}_{m,n} = \mathcal{M}_{m,n}(\mathbb{C})$ and $\mathcal{M}_n = \mathcal{M}_{n,n}$.

For each n, we fix a Hilbert space H_n of dimension n and identify $\mathcal{M}_{m,n}$ with the space $\mathcal{B}(H_n, H_m)$ of bounded operators from H_n to H_m.

Let m, n, p be nonzero cardinal numbers. For $x = [x_{ij}] \in \mathcal{M}_n(E)$, $\alpha = [\alpha_{ij}] \in \mathcal{M}_{m,n}$ and $\beta = [\beta_{ij}] \in \mathcal{M}_{n,p}$, the products $\alpha x \in \mathcal{M}_{m,n}(E)$, $x\beta \in \mathcal{M}_{n,p}(E)$ and $\alpha x \beta \in \mathcal{M}_{k,n}(E)$ can be defined as compositions under appropriate operator space embeddings. They can also be defined intrinsically by the formulas

$$[\alpha x]_{ij} = \sum_k \alpha_{ik} x_{kj}, \quad [x\beta]_{ij} = \sum_k x_{ik}\beta_{kj},$$

since the above series converge unconditionally, and

$$[\alpha x \beta]_{ij} = \lim_F \sum_{k \in F} \alpha_{ik}(x\beta)_{kj} = \lim_F \sum_{k \in F} (\alpha x)_{ik} \beta_{kj},$$

where the limits are taken over finite subsets F of n.

We let $\mathcal{M}(E) = \coprod_n \mathcal{M}_n(E)$, where the disjoint union is taken over all nonzero cardinal numbers $n \le \kappa$, where κ is a sufficiently large cardinal number. If $E = \mathbb{C}$, then we let $\mathcal{M} = \mathcal{M}(\mathbb{C})$. For a subset $X \subseteq \mathcal{M}(E)$ and a nonzero cardinal $n \le \kappa$, we will write $X_n = X \cap \mathcal{M}_n(E)$.

The existence of an upper bound κ is necessary to ensure that $\mathcal{M}(E)$ is a set. However, it will be convenient to allow κ to vary depending on the context. If we are considering finitely many operator spaces $E_1, \dots, E_n$, then it will suffice to take $\kappa = \dim H$, where H is a Hilbert space of minimal infinite dimension such that $E_1, \dots, E_n$ embed completely isometrically into $\mathcal{B}(H)$. In particular, if $E_1, \dots, E_n$ are separable, then it will suffice to take $\kappa = \aleph_0$. In practice, we will work with the understanding that κ exists and simply write e.g. "for all n" instead of "for all $n \le \kappa$."

If E is a dual operator space with a distinguished predual E_*, then there is a natural operator space isomorphism

$$\mathcal{M}_n(E) \cong \mathrm{CB}(E_*, \mathcal{M}_n),$$

where $\mathrm{CB}(E_*, \mathcal{M}_n)$ denotes the space of completely bounded maps from E_* to $\mathcal{M}_n$. The space $\mathrm{CB}(E_*, \mathcal{M}_n)$ is a dual operator space and the corresponding weak* topology is the point-weak* topology. We identify $\mathcal{M}_n(E)$ and $\mathrm{CB}(E_*, \mathcal{M}_n)$ and equip $\mathcal{M}_n(E)$ with the point-weak* topology. Note that this is the usual weak* topology on $\mathcal{M}_n$.

Unless otherwise specified, the convergence of a net or a series in $\mathcal{M}_n(E)$ will always be with respect to the point-weak* topology. For example, we will frequently use the fact that for any bounded family $\{x_i \in \mathcal{M}_n(E)\}$ and any family $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$, the sum $\sum \alpha_i^* x_i \alpha_i$ converges in $\mathcal{M}_n(E)$.

If E and F are operator spaces, then the product $E \times F$ is an operator space. If E and F are dual operator spaces with distinguished preduals E_* and F_* respectively, then $E \times F = (E_* \times_1 F_*)^*$, so that $E \times F$ is a dual operator space with the distinguished predual $E_* \times_1 F_*$.

At various points throughout this paper we will review results from classical convexity theory and classical Choquet theory. In particular, we will discuss function systems, also known as archimedean order unit spaces, which are classical precursors to operator systems. For a detailed reference on classical Choquet theory, we refer the reader to the books of Alfsen [1], Phelps [40] and Lukeš-Malý-Netuka-Spurný [34]. For a modern perspective on function systems, we refer the reader to the recent paper of Paulsen and Tomforde [38].

2.2. **Noncommutative convex sets.**

Definition 2.2.1. An *nc convex set* over an operator space E is a graded subset $K = \coprod K_n \subseteq \mathcal{M}(E)$ that is closed under direct sums and compressions, meaning that

(1) $\sum \alpha_i x_i \alpha_i^* \in K_n$ for every bounded family $\{x_i \in K_{n_i}\}$ and every family of isometries $\{\alpha_i \in \mathcal{M}_{n,n_i}\}$ satisfying $\sum \alpha_i \alpha_i^* = 1_n$,
(2) $\beta^* x \beta \in K_m$ for every $x \in K_n$ and every isometry $\beta \in \mathcal{M}_{n,m}$.

We will say that K is *closed* if E is a dual operator space and each K_n is closed in the topology on $\mathcal{M}_n(E)$. Similarly, we will say that K is *compact* if each K_n is compact in the topology on $\mathcal{M}_n(E)$.

Remark 2.2.2. Condition (1) is equivalent to the assertion that any unitary that conjugates $\oplus \mathcal{M}_{n_i}$ into $\mathcal{M}_n$ necessarily conjugates $\oplus K_{n_i}$ into K_n. Condition (2) implies that K is closed under compression to subspaces, and in particular that each K_n is closed under unitary conjugation. Note that each K_n is an (ordinary) convex set.

Remark 2.2.3. As discussed in Section 2.1, there is an infinite cardinal number κ such that $K = \coprod_{n \leq \kappa} K_n$. However, it will be convenient to work with the understanding that κ exists without necessarily mentioning it explicitly.

Example 2.2.4. A simple example of a compact nc convex set is a compact operator interval. Fix $c, d \in \mathbb{R}$ with $c < d$. For $n \in \mathbb{N}$, let

$K_n = [c1_n, d1_n]$, where

$$[c1_n, d1_n] = \{\alpha \in (\mathcal{M}_n)_{sa} : c1_n \le \alpha \le d1_n\}.$$

Then $K = \coprod_{n\in\mathbb{N}} K_n$ is a compact matrix convex set over $\mathbb{C}$. It is not difficult to show that if $L = \coprod_{n\in\mathbb{N}} L_n$ is a compact nc convex set with $K_1 = [c, d]$, then $L = K$.

Example 2.2.5. Let E be a dual operator space. The space $\mathcal{M}(E)$ is a closed nc convex set. For each n, let $\mathbb{B}_n(E)$ denote the unit ball of $\mathcal{M}_n(E)$ and let $\mathbb{B}(E) = \coprod_n \mathbb{B}_n(E)$, where the union is taken over cardinal numbers $n \le \kappa$ for a sufficiently large infinite cardinal number κ as discussed in Section 2.1. Each $\mathbb{B}_n(E)$ is compact in $\mathcal{M}_n(E)$, so $\mathbb{B}(E)$ is a compact nc convex set.

Example 2.2.6. Let S be an operator system, i.e. a unital self-adjoint subspace of a unital C*-algebra. The *nc state space* of S is the nc convex set $K = \coprod_n K_n$, where $K_n = \mathrm{UCP}(S, \mathcal{M}_n)$ and the union is taken over cardinal numbers $n \le \kappa$ for a sufficiently large infinite cardinal number κ as discussed in Section 2.1. Recall that for each n, we have identified $\mathcal{M}_n(S^*)$ with the space $\mathrm{CB}(S, \mathcal{M}_n)$. Hence the inclusion $K_n = \mathrm{UCP}(S, \mathcal{M}_n) \subseteq \mathrm{CB}(S, \mathcal{M}_n)$ implies the inclusion $K_n \subseteq \mathcal{M}_n(S^*)$. Moreover, K_n is compact in the point-weak* topology. So K is a compact nc convex set over S^*.

Definition 2.2.7. Let E be a dual operator space. For a bounded family $\{x_i \in \mathcal{M}_{n_i}(E)\}$ and a family $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$, we will refer to the element $\sum \alpha_i^* x_i \alpha_i \in \mathcal{M}_n$ as an *nc convex combination* of elements in K. We will say that a subset $K \subseteq \mathcal{M}(E)$ is closed under nc convex combinations if every nc convex combination of elements in K belongs to K.

Proposition 2.2.8. *Let E be a dual operator space. A subset $K \subseteq \mathcal{M}(E)$ is an nc convex set if and only if it is closed under nc convex combinations.*

Proof. Since the expressions in conditions (1) and (2) in Definition 2.2.1 are special cases of nc convex combinations, if K is closed under nc convex combinations, then it is clearly an nc convex set.

Conversely, suppose that K is an nc convex set and consider a bounded family $\{x_i \in K_{n_i}\}$ and a family $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$. Let $m = \sum n_i$ and let $\beta_i \in \mathcal{M}_{m,n_i}$ be isometries such that $\sum \beta_i \beta_i^* = 1_m$. Let $\gamma = \sum \beta_i \alpha_i \in \mathcal{M}_{m,n}$. Then $\gamma^*\gamma = \sum \alpha_i^* \alpha_i = 1_n$, so γ is an isometry. By (1), $y = \sum \beta_i x_i \beta_i^* \in K_m$. Hence by (2),

$$x = \gamma^* y \gamma = \sum_i \alpha_i^* \beta_i^* \Big(\sum_j \beta_j x_j \beta_j^* \Big) \sum_k \beta_k \alpha_k = \sum_i \alpha_i^* x_i \alpha_i \in K_n.$$

Here we used the fact that $\beta_i^* \beta_j = 0$ if $i \neq j$. □

The next result shows that while closed nc convex sets are closed on each level, they are also closed in a stronger sense.

Proposition 2.2.9. *Let K be a closed nc convex set over a dual operator space E. Suppose there is a net $\{x_i \in K_{n_i}\}$ and a net of isometries $\{\alpha_i \in \mathcal{M}_{n,n_i}\}$ satisfying $\lim \alpha_i \alpha_i^* = 1_n$ such that $\lim \alpha_i x_i \alpha_i^* = x \in \mathcal{M}_n(E)$. Then $x \in K_n$.*

Proof. Let $\{\beta_i \in \mathcal{M}_{n,m_i}\}$ be a net of isometries satisfying $\beta_i \beta_i^* = 1_n - \alpha_i \alpha_i^*$. Then $\lim \beta_i \beta_i^* = 0$. Fix $y \in K_1$ and let $z_i = \alpha_i x_i \alpha_i^* + \beta_i (y \otimes 1_{m_i}) \beta_i^*$. Then by (1) of Definition 2.2.1, $z_i \in K_n$, and by construction, $\lim z_i = x$. Hence $x \in K_n$ since K is closed. □

The next result shows that closed nc convex sets are completely determined by their finite levels.

Proposition 2.2.10. *Let K and L be closed nc convex sets over an operator system E. If $K_n = L_n$ for $n < \infty$, then $K = L$.*

Proof. For arbitrary n and $x \in K_n$, choose a net of finite rank isometries $\{\alpha_i \in \mathcal{M}_{n,n_i}\}$ such that $\lim \alpha_i \alpha_i^* = 1_n$ and let $x_i = \alpha_i^* x \alpha_i \in K_{n_i}$. Then $\lim \alpha_i x_i \alpha_i^* = x$. Since $K_{n_i} = L_{n_i}$ for each i, Proposition 2.2.9 implies $x \in L$. Hence $K \subseteq L$. By symmetry, $K = L$. □

2.3. **Matrix convexity.** In this section we briefly pause to discuss the relationship between the theory of noncommutative convexity and the theory of matrix convexity introduced by Wittstock [50]. We will also briefly mention the theory of C*-convexity introduced by Hoppenwasser, Moore and Paulsen [28].

At least on the surface, the definition of a matrix convex set is similar to the definition of an nc convex set. The key distinction is that matrix convex sets do not contain points corresponding to infinite matrices. Specifically, a matrix convex M set over an operator space E is a graded subset $M = \coprod_{n \in \mathbb{N}} M_n \subseteq \mathcal{M}(E)$. If E is a dual operator space, then M is said to be closed (resp. compact) if each M_n is closed (resp. compact) in the topology on $\mathcal{M}_n(E)$.

If K is a nc convex set, then the finite part $K_f := \coprod_{n \in \mathbb{N}} K_n$ is a matrix convex set. On the other hand, if M is a closed matrix convex set, then Proposition 2.2.10 implies that M determines a unique closed nc convex set. In fact, we will obtain results in Section 3 that imply the category of compact matrix convex sets is equivalent to the category of compact nc convex sets.

Nevertheless, we will see that there are major differences between the theory of noncommutative convexity and the theory of matrix convexity. This will become particularly apparent when we begin to develop noncommutative Choquet theory, where it will be essential to consider points corresponding to infinite matrices as first class objects.

There are two key reasons for this. First, beginning in Section 4, a major part of the noncommutative theory will involve the study of functions on nc convex sets. We will see that, even for reasonably nice functions, the restriction to the finite part of the set will not necessarily completely determine the function.

Second, when we introduce the notion of an extreme point for an nc convex set in Section 6, we will establish a noncommutative Krein-Milman theorem, along with an analogue of Milman's partial converse to the Krein-Milman theorem, showing that the set of extreme points in a compact nc convex set is a minimal generating set in a very strong sense. However, we will also see that the finite part of a compact nc convex set, even for simple examples, may not contain any extreme points at all.

To be more specific, in classical convexity theory, the set of extreme points in a compact convex set is a minimal generating set in a sense that can be made precise using the Krein-Milman theorem and Milman's partial converse to the Krein-Milman theorem. If C is a compact convex set and ∂C denotes the extreme points of C, then the Krein-Milman theorem asserts that C is the closed convex hull of ∂C. If $D \subseteq C$ is a closed subset with the property that the closed convex hull of D is C, then Milman's partial converse to the Krein-Milman theorem asserts that $\partial C \subseteq D$. This property of minimality underlies much of classical convexity theory, and is absolutely essential for the development of classical Choquet theory.

There is a notion of "matrix extreme point" in the theory of matrix convexity for which a Krein-Milman theorem holds. This was proved by Webster and Winkler [48] (see [23] for another proof), along with an analogue of Milman's partial converse to the Krein-Milman theorem. Their result extended a Krein-Milman theorem for C*-extreme points in the matrix state space of a C*-algebra proved earlier by Morenz [36] and Farenick-Morenz [24].

However, the set of matrix extreme points in a compact matrix convex set is generally not a minimal generating set in any meaningful sense. Simple examples show that if M is a compact matrix convex set and ∂M denotes the set of matrix extreme points in M, then it is possible for the closed convex hull of a much smaller subset of ∂M to be equal to M. The main problem is that it is possible for the matrix convex hull of a single matrix extreme point in M_n to contain matrix extreme points in M_m for $m < n$.

There have been attempts to work with a more restricted notion of extreme point in the matrix convex setting. For example, Kleski [32] defined a notion of "absolute extreme point" for matrix convex sets, and proved a corresponding Krein-Milman theorem for state spaces of operator systems that can be represented on finite dimensional Hilbert space. More recently, Evert, Helton, Klep and McCullough [22] proved a similar result for a special class of compact matrix convex sets called real spectrahedra.

In fact, we will see that these results are a special case of the noncommutative Krein-Milman theorem for extreme points in compact nc convex sets. In particular, the fact that the finite part of a compact nc convex set does not necessarily contain any extreme points implies that no general version of these results can hold within the framework of matrix convexity. Instead, it is necessary to work with the framework of noncommutative convexity.

2.4. **Noncommutative separation theorem.** In this section we prove a separation theorem for nc convex sets that extends the separation theorem for matrix convex sets of Effros and Winkler [21].

Let E and F be operator spaces and let $\varphi : E \to F$ be a linear map. We write φ_n for the induced map $\varphi_n : \mathcal{M}_n(E) \to \mathcal{M}_n(F)$ defined by

$$\varphi_n([e_{ij}]) = [\varphi(e_{ij})], \quad \text{for} \quad [e_{ij}] \in \mathcal{M}_n(E).$$

If E and F are operator systems, then the adjoint $\varphi^* : E \to F$ is defined by $\varphi^*(e) = \varphi(e^*)^*$. We say that φ is self-adjoint if $\varphi = \varphi^*$. If φ is self-adjoint then it maps self-adjoint elements to self-adjoint elements.

Theorem 2.4.1 (Noncommutative separation theorem)**.**
Let K be a closed nc convex set over a dual operator space E with $0_E \in K$. Suppose there is n and $y \in \mathcal{M}_n(E)$ such that $y \notin K_n$. Then there is a normal completely bounded linear map $\varphi : E \to \mathcal{M}_n$ such that

$$\operatorname{Re} \varphi_n(y) \not\leq 1_n \otimes 1_n \quad \textit{but} \quad \operatorname{Re} \varphi_p(x) \leq 1_p \otimes 1_n$$

for all p and $x \in K_p$. Moreover, if E is an operator system and $K \cup \{y\}$ consists of self-adjoint elements, then φ can be chosen self-adjoint.

Proof. First suppose $n \in \mathbb{N}$. Then by the Effros-Winkler separation theorem [21, Theorem 5.4], there is a continuous linear map $\varphi : E \to M_n$ such that $\operatorname{Re} \varphi_m(x) \leq 1_m \otimes 1_n$ for all $m \in \mathbb{N}$ and $x \in K_p$ but $\operatorname{Re} \varphi_n(y) \not\leq 1_n \otimes 1_n$. For arbitrary p and $x \in K_p$, it follows from above that for $m \in \mathbb{N}$ and an isometry $\alpha \in \mathcal{M}_{p,m}$,

$$\alpha^* \varphi_p(x) \alpha = \varphi_m(\alpha^* x \alpha) \leq 1_m \otimes 1_n.$$

Hence $\varphi_p(x) \leq 1_p \otimes 1_n$.

Since φ is continuous with respect to the weak* topology on E, it is bounded. Furthermore, since n is finite, it follows from a result of Smith [43, Theorem 2.10] that φ is completely bounded.

For infinite n, we can consider y as the point-weak-$*$ limit of the net of finite dimensional compressions. If each of these compressions was in K, then arguing as in Proposition 2.2.9 would imply $y \in K$. Since this is not the case, there is $m \in \mathbb{N}$ and a compression $z \in \mathcal{M}_m$ such that $z \notin K_m$.

Applying the above construction to z, we obtain a map $\psi : E \to M_m$ such that $\operatorname{Re} \psi_p(x) \leq 1_p \otimes 1_m$ for all p and $x \in K_p$, but $\operatorname{Re} \psi_m(z) \not\leq 1_m \otimes 1_m$. Then $\psi_n(y) \not\leq 1_n \otimes 1_m$ since $\psi_m(z)$ is a compression of $\psi_n(y)$. Hence we can take φ to be an infinite amplification of ψ.

If E is self-adjoint and $K \cup \{y\}$ consist of self-adjoint elements, then we can replace φ with $\frac{1}{2}(\varphi + \varphi^*)$. $\square$

The next result follows immediately from Theorem 2.4.1 by applying a translation.

Corollary 2.4.2. *Let K be a closed nc convex set over a dual operator space E. Suppose there is n and $y \in \mathcal{M}_n(E)$ such that $y \notin K_n$. Then there is a normal completely bounded linear map $\varphi : E \to \mathcal{M}_n$ and self-adjoint $\gamma \in \mathcal{M}_n$ such that*

$$\operatorname{Re} \varphi_n(y) \not\leq \gamma \otimes 1_n \quad \textit{but} \quad \operatorname{Re} \varphi_p(x) \leq \gamma \otimes 1_p$$

for all p and $x \in K_p$. Furthermore, if E is an operator system and $K \cup \{y\}$ consists of self-adjoint elements, then φ can be chosen self-adjoint.

2.5. **Noncommutative affine maps.**

Definition 2.5.1. Let K and L be nc convex sets over operator spaces E and F respectively. We say that a map $\theta : K \to L$ is an *affine nc map* if it is graded, respects direct sums and is equivariant with respect to isometries, meaning that

(1) $\theta(K_n) \subseteq L_n$ for all n,
(2) $\theta(\sum \alpha_i x_i \alpha_i^*) = \sum \alpha_i \theta(x_i) \alpha_i^*$ for every bounded family $\{x_i \in K_{n_i}\}$ and every family of isometries $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i \alpha_i^* = 1_n$,
(3) $\theta(\alpha^* x \alpha) = \alpha^* \theta(x) \alpha$ for every $x \in K_m$ and every isometry $\alpha \in \mathcal{M}_{m,n}$.

We say that θ is *continuous* if the restriction $\theta|_{K_n}$ is continuous for every n, and we say that θ is *bounded* if $\|\theta\|_\infty < \infty$, where $\|\theta\|_\infty$ is the *uniform norm* defined by

$$\|\theta\|_\infty = \sup_{x \in K} \|\theta(x)\|.$$

We say that θ is a *homeomorphism* and that K and L are *affinely homeomorphic* if θ has a continuous affine inverse. We let $\mathrm{A}(K, L)$ denote the space of continuous affine nc maps from K to L. We let $\mathrm{A}(K) = \mathrm{A}(K, \mathcal{M})$, and we refer to $\mathrm{A}(K)$ as the space of *continuous affine nc functions* on K.

Remark 2.5.2. Arguing as in Proposition 2.2.8, we see that continuous affine nc maps between closed nc convex sets respect nc convex combinations. Specifically, if K and L are closed nc convex sets and $\theta : K \to L$ is a continuous affine nc map, then

$$\theta\left(\sum \alpha_i^* x_i \alpha_i\right) = \sum \alpha_i^* \theta(x_i) \alpha_i.$$

for a bounded family $\{x_i \in K_{n_i}\}$ and a family $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$.

Proposition 2.5.3. *Let K and L be compact nc convex sets and let $\theta : K \to L$ be a continuous affine nc map. Then θ is bounded with*

$$\|\theta\| = \|\theta|_{K_{\aleph_0}}\| = \sup_{n < \infty} \|\theta|_{K_n}\|.$$

Proof. For each n, since K_n is compact and $\theta|_{K_n}$ is continuous and affine, $\|\theta|_{K_n}\| < \infty$. Moreover, it is clear that $\|\theta|_{K_n}\|$ is an increasing function of n. Hence $\sup_{m<\infty} \|\theta|_{K_m}\| \leq \|\theta|_{K_{\aleph_0}}\| \leq \|\theta\|$.

To obtain the reverse inequalities, we argue as in the proof of Proposition 2.2.9. Fix arbitrary n and $x \in K_n$. Let $\{\alpha_i \in \mathcal{M}_{n,n_i}\}$ be a net of finite rank isometries satisfying $\lim \alpha_i \alpha_i^* = 1_n$. Let $\{\beta_i \in \mathcal{M}_{n,m_i}\}$ be a net of isometries satisfying $\beta_i \beta_i^* = 1_n - \alpha_i \alpha_i^*$. Then $\lim \beta_i \beta_i^* = 0$.

Let $x_i = \alpha_i^* x \alpha_i \in K_{n_i}$. Fix $y \in K_0$ and let $z_i = \alpha_i x_i \alpha_i^* + \beta_i (y \otimes 1_{m_i}) \beta_i^*$. Then by (1) of Definition 2.2.1, $z_i \in K_n$, and from above, $\lim z_i = x$. Hence by (2) of Definition 2.5.1,

$$\theta(x) = \lim \theta(z_i) = \lim \alpha_i \theta(x_i) \alpha_i^* + \beta_i \theta(y \otimes 1_{m_i}) \beta_i^* = \lim \alpha_i \theta(x_i) \alpha_i^*.$$

Therefore, $\|\theta|_{K_n}\| \leq \sup_{m<\infty} \|\theta|_{K_m}\| \leq \|\theta|_{K_{\aleph_0}}\|$. $\square$

We will now make the assumption that $K \subset \mathcal{M}(E)$ where E is a dual operator system, and that K is compact with respect to the weak-$*$ topology induced from $\mathcal{M}(E)$.

Lemma 2.5.4. *Let K be a compact nc convex. Then $\mathrm{A}(K)$ separates the points of K.*

Proof. By the remarks in Section 2.1, the weak-$*$ topology corresponds to the point-weak-$*$ topology induced by $\mathcal{M}(E_*)$ obtained by using the identification between $\mathcal{M}_n(E)$ and $\mathrm{CB}(E_*, \mathcal{M}_n)$. In particular, $\mathcal{M}(E_*)$ is contained in $\mathrm{A}(K)$. Clearly E_* separates points on E and hence on K_1. But two elements of $x, y \in K_n$ are equal if and only if $\alpha^* x\alpha = \alpha^* y\alpha$ for all $\alpha \in \mathcal{M}_{1,n}$. Since these are separated by E_*, it follows that $M_n(E_*)$ separates points of K_n for each n. □

We will need to consider two natural topologies on K induced by the functions in $\mathrm{A}(K)$.

Definition 2.5.5. The *point-weak* topology* on K is the weakest topology that makes every affine nc function $a \in \mathrm{A}(K)$ continuous. Since each $\mathcal{M}_n$ is equipped with the weak-$*$ topology, this is the weakest topology that makes the maps $K_n \to \mathbb{C} : x \to \langle a(x)\xi, \eta\rangle$ continuous on K_n for all $a \in \mathrm{A}(K)$, $n \le \kappa$ and $\xi, \eta \in H_n$.

The *point-strong topology* on K is the weakest topology that makes the maps $K_n \to H_n : x \to a(x)\xi$ continuous on K_n for all $a \in \mathrm{A}(K)$, $n \le \kappa$ and $\xi \in H_n$.

Remark 2.5.6. Since K is bounded, the point-weak* and point-weak operator topologies will coincide. Similarly, the point-strong topology will coincide with the point-ultrastrong topology and, since $\mathrm{A}(K)$ is self-adjoint, the point-ultrastrong* topology. The point-ultrastrong* topology will come up when we discuss the C*-algebra generated by $\mathrm{A}(K)$.

Lemma 2.5.7. *Let K be a compact nc convex set over a dual operator space E. The topology on K coincides with the point-weak* topology.*

Proof. Since $\mathrm{A}(K)$ consists of continuous functions on $(K, \text{weak*})$ and the point-weak* topology is the weakest topology making these functions continuous, the identity map on K is continuous from the weak* topology to the point-weak* topology. The continuous affine nc functions separate points of K by Lemma 2.5.4, and thus the point-weak* topology is Hausdorff. Since K is compact in the weak* topology, it follows that this map is a homeomorphism. □

We now make a useful observation about $\mathrm{A}(K)$.

Theorem 2.5.8. *Let K be a compact nc convex set over a dual operator space E. An affine nc function on K is continuous if and only if it is continuous on K_1.*

Proof. An affine nc function a on K satisfies $a(\alpha^* x\alpha) = \alpha^* a(x)\alpha$ for an isometry $\alpha \in \mathcal{M}_{n,m}$ and $x \in K_n$. The since a unit vector $\xi \in H_n$ can be viewed as an isometry in $M_{1,n}$,

$$\langle a(x)\xi, \xi \rangle = \xi^* a(x) \xi = a(\xi^* x \xi).$$

If a is continuous on K_1, then the function $K_n \to \mathbb{C} : x \to a(\xi^* x \xi)$ is continuous for all $n \leq \kappa$ and $\xi \in H_n$. Hence it follows from the polarization identity that a is point-weak* continuous on each K_n. By Lemma 2.5.7, the point-weak* topology coincides with the weak* topology. Therefore, a is continuous in the weak* topology. $\square$

3. Categorical duality

3.1. **Convex sets and function systems.** If C is a compact convex set, then the space $\mathrm{A}(C)$ of complex-valued continuous affine functions on C is a function system, also referred to as an archimedean order unit space in the literature. This means that it is an ordered complex $*$-vector space with a distinguished archimedean order unit [38]. Specifically, the order on $\mathrm{A}(C)$ is determined by the positive cone $\mathrm{A}(C)^+$ consisting of positive continuous affine functions on C. The *-operation on $\mathrm{A}(C)$ is defined by conjugation and the order unit on $\mathrm{A}(C)$ is the constant function $1_{\mathrm{A}(C)}$. The state space of $\mathrm{A}(C)$, consisting of positive unital functionals in the dual of $\mathrm{A}(C)$, is compact with respect to the weak* topology and affinely homeomorphic to C via the evaluation map.

On the other hand, let V be a (closed) function system with state space C equipped with the weak* topology. For $v \in V$, the function $\hat{v} : C \to \mathbb{C}$ defined by $\hat{v}(x) = x(v)$ is a continuous affine function on C. Kadison's representation theorem [29] asserts that the unital map $V \to \mathrm{A}(C) : v \to \hat{v}$ is an order isomorphism. Hence every function system is order isomorphic to a function system of continuous affine functions on a compact convex set.

The above results can be conveniently expressed in the language of category theory. Let Conv denote the category of compact convex sets with morphisms consisting of continuous affine maps and let FuncSys denote the category of function systems with morphisms consisting of unital order homomorphisms. The above results are equivalent to the statement that Conv and FuncSys are dually equivalent via the contravariant functor $A : \mathrm{Conv} \to \mathrm{FuncSys}$.

For a compact convex set C, $\mathrm{A}(C)$ is the function system of continuous affine functions on C as above. If D is a compact convex set and $\varphi : C \to D$ is a continuous affine map, then the unital order homomorphism $A\varphi : \mathrm{A}(D) \to \mathrm{A}(C)$ is defined by $A\varphi(b)(x) = b(\varphi(x))$ for $b \in \mathrm{A}(D)$ and $x \in C$.

The inverse functor $A^{-1} : \mathrm{FuncSys} \to \mathrm{Conv}$ is defined similarly. For a function system V with state space C, $A^{-1}V = C$. If W is a function system with state space D and $\psi : W \to V$ is a unital order homomorphism, then the continuous affine map $A^{-1}\psi : C \to D$ is defined by $b((A^{-1}(\psi)(x)) = \psi(b)(x)$ for $b \in W$ and $x \in C$.

3.2. **Noncommutative convex sets and operator systems.** In this section we will show that the category of compact nc convex sets is dually equivalent to the category of operator systems.

The arguments in this section are similar to arguments of Webster and Winkler [48, Proposition 3.5]. They proved that the category of compact matrix convex sets is dually equivalent to the category of operator systems. This is not surprising in light of Proposition 2.2.10, which implies that a compact nc convex set is determined by its finite levels. Major differences between the theory of noncommutative convexity and the theory of matrix convexity will only begin to appear in the next section.

For a compact nc convex set K, the space $\mathrm{A}(K)$ of continuous affine nc functions on K is an operator system. This means that it is a matrix ordered complex $*$-vector space with a distinguished archimedean matrix order unit [9]. To see this, it will be convenient for $n \in \mathbb{N}$ to identify the space $\mathcal{M}_n(\mathrm{A}(K))$ with the space of continuous affine nc maps $\mathrm{A}(K, \mathcal{M}_n(\mathcal{M}))$, so that elements in $\mathcal{M}_n(\mathrm{A}(K))$ can be viewed as functions taking values in $\mathcal{M}_n(\mathcal{M})$.

For $a \in \mathcal{M}_n(\mathrm{A}(K))$, the *adjoint* $a^* \in \mathcal{M}_n(\mathrm{A}(K))$ is defined by $a^*(x) = a(x)^*$ for $x \in K$. We say that a is *self-adjoint* if $a = a^*$. If a is self-adjoint, then we say that it is *positive* and write $a \geq 0$ if $a(x) \geq 0$ for all $x \in K$. Letting $\mathcal{M}_n(\mathrm{A}(K))^+$ denote the positive elements in $\mathcal{M}_n(\mathrm{A}(K))$, the sequence of positive cones $(\mathcal{M}_n(\mathrm{A}(K))^+)_{n\in\mathbb{N}}$ determines the matrix order on $\mathrm{A}(K)$. Together, this gives $\mathrm{A}(K)$ the structure of a matrix ordered $*$-vector space.

Since K is compact, elements in $\mathcal{M}_n(\mathrm{A}(K))$ are bounded by Proposition 2.5.3. This implies that the constant function $1_{\mathrm{A}(K)} \in \mathrm{A}(K)$ defined by $1_{\mathrm{A}(K)}(x) = 1_n$ for $x \in K_n$ is an archimedean matrix order unit for $\mathrm{A}(K)$.

The operator system structure on $\mathrm{A}(K)$ induces a matrix norm on $\mathrm{A}(K)$, i.e. a norm on $\mathcal{M}_n(\mathrm{A}(K))$ for each $n \in \mathbb{N}$. This norm agrees with the uniform norm on $\mathcal{M}_n(\mathrm{A}(K))$.

Definition 3.2.1. Let K be a compact nc convex set. The *operator system of continuous affine nc functions on* K is the space $\mathrm{A}(K)$ equipped with the operator system structure defined above.

Theorem 3.2.2. *Let K be a compact nc convex set and let L denote the nc state space of* $\mathrm{A}(K)$. *Then K and L are affinely homeomorphic* *via the affine nc map $\theta : K \to L$ defined by*

$$\theta(x)(a) = a(x), \quad \textit{for} \quad x \in K.$$

Proof. The nc state space L of $\mathrm{A}(K)$ is a compact nc convex set over $\mathrm{A}(K)^*$ as in Example 2.2.6. It is clear that θ is a continuous affine nc map by the definition of $\mathrm{A}(K)$. We must show that θ is a homeomorphism. Since each K_n is compact, it suffices to show that θ is a bijection.

Suppose that K is a compact nc convex set over a dual operator system E. Then elements in E_* give rise to continuous nc affine functions in $\mathrm{A}(K)$ via the map $\varphi : E_* \to \mathrm{A}(K)$ defined by $\varphi(b)(x) = x(b)$ for $b \in E_*$. The injectivity of θ follows from the fact that E_* separates points in K.

For the surjectivity of θ, first note that $\theta(K) \subseteq L$ is a compact nc convex set. Suppose for the sake of contradiction there is $y_0 \in L_n \backslash \theta(K)_n$. Then by Corollary 2.4.2, there is a normal completely bounded linear map $\varphi : A(K)^* \to \mathcal{M}_n$ and self-adjoint $\gamma \in \mathcal{M}_n$ such that

$$\operatorname{Re} \varphi_n(y_0) \not\leq \gamma \otimes 1_n \quad \text{but} \quad \operatorname{Re} \varphi_p(y) \leq \gamma \otimes 1_p$$

for every p and $y \in \theta(K)_p$. Since φ is normal, we can identify φ with a continuous nc affine function $a \in \mathcal{M}_n(\mathrm{A}(K))$. By the definition of the operator system structure on $\mathrm{A}(K)$, the second inequality implies $\operatorname{Re} a \leq \gamma \otimes 1_{\mathrm{A}(K)}$. Since y_0 is unital and completely positive, this implies

$$\operatorname{Re} y_0(a) \leq y_0(a) \leq \gamma \otimes 1_n,$$

giving a contradiction. □

The next result is a noncommutative analogue of Kadison's representation theorem.

Theorem 3.2.3. *Let S be a closed operator system with nc state space K. For $s \in S$, the function $\hat{s} : K \to \mathcal{M}$ defined by*

$$\hat{s}(x) = x(s), \quad \textit{for} \quad s \in S,\ x \in K$$

is a continuous affine nc function on K. The map $S \to \mathrm{A}(K) : s \to \hat{s}$ is a complete order isomorphism.

Proof. For $s \in S$, it is clear that $\hat{s}$ is a continuous affine function on K. Kadison's representation theorem implies that the map $S \to \mathrm{A}(K) : s \to \hat{s}$ is an order isomorphism, so it remains to show that it is a complete order isomorphism. For this, it suffices to show that it preserves the matrix order, meaning that for $n \in \mathbb{N}$ and $s \in \mathcal{M}_n(S)$, if $s \geq 0$ then $\hat{s} \geq 0$. But this follows immediately from the fact that K consists of completely positive maps on S. □

Definition 3.2.4. We let NCConv denote the category with objects consisting of compact nc convex sets and morphisms consisting of continuous affine nc maps. We will refer to this as the category of *compact nc convex sets.* We let OpSys denote the category with objects consisting of closed operator systems and morphisms consisting of unital complete positive maps. We will refer to this as the category of *closed operator systems.*

We now define the functor $\mathrm{A} : \mathrm{NCConv} \to \mathrm{OpSys}$ implementing the dual equivalence between NCConv and OpSys. For a compact nc convex set K, $\mathrm{A}(K)$ is the operator system of continuous affine nc functions on K as in Definition 3.2.1. For compact nc convex sets K and L and a continuous affine map $\theta : K \to L$, $\mathrm{A}(\theta) : \mathrm{A}(L) \to \mathrm{A}(K)$ is a unital completely positive map defined by

$$\mathrm{A}(\theta)(b)(x) = b(\theta(x)), \quad \text{for} \quad b \in \mathrm{A}(L),\ x \in K.$$

The functor A has an inverse $\mathrm{A}^{-1} : \mathrm{OpSys} \to \mathrm{NCConv}$. For an operator system S with nc state space K, $\mathrm{A}^{-1}(S) = K$. For operator systems S and T with nc state spaces K and L respectively and a unital completely positive map $\varphi : T \to S$, $\mathrm{A}^{-1}(S) : K \to L$ is a continuous nc affine map defined by

$$b(\mathrm{A}^{-1}(\varphi)(x)) = \varphi(b)(x), \quad \text{for} \quad b \in T,\ x \in K.$$

Theorem 3.2.5. *The map $A : \mathrm{NCConv} \to \mathrm{OpSys}$ is a contravariant functor with inverse $A^{-1} : \mathrm{OpSys} \to \mathrm{NCConv}$. In particular, the categories* NCConv *and* OpSys *are dually equivalent.*

The next result follows immediately from Theorem 3.2.5.

Corollary 3.2.6. *Let K and L be compact nc convex sets. Then $\mathrm{A}(K)$ and $\mathrm{A}(L)$ are isomorphic if and only if K and L are affinely homeomorphic. Hence two operator systems are unitally completely order isomorphic if and only if their nc state spaces are affinely homeomorphic.*

4. Noncommutative functions

4.1. **Functions on compact convex sets.** An essential component of classical Choquet theory is the interplay between the space $\mathrm{A}(C)$ of continuous affine functions on a compact convex set C and the C*-algebra $\mathrm{C}(C)$ of continuous functions on C.

The Stone-Weierstrass theorem implies that the C*-algebra $\mathrm{C}(C)$ of continuous functions on C is generated by the function system $\mathrm{A}(C)$ of continuous affine functions on C. In fact, $\mathrm{C}(C)$ is uniquely determined by the following universal property: $\mathrm{C}(C)$ is generated by $\mathrm{A}(C)$ and for any unital commutative C*-algebra $\mathrm{C}(X)$ and unital order embedding $\varphi : \mathrm{A}(C) \to \mathrm{C}(X)$ satisfying $\mathrm{C}^*(\varphi(\mathrm{A}(C))) = \mathrm{C}(X)$, there is a surjective homomorphism $\pi : \mathrm{C}(C) \to \mathrm{C}(X)$ such that $\pi|_{\mathrm{A}(C)} = \varphi|_{\mathrm{A}(C)}$.

$$\begin{tikzcd} \mathrm{A}(C) \arrow[r, hook, "\mathrm{id}"] \arrow[dr, hook, "\varphi"'] & \mathrm{C}(C) \arrow[d, two heads, "\pi"] \\ & \mathrm{C}(X) = \mathrm{C}^*(\varphi(\mathrm{A}(C))) \end{tikzcd}$$

This says that $\mathrm{C}(C)$ is the maximal commutative C*-algebra generated by a unital order embedding of $\mathrm{A}(C)$.

The Riesz-Markov-Kakutani representation theorem implies that the state space of $\mathrm{C}(C)$ can be identified with the space $\mathrm{P}(C)$ of regular Borel probability measures on C. For a point $x \in C$, a measure $\mu \in \mathrm{P}(C)$ is said to represent x and x is said to be the barycenter of μ if $\mu|_{\mathrm{A}(C)} = x$. Since the point mass $\delta_x \in \mathrm{P}(C)$ represents x, every point in C has at least one representing probability measure. Moreover, x has a unique representing measure if and only if $x \in \partial C$, where ∂C denotes the extreme boundary of C. More generally, the Choquet-Bishop-de Leeuw integral representation theorem, which we will review later, asserts that for any $x \in C$, it is always possible to choose a representing measure that is supported on ∂C in an appropriate sense.

The closure of the extreme boundary $\overline{\partial C}$ is the Shilov boundary of the function system $\mathrm{A}(C)$. This means that the restriction map $\rho : \mathrm{C}(C) \to \mathrm{C}(\overline{\partial C})$ is a unital order embedding, and the C*-algebra $\mathrm{C}(\overline{\partial C})$ is uniquely determined by the following universal property: for any unital commutative C*-algebra $\mathrm{C}(X)$ and any unital order embedding $\varphi : \mathrm{A}(C) \to \mathrm{C}(X)$ satisfying $\mathrm{C}^*(\varphi(\mathrm{A}(C))) = \mathrm{C}(X)$, there is a surjective homomorphism $\pi : \mathrm{C}(X) \to \mathrm{C}(\overline{\partial C})$ satisfying $\pi \circ \varphi = \rho$.

$$\begin{tikzcd} & \mathrm{C}(X) = \mathrm{C}^*(\varphi(\mathrm{A}(C))) \arrow[d, two heads, "\pi"] \\ \mathrm{A}(C) \arrow[ur, hook, "\varphi"] \arrow[r, hook, "\rho"] & \mathrm{C}(\overline{\partial C}) \end{tikzcd}$$

This says that $\mathrm{C}(\overline{\partial C})$ is the minimal commutative C*-algebra generated by a unital order embedding of $\mathrm{A}(C)$.

4.2. **Noncommutative functions.** In this section we will introduce a definition of nc function on a compact nc convex set. We will associate a C*-algebra of nc functions to every compact nc convex set that plays a role in the noncommutative setting analogous to the role in the classical setting of the C*-algebra of continuous functions on a compact convex set. In Section 4.4, we will see that the elements in this C*-algebra are, in fact, precisely the continuous nc functions on K, when continuity is defined in an appropriate sense.

Definition 4.2.1. Let K be a compact nc convex set and let $f : K \to \mathcal{M}$ be a function. We say that f is an *nc function* if it is graded, respects direct sums and is unitarily equivariant, meaning that

(1) $f(K_n) \subseteq \mathcal{M}_n$ for all n,
(2) $f(\sum \alpha_i x_i \alpha_i^*) = \sum \alpha_i f(x_i) \alpha_i^*$ for every family $\{x_i \in K_{n_i}\}$ and every family of isometries $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i \alpha_i^* = 1_n$,
(3) $f(\beta x \beta^*) = \beta f(x) \beta^*$ for every $x \in K_n$ and every unitary $\beta \in \mathcal{M}_n$.

We say that f is *bounded* if $\|f\|_\infty < \infty$, where $\|f\|_\infty$ denotes the *uniform norm* defined by

$$\|f\|_\infty = \sup_{x \in K} \|f(x)\|.$$

We let $\mathrm{B}(K)$ denote the space of all bounded nc functions on K.

Remark 4.2.2. It is clear that nc affine functions on K are in particular nc functions on K. Moreover, by Proposition 2.5.3, functions in the space $\mathrm{A}(K)$ of continuous nc affine functions on K are bounded nc functions. Therefore, $\mathrm{B}(K)$ contains $\mathrm{A}(K)$.

Remark 4.2.3. The study of nc functions has had a large following in recent years. The book [30] lays out the fundamentals of this theory which has its roots in the work of Taylor [46] on a functional calculus for multivariable functions in non-commutating variables. Their theory is similarity invariant (in an appropriate restricted sense), not just unitarily invariant like our definition. In fact, this self-adjoint version has even older roots in the work of Takesaki [44] on a non-commutative Gelfand theory for C*-algebras. See § 4.3.

Let K be a compact nc convex set. For a nc function $f : K \to \mathcal{M}$, the adjoint $f^* : K \to \mathcal{M}$ is defined by $f^*(x) = f(x)^*$ for $x \in K$. Note that f^* is graded and respects direct sums. Also, for $x \in K_n$ and a unitary $\beta \in \mathcal{M}_n$,

$$f^*(\beta x \beta^*) = f(\beta x \beta^*)^* = (\beta f(x) \beta^*)^* = \beta f(x)^* \beta^* = \beta f^*(x) \beta^*,$$

so f^* is also unitarily equivariant. Hence f^* is an nc function. Furthermore, it is clear that if $f \in \mathrm{B}(K)$, then $f^* \in \mathrm{B}(K)$.

For $f, g \in \mathrm{B}(K)$, define the pointwise product fg by $fg(x) = f(x)g(x)$ for $x \in K$. Then $fg \in \mathrm{B}(K)$, so $\mathrm{B}(K)$ is closed under the pointwise product. Moreover, it is easy to check that $\mathrm{B}(K)$ is closed with respect to the uniform norm and satisfies the C*-identity. Therefore, B(K) is a C*-algebra.

Definition 4.2.4. Let K be a compact nc convex set. The C*-algebra of bounded nc functions on K is the C*-algebra B(K) equipped with the C*-algebra structure defined above. We will let C(K) denote C*-algebra subalgebra of B(K) generated by A(K).

The analogy with the classical setting suggests that the C*-algebra C(K) should consist precisely of the nc functions that are continuous on K. In Section 4.4, we will see that this is true when K is equipped with the point-strong topology, for which we will require Takesaki and Bichteler's noncommutative Gelfand representation of a C*-algebra. However, it turns out that elements in C(K) are not necessarily continuous on K with respect to the point-weak* topology, as we will now discuss.

Observe that for $n < \infty$, an nc function in A(K) is weak*-to-norm continuous on K_n since the weak* and norm topologies agree on $\mathcal{M}_n$. Since C(K) is the C*-algebra generated by A(K), and since multiplication on $\mathcal{M}_n$ is jointly continuous in the norm topology, it follows that for $n < \infty$, elements in C(K) are continuous on K_n with respect to the point-weak* topology. However, the next example shows that for infinite n, elements in C(K) are not necessarily continuous on K_n with respect to the point-weak* topology.

Example 4.2.5. Consider the function system $S = \operatorname{span}\{1, z, \bar{z}\}$ in C($\mathbb{T}$). For each n, every unital completely positive map $\varphi : S \to \mathcal{M}_n$ is determined by $\alpha := \varphi(z) \in \mathcal{M}_n$. Clearly $\|\alpha\| \leq 1$. On the other hand, for $\alpha \in \mathcal{M}_n$ with $\|\alpha\| \leq 1$, von Neumann's inequality implies the existence of a unital completely positive map $\varphi : S \to \mathcal{M}_n$ satisfying $\varphi(z) = \alpha$.

Let K denote the nc state space of S, so that S is isomorphic to A(K). Then it follows from above that for each n, K_n is affinely homeomorphic to the unit ball of $\mathcal{M}_n$. Let $a \in$ A(K) denote the nc function corresponding to $z \in S$. Then a^* corresponds to $\bar{z} \in S$. For an nc state $x \in K_n$ corresponding to a contraction $\alpha \in \mathcal{M}_n$ as above, $a(x) = \alpha$ and $a^*(x) = \alpha^*$. Let a^*a denote the pointwise product of a^* and a. Then a^*a is an nc function on K with $(a^*a)(x) = a^*(x)a(x)$.

For $t \in (0, 1]$, identify $\mathcal{M}_{\aleph_0}$ with $\mathcal{B}(L^2[0, 1])$ and let $x_t \in K_{\aleph_0}$ denote the nc state corresponding to the isometry $\alpha_t \in \mathcal{M}_{\aleph_0}$ defined by

$$\alpha_t(f)(x) = \begin{cases} t^{-1/2} f(t^{-1}x) & 0 < x < t, \\ 0 & t \leq x < 1. \end{cases}$$

Let $x_0 \in K_{\aleph_0}$ denote the point corresponding to $0_{\aleph_0}$. Then

$$\lim_{t\to 0} a(x_t) = \lim_{t\to 0} \alpha_t = 0.$$

Similarly, $\lim_{t\to 0} a^*(x_t) = 0$. So $\lim_{t\to 0} x_t = x_0$. However,

$$\lim_{t\to 0}(a^*a)(x_t) = \lim_{t\to 0} \alpha_t^*\alpha_t = 1_{\aleph_0} \neq 0 = (a^*a)(x_0).$$

It follows that a^*a is not continuous as a function on $K_{\aleph_0}$.

The next example shows that bounded nc functions are not necessarily determined by their values on finite levels.

Example 4.2.6. Consider the Cuntz operator system

$$S = \operatorname{span}\{1, s_1, s_2, s_1^*, s_2^*\} \subseteq \mathcal{O}_2,$$

where s_1, s_2 are the standard generators of the Cuntz C*-algebra $\mathcal{O}_2$ (see Example 6.6.2 for more details). Let K denote the nc state space of S so that S is completely order isomorphic to $\mathrm{A}(K)$. For each n, a point $x \in K_n$ determines a contractive 1×2 matrix $[x(s_1)\ x(s_2)]$ with entries in $\mathcal{M}_n$ (a row contraction). Conversely, a row contraction $[\alpha_1\ \alpha_2]$ with entries in $\mathcal{M}_n$ determines a point in $x \in K_n$. So K_n is affinely homeomorphic to the compact convex set of row contractions with entries in $\mathcal{M}_n$ (which we can identify with a subset of $\mathcal{M}_{n,2n}$). Since $\mathcal{O}_2$ is simple and infinite dimensional, every representation is infinite dimensional. The corresponding representation δ_x of $\mathrm{C}(K)$ factors through $\mathcal{O}_2$ if and only if $[\alpha_1\ \alpha_2]$ is a unitary. Define a function $f : K \to \mathcal{M}$ by

$$f(x) = \begin{cases} 1_n & \text{if } x \in K_n \text{ extends to a representation of } \mathcal{O}_2 \\ 0_n & \text{otherwise.} \end{cases}$$

Then f is evidently bounded, graded, preserves direct sums and is unitarily equivariant. So it is a bounded nc function (in fact a projection) in $\mathrm{B}(K)$. The significance of this example is that f vanishes at all finite levels, but is nonzero. So nc functions are not necessarily determined by their values on finite levels. Moreover this function is continuous on all finite levels but is not continuous on $K_{\aleph_0}$.

4.3. **Noncommutative Gelfand theory.** Takesaki [44] obtained a noncommutative version of the Gelfand representation for separable C*-algebras which was later extended to non-separable C*-algebras by Bichteler [5]. The basic idea is to represent a C*-algebra as a space of noncommutative functions on its representation space. In this section we will briefly review their results.

Let A be a C*-algebra, and let H be a Hilbert space of sufficiently large infinite dimension that every cyclic representation of A is unitarily equivalent to a representation of A on a subspace of H. Let $\operatorname{Rep}(A, H)$ denote the set of all such representations, equipped with the point-weak* topology. Note that elements in $\operatorname{Rep}(A, H)$ may be degenerate as representations on H.

Recall that the ultrastrong* topology on $\mathcal{B}(H)$ is determined by the seminorms $b \to (\sum \|b\xi_i\|)^{1/2}$ and $b \to (\sum \|b^*\xi_i\|)^{1/2}$ for $b \in \mathcal{B}(H)$ and $\xi_i \in H$ satisfying $\sum \|\xi_i\|^2 < \infty$ (see e.g. [16, Chapter 3, Part 1]). Since A is a C*-algebra, the point-weak* topology on $\operatorname{Rep}(A, H)$ agrees with the point-ultrastrong* topology. The advantage of the point-ultrastrong* topology is that the adjoint map is always point-ultrastrong*-continuous and multiplication is always jointly point-ultrastrong*-continuous. This ensures that the functions in the C*-algebra generated by a collection of point-ultrastrong* continuous functions are also point-ultrastrong* continuous.

For $\pi \in \operatorname{Rep}(A, H)$, let $p_\pi \in \mathcal{B}(H)$ denote the projection onto $\pi(A)H$. If $u \in \mathcal{B}(H)$ is a partial isometry satisfying $u^*u \geq p_\pi$, then $\pi^u = u\pi u^*$ is a representation of A unitarily equivalent to π. For $\pi, \sigma \in \operatorname{Rep}(A, H)$, $\pi \oplus \sigma$ can be identified with an element in $\operatorname{Rep}(A, H)$. The unitary group $\mathrm{U}(H)$ acts on $\operatorname{Rep}(A, H)$ by conjugation.

In this setting, the analogue of an nc function is an admissible operator field. This is a map $T : \operatorname{Rep}(A, H) \to \mathcal{B}(H)$ that is bounded, non-degenerate, respects direct sums and is equivariant with respect to partial isometries, meaning that

(i) $\|T\| := \sup\{\|T(\pi)\| : \pi \in \operatorname{Rep}(A, H)\} < \infty$,
(ii) $T(\pi) = p_\pi T(\pi) p_\pi$,
(iii) $T(\pi_1 \oplus \pi_2) = T(\pi_1) + T(\pi_2)$ when $p_{\pi_1} p_{\pi_2} = 0$,
(iv) $T(\pi^u) = u^* T(\pi) u$ when $u^*u \geq p_\pi$.

Each representation π of A has a unique extension to a normal representation of the bidual A^{**} (which we also denote by π). For $b \in A^{**}$, we can define an admissible operator field by

$$\hat{b}(\pi) := \pi(b) \quad \text{for} \quad \pi \in \operatorname{Rep}(A, H).$$

If $\mathcal{B}(H)$ is equipped with the ultrastrong* topology, then for $a \in A$, the map $\hat{a}$ is a point-ultrastrong* continuous admissible operator field. It is easy to check that the space of all admissible operator fields forms a C*-algebra.

The Takesaki-Bichteler theorem asserts that the C*-algebra of admissible operator fields is naturally isomorphic to the bidual A^{**} of A.

Moreover, the image of the point-ultrastrong* continuous admissible operator fields under this isomorphism is precisely A.

In the next section, we will apply the results of Takesaki and Bichteler to give another description of the C*-algebra of continuous functions on a compact nc convex set.

4.4. **Maximal C*-algebra.** In this section we will show that for a compact nc convex set K, the functions in the C*-algebra $\mathrm{C}(K)$ from Definition 4.2.4 are precisely the continuous nc functions on K when continuity is defined in an appropriate sense. We will first show that, as in the classical setting with the C*-algebra of continuous functions on a compact convex set, $\mathrm{C}(K)$ is uniquely determined by an important universal property.

Kirchberg and Wassermann [31] introduced the maximal C*-algebra $\mathrm{C}^*_{\max}(S)$ of an operator system S. This C*-algebra is uniquely determined up to isomorphism by the following universal property: there is a unital complete order embedding $\iota : S \to \mathrm{C}^*_{\max}(S)$ such that $\mathrm{C}^*(\iota(S)) = \mathrm{C}^*_{\max}(S)$ and for any C*-algebra A and unital complete order embedding $\varphi : S \to A$ satisfying $\mathrm{C}^*(\varphi(S)) = A$, there is a unique homomorphism $\pi : \mathrm{C}^*_{\max}(S) \to A$ satisfying $\pi \circ \iota = \varphi$.

$$\begin{tikzcd} S \arrow[r, hook, "\iota"] \arrow[dr, hook, "\varphi"'] & \mathrm{C}^*_{\max}(S) \arrow[d, two heads, "\pi"] \\ & A = \mathrm{C}^*(\varphi(S)) \end{tikzcd}$$

For a compact nc convex set K, we will show that the C*-algebra $\mathrm{C}(K)$ is naturally isomorphic to the maximal C*-algebra $\mathrm{C}^*_{\max}(\mathrm{A}(K))$, and hence that it satisfies the above universal property. The proof will require Takesaki and Bichteler's noncommutative Gelfand theorem from Section 4.3, along with some technical preliminaries.

Recall from Section 2.1 that for each n, the topology on $K_n \subseteq \mathcal{M}_n(E)$ is the point-weak* topology corresponding to the identification $\mathcal{M}_n(E) \cong \mathrm{CB}(E_*, \mathcal{M}_n)$. Example 4.2.5 shows that if n is infinite, then functions in $\mathrm{C}(K)$ are not necessarily continuous on K_n with respect to the point-weak* topology. The Takesaki-Bichteler theorem suggests that we should instead consider the point-ultrastrong* topology on K_n which, by Remark 2.5.6, agrees with the point-strong topology on K_n.

For a bounded nc function $f \in \mathrm{B}(K)$, we will say that f is continuous with respect to the point-strong topology on K if the restriction $f|_{K_n}$ is continuous with respect to the point-strong topology on K_n for each n.

Remark 4.4.1. For finite n, the point-weak* topology and the point-strong topology will agree on K_n, while for infinite n, K_n is not necessarily even compact in the point-strong topology. However, for every n the weak* and ultrastrong* topologies have the same continuous linear functionals. It follows from an easy separation argument that a convex subset of K_n is closed in the point-weak* topology if and only if it is closed in the point-ultrastrong* topology (equivalently, the point-strong topology).

However, in general K is not compact in point-strong topology. To see this, suppose for the sake of convenience that $\mathrm{A}(K)$ is separable. Let φ be a u.c.p. map of $\mathrm{C}^*_{\max}(\mathrm{A}(K))$ into H_ω which is not a $*$-homomorphism. By the Stinespring dilation theorem, there is a $*$-representation π of $\mathrm{C}^*_{\max}(\mathrm{A}(K))$ into H_ω and an isometry α such that $\varphi = \alpha^*\pi\alpha$. Choose a sequence of unitaries such that $\beta_n \to \alpha$ in the weak-$*$ topology. Then φ is the point-weak-$*$ limit *relative to* $\mathrm{C}(K)$ of the maps $\beta_n^*\pi\beta_n$. The sequence belongs to K_ω, but φ does not. Since the point-strong topology coincides with the point-weak-$*$ topology relative to $\mathrm{C}(K)$, we see that K is not compact in this topology. Note that in the weak-$*$ topology on K, this sequence converges to $x := \varphi|_{\mathrm{A}(K)}$. This extends to the $*$-homomorphism δ_x on $\mathrm{C}(K)$, which is distinct from φ.

Let κ be the largest cardinal required in the definition of K, so that every nondegenerate cyclic representation of $\mathrm{C}^*_{\max}(\mathrm{A}(K))$ acts on a Hilbert space of cardinality no larger than κ. For each $n \le \kappa$, let $\mathrm{Isom}(H_n, H_\kappa)$ denote the space of all isometries of H_n into H_κ equipped with the relative ultrastrong* topology obtained from the inclusion $\mathrm{Isom}(H_n, H_\kappa) \subseteq \mathcal{B}(H_n, H_\kappa)$. Then $\mathrm{Isom}(H_n, H_\kappa)$ is closed.

For $x \in K_n$, it follows from the universal property of $\mathrm{C}^*_{\max}(\mathrm{A}(K))$ that there is a unique homomorphism $\delta_x : \mathrm{C}^*_{\max}(\mathrm{A}(K)) \to \mathcal{M}_n$ satisfying $\delta_x \circ \iota = x$. Conversely, if $\pi : \mathrm{C}^*_{\max}(\mathrm{A}(K)) \to \mathcal{M}_n$ is a homomorphism and $x = \pi \circ \iota$, then $\pi = \delta_x$. We will say more about this in Section 4.5.

Let $\tilde{K} = \coprod ((K_n, \tau_{us*}) \times \mathrm{Isom}(H_n, H_\kappa))$ and define a map $\varepsilon : \tilde{K} \to \mathrm{Rep}(\mathrm{C}^*_{\max}(\mathrm{A}(K)), H_\kappa)$ by

$$\varepsilon(x, \alpha) = \alpha\delta_x\alpha^* \quad \text{for} \quad x \in K_n \text{ and } \alpha \in \mathrm{Isom}(H_n, H_\kappa).$$

For $x \in K_n$, let $\mathcal{U}(x) = \{u^*xu : u \in \mathcal{U}_n\}$ denote the unitary equivalence class of x. Similarly, for $\pi \in \mathrm{Rep}(\mathrm{C}^*_{\max}(\mathrm{A}(K)), H_\kappa)$, let

$$\mathcal{O}(\pi) = \{\pi^u : u^*u \ge p_\pi\}$$

denote the equivalence class of π under conjugation by appropriate partial isometries.

Proposition 4.4.2. *The map $\varepsilon : \tilde{K} \to \mathrm{Rep}(\mathrm{C}^*_{\max}(\mathrm{A}(K)), H_\kappa)$ is a continuous surjection that respects direct sums and satisfies*

$$\varepsilon\left(\mathcal{U}(x) \times \mathrm{Isom}(H_n, H_\kappa)\right) = \mathcal{O}(\delta_x).$$

There is a bijection τ between the C^*-algebra $\mathrm{B}(K)$ of bounded nc functions on K and admissible operator fields on $\mathrm{Rep}(\mathrm{C}^*_{\max}(\mathrm{A}(K)), H_\kappa)$ defined for $f \in \mathrm{B}(K)$, $x \in K_n$ and $\alpha \in \mathrm{Isom}(H_n, H_\kappa)$ by

$$\tau(f)(\alpha^*\delta_x\alpha) = \alpha^* f(x)\alpha.$$

A bounded nc function $f \in \mathrm{B}(K)$ is an ultrastrong continuous nc function if and only if $\tau(f)$ is an ultrastrong* continuous admissible operator field.*

Proof. For this proof it will be convenient to identify $\mathrm{A}(K)$ with its image $\iota(\mathrm{A}(K)) \subseteq \mathrm{C}^*_{\max}(\mathrm{A}(K))$.

We will first show that ε is continuous. Note that for a net $x_i \in K_n$, the statement that $\lim x_i = x$ in the point-ultrastrong* topology means that $\lim a(x_i) = a(x)$ in the ultrastrong* topology for each $a \in \mathrm{A}(K)$. In this case, since products and sums are ultrastrong* continuous, it follows that $\lim \delta_{x_i}(b) = \delta_x(b)$ in the ultrastrong* topology for all b in the $*$-algebra generated by $\mathrm{A}(K)$. Since the δ_{x_i} are all contractions, $\lim \delta_{x_i}(b) = \delta_x(b)$ in the ultrastrong* topology for all $b \in \mathrm{C}^*_{\max}(\mathrm{A}(K))$. Hence $\lim \delta_{x_i} = \delta_x$ in the point-ultrastrong* topology on $\mathrm{C}^*_{\max}(\mathrm{A}(K))$.

From above the map $x \to \delta_x$ is point-ultrastrong* to point-ultrastrong* continuous. Since multiplication is jointly ultrastrong* continuous, it follows that ε is continuous on $\tilde{K}$.

For a representation $\pi \in \mathrm{Rep}(\mathrm{C}^*_{\max}(\mathrm{A}(K)), H_\kappa)$, let n denote the rank of p_π and let $\alpha \in \mathcal{M}_{n,\kappa}$ be an isometry with range p_π. Then $\alpha^*\pi\alpha$ is a non-degenerate representation in $\mathrm{Rep}(\mathrm{C}^*_{\max}(\mathrm{A}(K)), H_n)$. Let $x = \alpha^*\pi\alpha|_{\mathrm{A}(K)} \in K_n$. Then evidently $\alpha^*\pi\alpha = \delta_x$ and $\pi = \varepsilon(x, \alpha)$. Hence ε is surjective.

It is clear that ε preserves direct sums. The argument above shows that if $\pi \simeq \delta_x$, then there is an isometry α such that $\pi = \varepsilon(x, \alpha)$. Hence ε maps $\mathcal{U}(x) \times \mathrm{Isom}(H_n, H_\kappa)$ onto $\mathcal{O}(\delta_x)$.

Next we show that τ is well defined and that for $f \in \mathrm{B}(K)$, $\tau(f)$ is an admissible operator field. Suppose that $\pi = \alpha\delta_x\alpha^* = \beta\delta_y\beta^*$. Then $u = \alpha^*\beta$ is a unitary on H_n and $\delta_y = u^*\delta_x u$. For $f \in \mathrm{B}(K)$, the unitary equivariance of f implies that

$$\beta^* f(y)\beta = \beta^* u^* f(x) u\beta = \alpha^* f(x)\alpha.$$

Hence τ is well defined.

It is evident that $\|\tau(f)\| = \|f\| < \infty$. The fact that $\tau(f)(\pi) = p_\pi\tau(f)(\pi)p_\pi$ is immediate from the definition. Furthermore, since f

preserves direct sums, so does $\tau(f)$. Arguing as in the last paragraph shows that the unitary equivariance of f corresponds to $\tau(f)$ being equivariant under partial isometries. Hence $\tau(f)$ is an admissible operator field.

Conversely, if T is an admissible operator field, define an nc function $f = \sigma(T)$ by $f(x) = \alpha^* T(\alpha \delta_x \alpha^*)\alpha$. Again the equivariance shows that this is a well defined nc function on K. It is clear that $\tau(f) = T$ and that σ provides the inverse of τ. Thus τ is a bijection.

Finally if f is ultrastrong* continuous, then the ultrastrong* continuity of ε shows that $\tau(f)$ is ultrastrong* continuous as well. The converse follows from the formula for the inverse map. $\square$

We now can establish the main result of this section.

Theorem 4.4.3. *Let K be a compact nc convex set. Then the map $\sigma : \mathrm{C}^*_{\max}(\mathrm{A}(K))^{**} \to \mathrm{B}(K)$ defined by*

$$\sigma(b)(x) = \delta_x(b) \quad \textit{for} \quad b \in \mathrm{C}^*_{\max}(\mathrm{A}(K))^{**},\ x \in K$$

is a normal $*$-isomorphism that restricts to a $*$-isomorphism from $\mathrm{C}^*_{\max}(\mathrm{A}(K))$ onto $\mathrm{C}(K)$. In particular, the elements in $\mathrm{C}(K)$ are precisely the point-strong continuous nc functions on K. Furthermore, $\sigma \circ \iota$ is the identity map on $\mathrm{A}(K)$.

Proof. We have shown that every representation of $\mathrm{C}^*_{\max}(\mathrm{A}(K))$ is of the form δ_x for $x \in K$. In the proof of the previous proposition, we showed that $\mathrm{Rep}(\mathrm{C}^*_{\max}(\mathrm{A}(K)), H_\kappa)$ is the continuous image of $\tilde{K}$. By Takesaki and Bichteler's theorem [5, 44], there is a normal isomorphism of the C*-algebra of admissible operator fields onto $\mathrm{C}^*_{\max}(\mathrm{A}(K))^{**}$ that carries the subalgebra of point-ultrastrong* continuous admissible operator fields onto $\mathrm{C}^*_{\max}(\mathrm{A}(K))$. By Proposition 4.4.2, the map τ identifies $\mathrm{B}(K)$ with the algebra of admissible operator fields. It is easy to see that τ is a $*$-isomorphism. Moreover, it preserves suprema, and hence is normal. It follows that $\mathrm{B}(K)$ is a von Neumann algebra.

The composition of these two normal isomorphisms yields the identification between $\mathrm{C}^*_{\max}(\mathrm{A}(K))^{**}$ and $\mathrm{B}(K)$. One can readily check that under this identification, $\sigma(b)(x) = \delta_x(b)$ for $b \in \mathrm{C}^*_{\max}(\mathrm{A}(K))^{**}$.

This isomorphism maps the subalgebra of point-ultrastrong* continuous maps on $\mathrm{Rep}(\mathrm{C}^*_{\max}(\mathrm{A}(K)), H_\kappa)$, which is precisely $\mathrm{C}^*_{\max}(\mathrm{A}(K))$, onto $\mathrm{C}(K)$. It follows from Proposition 4.4.2 that the elements of $\mathrm{C}(K)$ are precisely the point-ultrastrong* continuous (equivalently, point-strong continuous) nc functions on K. Finally, it is now clear that $\sigma \circ \iota$ is the identity map on $\mathrm{A}(K)$. $\square$

Corollary 4.4.4. *Let K be a compact nc convex set. The enveloping von Neumann algebra $\mathrm{C}(K)^{**}$ of $\mathrm{C}(K)$ is isomorphic to the C*-algebra $\mathrm{B}(K)$ of bounded nc functions on K. The dual operator system $\mathrm{A}(K)^{**}$ is completely order isomorphic to the operator system of bounded nc affine functions on K.*

The proof of Proposition 2.5.3 can be applied verbatim to prove the next result.

Proposition 4.4.5. *Let K be a compact nc convex set and let $f \in \mathrm{C}(K)$ be a continuous nc function. Then f is bounded with*

$$\|f\| = \|f|_{K_{\aleph_0}}\| = \sup_{n<\infty} \|f|_{K_n}\|.$$

For the remainder of this paper, we identify $\mathrm{C}(K)$ with $\mathrm{C}^*_{\max}(\mathrm{A}(K))$ and refer to elements in $\mathrm{C}(K)$ as continuous nc functions. Similarly, we will identify $\mathrm{B}(K)$ with $\mathrm{C}^*_{\max}(\mathrm{A}(K))^{**}$ and refer to elements in $\mathrm{B}(K)$ as bounded nc functions. In particular we will identify $\mathrm{A}(K)$ with its image in $\mathrm{C}^*_{\max}(\mathrm{A}(K))$ and identify $\mathrm{A}(K)^{**}$ with its image in $\mathrm{C}^*_{\max}(\mathrm{A}(K))^{**}$.

4.5. **Representing maps.** For a compact nc convex set K, unital completely positive maps $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ play the role of probability measures in the classical setting. In this section we will introduce a natural notion of representing maps for points in K.

Definition 4.5.1. Let K be a compact nc convex set. For $x \in K_n$, we say that a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ *represents* x and that x is the *barycenter* of μ if μ restricts to x on the function system $\mathrm{A}(K)$ of continuous affine functions on K, i.e. if $\mu|_{\mathrm{A}(K)} = x$. If δ_x is the unique representing map for x, then we will say that x has a *unique representing map*.

It will be important to determine the points in K that have unique representing maps. We will revisit this in Section 5.2.

Because of the identification of $\mathrm{B}(K)$ with the enveloping von Neumann algebra of $\mathrm{C}(K)$ in Section 4.4, every unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ has a unique weak*-continuous extension from $\mathrm{B}(K)$ to $\mathcal{M}_n$. We will continue to denote this extension by μ. Note that for $f \in \mathrm{B}(K)$ and $x \in K$, $f(x) = \delta_x(f)$.

4.6. **Minimal C*-algebra.** In this section we will review the notion of the Shilov boundary of an operator system along with the corresponding notion of minimal C*-algebra of an operator system which, as in the classical setting with the C*-algebra of continuous functions on the Shilov boundary, satisfies an important universal property.

The existence of a noncommutative analogue of the Shilov boundary was conjectured by Arveson [2], and the existence and uniqueness was proved by Hamana [26]. For an operator system S, the *minimal C*-algebra* $\mathrm{C}^*_{\min}(S)$ is uniquely determined up to isomorphism by the following universal property: there is a unital complete order embedding $\iota : S \to \mathrm{C}^*_{\min}(S)$ such that $\mathrm{C}^*(\iota(S)) = \mathrm{C}^*_{\min}(S)$ and for any unital C*-algebra A and unital complete order embedding $\varphi : S \to A$ satisfying $\mathrm{C}^*(\varphi(S)) = A$, there is a surjective homomorphism $\pi : A \to \mathrm{C}^*_{\min}(S)$ satisfying $\pi \circ \varphi = \iota$.

$$\begin{tikzcd} & A = \mathrm{C}^*(\varphi(S)) \arrow[d, two heads, "\pi"] \\ S \arrow[ur, hook, "\varphi"] \arrow[r, hook, "\iota"] & \mathrm{C}^*_{\min}(S) \end{tikzcd}$$

In the literature, $\mathrm{C}^*_{\min}(S)$ is often referred to as the *C*-envelope* of S.

The minimal C*-algebra has been computed for many operator systems in the literature. For now, we give two simple examples. We will consider more examples in Section 6.6.

Example 4.6.1. If A is a unital C*-algebra, then it is clear that $\mathrm{C}^*_{\min}(A) = A$.

Example 4.6.2. Let A be a simple unital C*-algebra and let $S \subseteq A$ be an operator system such that $\mathrm{C}^*(S) = A$. Since $\mathrm{C}^*_{\min}(S)$ is a quotient of A, the simplicity of A implies that $\mathrm{C}^*_{\min}(S) = A$.

Let K be a compact nc convex set. Then it follows from the universal properties of the maximal C*-algebra $\mathrm{C}(K)$ and the minimal C*-algebra $\mathrm{C}^*_{\min}(\mathrm{A}(K))$ that there is a unique surjective homomorphism $\pi : \mathrm{C}(K) \to \mathrm{C}^*_{\min}(\mathrm{A}(K))$ such that $\pi|_{\mathrm{A}(K)} = \iota$, where $\iota : \mathrm{A}(K) \to \mathrm{C}^*_{\min}(\mathrm{A}(K))$ denotes the canonical unital complete order embedding. We will say more about the relationship between K and the structure of $\mathrm{C}^*_{\min}(\mathrm{A}(K))$ in Section 6.5.

5. Dilations of points and representations of maps

5.1. **Dilations, compressions and maximal points.** For a compact nc convex set K, unital completely positive maps on $\mathrm{C}(K)$ play the role of probability measures in the classical setting. The nc state space of $\mathrm{C}(K)$ is a compact nc convex set, and relationships between the graded components of this space provide it with a rich structure that has no classical counterpart.

Definition 5.1.1. Let K be an nc convex set. We will say that a point $x \in K_m$ is *dilated* by a point $y \in K_n$ and refer to y as a *dilation* of x

if there is an isometry $\alpha \in \mathcal{M}_{n,m}$ such that $x = \alpha^* y \alpha$. In this case we will say that x is a *compression* of y. If y decomposes with respect to the range of α as $y = x \oplus z$ for some $z \in K$, then we will say that the dilation is *trivial.* We will say that x is *maximal* if it has no non-trivial dilations.

Remark 5.1.2. Suppose that $x \in K_n$ can be written as a finite nc convex combination $x = \sum \alpha_i^* x_i \alpha_i$ for $\{x_i \in K_{n_i}\}$ and $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$. Let $y = \oplus_{i=1}^k x_i$ and let $\alpha = [\alpha_1 \ \cdots \ \alpha_k]^t$. Then α is an isometry and $x = \alpha^* y \alpha$, so x is a compression of y. Hence if x is maximal, then $y \simeq x \oplus z$ for some $z \in K$.

The next result is a restatement of an important result of Dritschel and McCullough [17, Theorem 1.2].

Theorem 5.1.3. *Let K be a compact nc convex set. Then every point in K has a maximal dilation.*

We will give a new proof of Theorem 5.1.3 in Section 8.7 using ideas from this paper.

5.2. **Representations of maps.** Stinespring's dilation theorem asserts that completely positive maps on C*-algebras dilate to representations. However, understanding the dilation theory of completely positive maps on more general operator systems is a much more difficult problem. The framework of noncommutative convexity provides a powerful new perspective on this issue.

Let K be a compact nc convex set. In this section we will begin to see how questions about unital completely positive maps on $\mathrm{C}(K)$ can be reduced to questions about points in K.

If $\pi : \mathrm{C}(K) \to \mathcal{M}_n$ is a representation, then there is an nc state $x \in K_n$ such that $\pi = \delta_x$. Specifically, $x = \pi|_{\mathrm{A}(K)}$ is the barycenter of π. Therefore, if $\mu : \mathrm{C}(K) \to \mathcal{M}_m$ is a unital completely positive map, then Stinespring's theorem implies there is a point $x \in K_n$ and an isometry $\alpha \in \mathcal{M}_{m,n}$ such that $\mu = \alpha^* \delta_x \alpha$. Considered as points in the nc state space of $\mathrm{C}(K)$, μ is dilated by δ_x in the terminology of Definition 5.1.1.

Definition 5.2.1. Let K be a compact nc convex set and let $\mu : \mathrm{C}(K) \to \mathcal{M}_m$ be a unital completely positive map. We will say that a pair (x, α) consisting of a point $x \in K_n$ and an isometry $\alpha \in \mathcal{M}_{n,m}$ *is a representation of* μ if $\mu = \alpha^* \delta_x \alpha$. We will say that the representation (x, α) of μ is *minimal* if $\{f(x)\alpha H_m : f \in \mathrm{C}(K)\}$ is dense in H_n.

Remark 5.2.2. By Stinespring's theorem, a minimal representation $(x, \alpha) \in K_n \times \mathcal{M}_{n,m}$ of μ is unique in the sense that if $(y, \beta) \in K_p \times \mathcal{M}_{p,m}$

is another minimal representation of μ, then $n = p$ and there is a unitary $\gamma \in \mathcal{M}_n$ such that $x = \gamma y \gamma^*$ and $\alpha = \gamma^* \beta$.

In Section 4.5, we observed that every unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_m$ extends to a unital completely positive map $\mu : \mathrm{B}(K) \to \mathcal{M}_m$ using the fact that $\mathrm{B}(K)$ is the enveloping von Neumann algebra of $\mathrm{C}(K)$. This extension can be described more concretely in the following way: Let $(x, \alpha) \in K_n \times \mathcal{M}_{n,m}$ be a minimal representation of μ. Then μ can be extended by defining

$$\mu(f) = \alpha^* f(x) \alpha, \quad \text{for} \quad f \in \mathrm{B}(K).$$

To see that this extension is well defined, let $(y, \beta) \in K_p \times \mathcal{M}_{p,m}$ be another minimal representation. Then from above, there is a unitary $\gamma \in \mathcal{M}_n$ such that $x = \gamma y \gamma^*$ and $\alpha = \gamma^* \beta$. Then by the unitary equivariance of f,

$$\beta^* f(y) \beta = \beta^* f(\gamma^* y \gamma) \beta = \beta^* \gamma^* f(y) \gamma \beta = \alpha^* f(y) \alpha.$$

The map δ_x is normal on $\mathrm{B}(K)$, so μ is the composition of normal maps, and hence is itself normal. The fact that this definition of μ agrees with the previous definition now follows from the uniqueness of the normal extension of a unital completely positive map to the enveloping von Neumann algebra.

Using the notion of maximal points, we can now characterize points with unique representing maps in the sense of Section 4.5.

Proposition 5.2.3. *Let K be a compact nc convex set. A point in K has a unique representing map if and only if it is maximal.*

Proof. Suppose $x \in K_m$ has a unique representing map. Let $y \in K_n$ be a maximal dilation of x. Then there is an isometry $\alpha \in \mathcal{M}_{n,m}$ such that $x = \alpha^* y \alpha$. Define a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_m$ by $\mu = \alpha^* \delta_y \alpha$. Then μ has barycenter x. Since x has a unique representing map, it follows that $\mu = \delta_x$. Therefore, $\delta_y \cong \delta_x \oplus \delta_z$ for some $z \in K_p$, where the decomposition is taken with respect to the range of α. In particular, $y \cong x \oplus z$. Since the summands of a maximal point in K are maximal, it follows that x is maximal.

Conversely, suppose that $x \in K_m$ is maximal. Let $\mu : \mathrm{C}(K) \to \mathcal{M}_m$ be a unital completely positive map with barycenter x. Let $(y, \alpha) \in K_n \times \mathcal{M}_{n,m}$ be a representation of μ. Then $x = \alpha^* y \alpha$, so y is a dilation of x. The fact that x is maximal implies that $y \cong x \oplus z$ for some $z \in K_p$, where the decomposition is taken with respect to the range of α. Hence $\delta_y = \delta_x \oplus \delta_z$, so $\mu = \delta_x$. □

Proposition 5.2.4. *Let K be a compact nc convex set. If $x \in K_n$ is maximal, then the corresponding representation $\delta_x : \mathrm{C}(K) \to \mathcal{M}_n$*

factors through $\mathrm{C}^*_{\min}(\mathrm{A}(K))$. *Conversely, if the only representing map for* x *that factors through* $\mathrm{C}^*_{\min}(\mathrm{A}(K))$ *is* δ_x*, then* x *is maximal.*

Proof. Suppose $x \in K_n$ is maximal. Let $\iota : \mathrm{A}(K) \to \mathrm{C}^*_{\min}(\mathrm{A}(K))$ denote the canonical embedding and define $\varphi : \iota(\mathrm{A}(K)) \to \mathcal{M}_n$ by $\varphi = x \circ \iota^{-1}$. By Arveson's extension theorem we can extend φ to a unital completely positive map $\psi : \mathrm{C}^*_{\min}(\mathrm{A}(K)) \to \mathcal{M}_n$. Let $q : \mathrm{C}(K) \to \mathrm{C}^*_{\min}(\mathrm{A}(K))$ denote the canonical quotient map. Then $(\varphi \circ q)|_{\mathrm{A}(K)} = x$. Since x has a unique representing map, it follows that $\varphi \circ q = \delta_x$. In particular, $\ker \delta_x \supseteq \ker q$.

Conversely, suppose that the only representing map for x that factors through $\mathrm{C}^*_{\min}(\mathrm{A}(K))$ is δ_x. Let $y \in K_p$ be a maximal dilation of x and let $\alpha \in \mathcal{M}_{p,n}$ be an isometry such that $x = \alpha^* y \alpha$. Define a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ by $\mu = \alpha^* \delta_y \alpha$. Then μ has barycenter x. From above, δ_y factors through $\mathrm{C}^*_{\min}(\mathrm{A}(K))$. Hence μ also factors through $\mathrm{C}^*_{\min}(\mathrm{A}(K))$. Therefore, by assumption $\mu = \delta_x$ and arguing as in the proof of Proposition 5.2.3 implies that x is maximal. □

6. Extreme points

6.1. **Extreme points.** In this section we will introduce the definition of extreme point for an nc convex set. The basic idea is that there should be no way of expressing an extreme point as a non-trivial nc convex combination.

Definition 6.1.1. Let K be an nc convex set. We will say that a point $x \in K_n$ is *extreme* if whenever x is written as a finite nc convex combination $x = \sum \alpha_i^* x_i \alpha_i$ for $\{x_i \in K_{n_i}\}$ and nonzero $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$, then each α_i is a positive scalar multiple of an isometry $\beta_i \in \mathcal{M}_{n_i,n}$ satisfying $\beta_i^* x_i \beta_i = x$ and each x_i decomposes with respect to the range of α_i as a direct sum $x_i = y_i \oplus z_i$ for $y_i, z_i \in K$ with y_i unitarily equivalent to x. The set of all extreme points is denoted $\partial K = \coprod_n (\partial K)_n$.

We will occasionally be interested in the (classical) extreme points of the compact convex set K_n for some n, which we will denote by ∂K_n.

We also define a notion of pure point, which more closely resembles the classical notion of extreme point. We are grateful to Bojan Magajna for suggesting a definition that is preserved by affine nc homeomorphisms (see Proposition 6.1.5) inspired by [35].

Definition 6.1.2. Let K be an nc convex set. We will say that a point $x \in K_n$ is *pure* if whenever x is written as a finite nc convex combination $x = \sum \alpha_i^* x_i \alpha_i$ for $\{x_i \in K_{n_i}\}$ and nonzero $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$

satisfying $\sum \alpha_i^* \alpha_i = 1_n$, then each α_i is a positive scalar multiple of an isometry $\beta_i \in \mathcal{M}_{n_i,n}$ satisfying $\beta_i^* x_i \beta_i = x$.

Remark 6.1.3. A pure point $x \in K_n$ is a (classical) extreme point of the compact convex set K_n. However, we will see in the next proposition that a (classical) extreme point of K_n is not necessarily pure. If x is pure, then it cannot be decomposed as a (non-trivial) direct sum, so the corresponding representation $\delta_x : \mathrm{C}(K) \to \mathcal{M}_n$ is irreducible. Note however that even if δ_x is irreducible, it is not necessarily true that x is pure. For example, for any $x \in K_1$, δ_x is a character on $\mathrm{C}(K)$, and in particular is irreducible.

Proposition 6.1.4. *Let K be an nc convex set. A point $x \in K$ is extreme if and only if it is both pure and maximal.*

Proof. Suppose x can be written as a finite nc convex combination $x = \sum \alpha_i^* x_i \alpha_i$ for $\{x_i \in K_{n_i}\}$ and nonzero $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$. The condition that each α_i is a positive scalar multiple of an isometry $\beta_i \in \mathcal{M}_{n_i,n}$ satisfying $\beta_i^* x_i \beta_i = x$ is equivalent to x being pure. The condition that each x_i decomposes with respect to the range of α_i as a direct sum $x_i = y_i \oplus z_i$ for $y_i, z_i \in K$ with y_i unitarily equivalent to x, combined with the preceding condition, is equivalent to the maximality of x. ☐

The next result will be (implicitly) invoked when we apply the dual equivalence between compact nc convex sets and operator systems from Section 3.

Proposition 6.1.5. *Let K and L be nc convex sets and let $\theta : K \to L$ be an affine nc homeomorphism. Then θ maps pure points in K to pure points in L and maximal points in K to maximal points in L. Hence θ maps extreme points in K to extreme points in L.*

Proof. Let K and L be nc convex sets and let $\theta : K \to L$ be an affine nc homeomorphism. Then there is an inverse affine nc homeomorphism $\theta^{-1} : L \to K$.

Let $x \in K_n$ be a pure point and suppose that $\theta(x)$ can be written as a finite nc convex combination $\theta(x) = \sum \alpha_i^* y_i \alpha_i$ for $\{y_i \in L_{n_i}\}$ and nonzero $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$. Then applying θ^{-1} to both sides implies $x = \sum \alpha_i^* \theta^{-1}(y_i) \alpha_i$. Since x is pure, each α_i is a positive scalar multiple of an isometry $\beta_i \in \mathcal{M}_{n_i,n}$ satisfying $\beta_i^* \theta^{-1}(y_i) \beta_i = x$, say $\alpha_i = \gamma_i \beta_i$ for $\gamma_i > 0$. Applying θ to both sides implies $\beta_i^* y_i \beta_i = \theta(x)$. Hence $\theta(x)$ is pure.

Now let $z \in K_m$ be a maximal point and let $u \in L_n$ be a dilation of $\theta(z)$. Then there is an isometry $\xi \in \mathcal{M}_{n,m}$ such that $\theta(z) = \xi^* u \xi$.

Applying θ^{-1} to both sides implies $z = \xi^* \theta^{-1}(u) \xi$. Hence $\theta^{-1}(u)$ is a dilation of z. Since z is maximal, $\theta^{-1}(u)$ decomposes with respect to the range of ξ as $\theta^{-1}(u) = z \oplus w$ for some $w \in K$. Since θ respects direct sums, applying θ to both sides implies u decomposes with respect to the range of ξ as $u = \theta(z) \oplus \theta(v)$. Hence $\theta(z)$ is maximal.

The fact that θ maps extreme points in K to extreme points in L now follows from Proposition 6.1.4. $\square$

Remark 6.1.6. Say that a compact nc convex set K over a dual operator space E is *regularly embedded* if there is an *nc hyperplane* $H \subseteq \mathcal{M}(E)$ of the form

$$H_n = \{x \in E_n : \theta(x) = \gamma 1_n\}$$

for a continuous affine nc map $\theta : E \to \mathcal{M}$ and a constant $\gamma \in \mathbb{R}$ such that $K \subseteq H$ and $0_n \notin H_n$ for all n. This is a noncommutative analogue of the notion of a regular embedding of a compact convex set (see [1, Chapter 2]).

If K is regularly embedded, then a point $x \in K_n$ is pure in the sense of Definition 6.1.2 if and only if whenever x is written as a finite nc convex combination $x = \sum \alpha_i^* x_i \alpha_i$ for $\{x_i \in K_{n_i}\}$ and nonzero $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$, then each $\alpha_i^* x_i \alpha_i$ is a positive scalar multiple of x. This is analogous to the definition of a pure unital completely positive map (see e.g. [12]).

To see this, suppose that $\alpha_i^* x_i \alpha_i = \delta_i x$ for $\delta_i > 0$ and let $\beta_i = \delta_i^{-1/2} \alpha_i$. Then $\beta_i^* x_i \beta_i = x$. Applying θ to both sides implies $\gamma \beta_i^* \beta_i = \gamma 1_n$, i.e. $\beta_i^* \beta_i = 1_n$. Hence β_i is an isometry satisfying $\beta_i^* x_i \beta_i = x$.

The canonical affine nc homeomorphism from K to the nc state space of the operator system $\mathrm{A}(K)$ of affine nc functions on K is a regular embedding of K into the dual operator space $\mathrm{A}(K)^*$ with respect to the nc hyperplane $H \subseteq \mathcal{M}(\mathrm{A}(K)^*)$ defined by

$$H_n = \{x \in \mathcal{M}_n(\mathrm{A}(K)^*) : 1_{\mathrm{A}(K)}(x) = 1_n\},$$

where $1_{\mathrm{A}(K)} \in \mathrm{A}(K)$ denotes the unit. Hence in this case, the points in K that are pure in the sense of Definition 6.1.2 are precisely the points in K that are pure unital completely positive maps.

Example 6.1.7. Let A be a C*-algebra with nc state space K. Arveson [2, Corollary 1.4.3] showed that a point $x \in K_n$ is pure if and only if x is a compression of an irreducible representation of A. In particular, if A is commutative so that every irreducible representation of A is a character, then for $n \geq 2$ no point of K_n is pure.

Example 6.1.8. Let A be a C*-algebra with nc state space K so that A is completely order isomorphic to $\mathrm{A}(K)$. If $x \in K$ is a representation of A, then it is clear that x is necessarily maximal. On the other hand, if x is maximal, then by Proposition 5.2.3, the representation δ_x is the unique representing map for x. Moreover, by Proposition 5.2.4, δ_x factors through $\mathrm{C}^*_{\min}(\mathrm{A}(K)) = A$. So x is a representation of A. Therefore, x is maximal precisely when it is a representation of A. If $x \in K$ is a representation, then Example 6.1.7 implies that it is pure if and only if it is irreducible. It follows that the extreme points ∂K of K are precisely the irreducible representations of A.

Theorem 6.1.9. *Let K be a compact nc convex set. A point $x \in K_n$ is an extreme point if and only if the representation $\delta_x : \mathrm{C}(K) \to M_n$ is both irreducible and the unique representing map for x.*

Proof. If x is extreme, then by Proposition 6.1.4 it is pure and maximal. In this case, Remark 6.1.3 implies that δ_x is irreducible and Proposition 5.2.3 implies that δ_x is the unique representing map for x.

For the converse, suppose that δ_x is both irreducible and the unique representing map for $x \in K_n$. By Proposition 6.1.4, to show that x is extreme it suffices to show that x is pure and maximal. Proposition 5.2.3 implies that x is maximal.

To see that x is pure, suppose that x can be written as a finite nc convex combination $x = \sum \alpha_i^* x_i \alpha_i$ for $\{x_i \in K_{n_i}\}$ and nonzero $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$. Define a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ by $\mu = \sum \alpha_i^* \delta_{x_i} \alpha_i$. Then μ has barycenter x, and hence represents x. Since x has a unique representing map, this implies $\mu = \delta_x$. Since δ_x is irreducible, it follows from Example 6.1.7 that it is a pure point in the nc state space of $\mathrm{C}(K)$. Hence each α_i is a scalar multiple of an isometry β_i satisfying $\beta_i^* \delta_{x_i} \beta_i = \delta_x$, implying $\beta_i^* x_i \beta_i = x$. Hence x is pure. $\square$

Example 6.1.10. Let C be a compact convex set and let $\mathrm{A}(C)$ denote the function system of continuous affine functions on C, considered as an operator subsystem of the C*-algebra $\mathrm{C}(C)$ of continuous functions on C. Let K denote the nc state space of $\mathrm{A}(C)$, so that $\mathrm{A}(C)$ is completely order isomorphic to $\mathrm{A}(K)$. Then $K_1 = C$ and $\mathrm{C}^*_{\min}(\mathrm{A}(K)) = \mathrm{C}(\overline{\partial C})$ (see the beginning of Section 4.6). We will show that $\partial K = \partial C$.

For $x \in \partial K$, Theorem 6.1.9 implies that the representation δ_x is both irreducible and maximal. In this case, Proposition 5.2.4 implies that δ_x factors through $\mathrm{C}(\overline{\partial C})$. Since $\mathrm{C}(\overline{\partial C})$ is commutative, it follows that $x \in K_1$. Hence $x \in (\partial K)_1$ and it is clear that $x \in \partial C$.

On the other hand, suppose $x \in \partial C$. If $y \in K_n$ dilates x, then there is an isometry $\alpha \in \mathcal{M}_{n,1}$ such that $x = \alpha^* y \alpha$. Define a state $\mu : \mathrm{C}(K) \to \mathbb{C}$ by $\mu = \alpha^* \delta_y \alpha$. By Proposition 5.2.4, δ_y factors through

$\mathrm{C}(\overline{\partial C})$. Hence μ factors through $\mathrm{C}(\overline{\partial C})$. So by the Riesz-Markov-Kakutani representation theorem, μ can be identified with a probability measure on $\overline{\partial C}$ with barycenter x. Since x is an extreme point in C, it follows that $\mu = \delta_x$. Hence y is a trivial dilation of x, implying that x is maximal. Since δ_x is irreducible, it follows from Theorem 6.1.9 that $x \in (\partial K)_1$. Therefore $\partial K = \partial C$.

6.2. **Existence of extreme points.** The fact that every compact nc convex set has extreme points is highly non-trivial. In fact, it is equivalent to a conjecture of Arveson [2] about the existence of boundary representations for operator systems, which was open for over 45 years. The conjecture was eventually verified by Arveson himself [3] in the separable case and by the authors [12] in the general case.

Let S be an operator system. An irreducible representation $\pi : \mathrm{C}^*_{\min}(S) \to \mathcal{B}(H)$ is said to be a boundary representation for S if whenever $\varphi : \mathrm{C}^*_{\min}(S) \to \mathcal{B}(H)$ is a unital completely positive map satisfying $\varphi|_S = \pi|_S$, then $\varphi = \pi$. In other words, π is a boundary representation for S if the restriction $\pi|_S$ has a unique extension to a unital completely positive map on $\mathrm{C}^*_{\min}(S)$. The next result is an immediate consequence of Theorem 6.1.9.

Corollary 6.2.1. *Let S be an operator system with nc state space K. The extreme points ∂K of K are precisely the restrictions of boundary representations of S.*

The next result is a restatement of [12, Theorem 2.4]. It asserts that the existence of extreme points in a compact nc convex set,

Theorem 6.2.2. *Let K be a compact nc convex set. Then every pure point in K has an extreme dilation.*

We will give a new proof of Theorem 6.2.2 using ideas from this paper in Section 8.7.

The next result is a restatement of [12, Theorem 3.1]. It asserts that the extreme points in a compact nc convex set completely norm the nc affine functions on the set.

Theorem 6.2.3. *Let K be a compact nc convex set. For every $n \in \mathbb{N}$ and $a \in \mathcal{M}_n(\mathrm{A}(K))$ there is an extreme point $x \in \partial K$ such that $\|a\| = \|a(x)\|$.*

The next result is a restatement of [12, Theorem 3.4]. It asserts that the extreme points in a compact nc convex set give rise to a faithful representation of the minimal C*-algebra of the corresponding operator system of continuous affine nc functions.

Theorem 6.2.4. *Let K be a compact nc convex set. Define a Hilbert space $H = \oplus_n \oplus_{x \in (\partial K)_n} H_n$ and let $\pi : \mathrm{C}(K) \to \mathcal{B}(H)$ denote the representation defined by $\pi = \oplus_{x \in \partial K} \delta_x$. Then the restriction $\pi|_{\mathrm{A}(K)}$ is a complete order monomorphism and the image $\pi(\mathrm{C}(K))$ is isomorphic to $\mathrm{C}^*_{\min}(\mathrm{A}(K))$.*

6.3. **Accessibility of extreme points.** Let K be a compact nc convex set. It was shown in [12, Theorem 3.4] that the map restricting functions in $\mathrm{A}(K)$ to ∂K is completely isometric. In particular, this implies that the minimal C*-algebra $\mathrm{C}^*_{\min}(\mathrm{A}(K))$ is completely determined by this restriction. However, the set of points in ∂K that can be obtained by dilating pure points in finite components of K as in Theorem 6.2.2 can be a proper subset of ∂K. The corresponding boundary representations are called accessible in [33]. We thank Ben Passer for providing some examples, of which the following is a variant.

Example 6.3.1. Let $\mathbb{F}_2 = \langle u, v \rangle$ denote the free group on two generators and let $S = \mathrm{span}\{1, u, u^*, v, v^*\}$ in $\mathrm{C}^*(\mathbb{F}_2)$. Let K denote the nc state space of S, so that $\mathrm{A}(K)$ is completely order isomorphic to S. Let $a, b \in \mathrm{A}(K)$ denote the continuous affine nc functions corresponding to $u, v \in S$ respectively.

A point $x \in K_m$ is completely determined by the pair of contractions $(a(x), b(x))$. If $(a(x), b(x))$ is a pair of unitaries, then x is maximal, since it does not have non-trivial dilations. Since unitaries are extreme points in the unit ball of $\mathcal{M}_m$, it follows that if $(a(x), b(x))$ is an irreducible pair of unitaries, in the sense that they do not have any common non-trivial invariant subspaces, then x is an extreme point. There are many such examples for any m.

For $m \in \mathbb{N}$, the extreme points of the unit ball of $\mathcal{M}_m$ are precisely the unitaries. Hence if $x \in K_m$ is pure, then $(a(x), b(x))$ is an irreducible pair of unitaries, implying that x is an extreme point. In particular, pure points in K_m do not have non-trivial pure dilations.

It follows that for m infinite, an extreme point in K_m cannot be obtained as the limit of an increasing sequence of finite dimensional pure compressions.

6.4. **Noncommutative Krein-Milman theorem.** In this section we will prove a noncommutative analogue of the Krein-Milman theorem asserting that every compact nc convex set is the closed nc convex hull of its extreme points. We will also prove an analogue of Milman's partial converse to the Krein-Milman theorem.

Definition 6.4.1. For a dual operator space E and a subset $X \subseteq \mathcal{M}(E)$, the *closed nc convex hull* $\overline{\mathrm{ncconv}}(X)$ of X is the intersection of all closed nc convex sets over E that contain X. Equivalently, the closed nc convex hull of X is the the closure of the set of all nc convex combinations of elements in X.

Theorem 6.4.2 (Noncommutative Krein-Milman theorem)**.**
A compact nc convex set is the closed nc convex hull of its extreme points.

Proof. Let L be a compact nc convex set. By the results in Section 3, we can identify L with the nc state space of the operator system $\mathrm{A}(L)$ of continuous affine nc functions on L. In particular, L is a compact nc convex set over the dual operator space $\mathrm{A}(L)^*$.

Let K denote the closed nc convex hull of the extreme points of L. Clearly $K \subseteq L$. Suppose for the sake of contradiction there is n and $y \in L_n \setminus K_n$. Then by Corollary 2.4.2, there is a normal completely bounded linear map $\varphi : A(L)^* \to \mathcal{M}_n$ and self-adjoint $\gamma \in \mathcal{M}_n$ such that

$$\operatorname{Re}\varphi_n(y) \not\leq \gamma \otimes 1_n \quad \text{but} \quad \operatorname{Re}\varphi_p(x) \leq \gamma \otimes 1_p$$

for every p and $x \in K_p$.

Since φ is normal, there is $a \in \mathcal{M}_n(\mathrm{A}(L))$ such that $\varphi_n(x) = a(x)$ for all $x \in L_n$. Let $b = (a + a^*)/2$. Then from above, $b(y) \not\leq \gamma \otimes 1_n$ but $b(x) \leq \gamma \otimes 1_p$ for all p and $x \in K_p$.

Let $H = \oplus_n \oplus_{x \in (\partial L)_n} H_n$ and let $\pi : \mathrm{C}(L) \to \mathcal{B}(H)$ denote the representation defined by $\pi = \oplus_{x \in \partial L} \delta_x$. Then by Theorem 6.2.4, the restriction $\pi|_{\mathrm{A}(L)}$ is a complete order monomorphism. Hence from above, $\pi(b) \leq \gamma \otimes 1_H$. It follows that $b \leq \gamma \otimes 1_{\mathrm{A}(L)}$, so in particular $b(y) \leq \gamma \otimes 1_n$, giving a contradiction. □

The next result is a noncommutative analogue of Milman's partial converse to the Krein-Milman theorem.

Theorem 6.4.3. *Let K be a compact nc convex set. Let $X \subseteq K$ be a closed subset that is closed under compressions, meaning that*

$$\alpha^* X_n \alpha \subseteq X_m$$

for every isometry $\alpha \in \mathcal{M}_{n,m}$. If the closed nc convex hull of X is K, then $\partial K \subseteq X$.

Proof. Let L denote the nc state space of $\mathrm{C}(K)$ and let

$$Z = \overline{\operatorname{ncconv}}\{\delta_x : x \in X\} \subseteq L.$$

Since $K = \overline{\operatorname{ncconv}}(X)$ and the barycenter map from L onto K is continuous and affine, it follows that for every $y \in K$ there is $\mu \in Z$ with barycenter y.

Fix $y \in \partial K$ and $\mu \in Z$ with barycenter y. Since y is extreme, Theorem 6.1.9 implies that $\mu = \delta_y$. For $n \in \mathbb{N}$, there is a standard trick to identify an n-dimensional compression of δ_y with a state supported on the representation $\delta_y \otimes \mathrm{id}_n$ on $\mathcal{M}_n(\mathrm{C}(K))$. Since δ_y is irreducible, so is $\delta_y \otimes \mathrm{id}_n$. In particular, every state supported on $\delta_y \otimes \mathrm{id}_n$ is a pure state. By construction $\ker \delta_y \supseteq \cap_{x \in X} \ker \delta_x$, so $\ker \delta_y \otimes \mathrm{id}_n \supseteq \cap_{x \in X} \ker \delta_x \otimes \mathrm{id}_n$. Hence by [15, Proposition 3.4.2 (ii)], every state that factors through $\delta_y \otimes \mathrm{id}_n$ is a limit of a net of pure states, each of which is supported on some $\delta_x \otimes \mathrm{id}_n$ for $x \in X$.

This translates to the statement that every n-dimensional compression of δ_y is a point-weak* limit of compressions of $\{\delta_x : x \in X\}$. Since the barycenter map is continuous and affine, and since X is both closed and closed under compressions, arguing as in the proof of Proposition 2.2.9 implies that $y \in X$. □

Remark 6.4.4. Simple examples demonstrate that the assumption that X is closed under compressions is necessary. For instance, consider Example 6.1.10. Fix any point $y \in K_n$ and let X denote the closure of the set $\{x \oplus y : x \in \partial K\}$. This is contained in K_{n+1}, so it is disjoint from $\partial K = \partial C \subset K_1$. Nevertheless, it follows from Theorem 6.4.2 that K is the closed nc convex hull of X.

This trick fails for certain infinite dimensional examples like the Cuntz system of Examples 4.2.6 and 6.6.2. It follows from a version of Voiculescu's non-commutative Weyl-von Neumann theorem (see [7, Corollary 1.7.7]) that for any $y \in K_n$ with $n < \infty$, the point-norm closure of $\{x \oplus y : x \in \partial K\}$ contains all representations of $\mathcal{O}_n$ (restricted to S). So in particular, the point-weak-$*$ closure contains ∂K.

6.5. **Extreme points and the minimal C*-algebra.** In this section we relate the extreme points of a compact nc convex set to the nc state space of the corresponding minimal C*-algebra.

For a compact nc convex set K, the universal properties of $\mathrm{C}(K)$ and $\mathrm{C}^*_{\min}(\mathrm{A}(K))$ imply the existence of a unique surjective homomorphism $\pi : \mathrm{C}(K) \to \mathrm{C}^*_{\min}(\mathrm{A}(K))$ such that $\pi|_{\mathrm{A}(K)} = \iota$, where $\iota : \mathrm{A}(K) \to \mathrm{C}^*_{\min}(\mathrm{A}(K))$ denotes the canonical unital complete order embedding. It follows from the results in Section 3 that the nc state space of $\mathrm{C}^*_{\min}(\mathrm{A}(K))$ is affinely homeomorphic to a closed subset of nc states on $\mathrm{C}(K)$; namely, the set of nc states on $\mathrm{C}(K)$ that factor through $\mathrm{C}^*_{\min}(\mathrm{A}(K))$.

For $x \in \partial K$, Proposition 5.2.3 implies that the corresponding representation δ_x factors through $\mathrm{C}^*_{\min}(\mathrm{A}(K))$. Hence the extreme boundary ∂K corresponds to a subset of the irreducible representations of

$\mathrm{C}^*_{\min}(\mathrm{A}(K))$. However, we have seen that, as in the classical setting, this subset will often be proper (see e.g. Example 6.6.3).

Motivated by the classical setting, we can think of the set of irreducible representations of $\mathrm{C}^*_{\min}(\mathrm{A}(K))$ as the Shilov boundary of $\mathrm{A}(K)$, and ∂K as the Choquet boundary of $\mathrm{A}(K)$. The next result describes the precise relationship between these sets. It can be viewed as a noncommutative analogue of the fact that in the classical setting, the Shilov boundary is the closure of the Choquet boundary.

Theorem 6.5.1. *Let K be a compact nc convex set and let L denote the nc state space of $\mathrm{C}^*_{\min}(\mathrm{A}(K))$, identified with the set of nc states on $\mathrm{C}(K)$ that factor through $\mathrm{C}^*_{\min}(\mathrm{A}(K))$. Then L is the closed nc convex hull of the set $\{\delta_x \in \partial K\}$.*

Proof. Let $X = \{\delta_x \in \partial K\}$. For $x \in \partial K$, Proposition 5.2.4 implies that the corresponding representation δ_x factors through $\mathrm{C}^*_{\min}(\mathrm{A}(K))$. Hence the representation $\sigma := \oplus_{x\in\partial K}\delta_x$ factors through $\mathrm{C}^*_{\min}(\mathrm{A}(K))$, and we can view it as a representation of $\mathrm{C}^*_{\min}(\mathrm{A}(K))$. Theorem 6.2.3 implies that the restriction $\sigma|_{\mathrm{A}(K)}$ is a unital complete order embedding. Hence by the universal property of $\mathrm{C}^*_{\min}(\mathrm{A}(K))$, σ is faithful.

Let $y \in K$ be a point such that the corresponding representation δ_y factors through $\mathrm{C}^*_{\min}(\mathrm{A}(K))$. Then

$$\ker \delta_y \supseteq \bigcap_{x\in\partial K} \ker \delta_x = \ker \sigma.$$

Hence an argument similar to the proof of Theorem 6.4.3 implies that δ_y is contained in the closed nc convex hull of X. Every irreducible representation of $\mathrm{C}^*_{\min}(\mathrm{A}(K))$ is of this form, and by Example 6.1.8, these are precisely the extreme points of L. Therefore, it follows from Theorem 6.4.2 that L is the closed nc convex hull of X. □

6.6. **Examples.** In this section we will illustrate the results we have obtained so far with some examples.

Example 6.6.1. Let $S \subseteq \mathcal{K}(H)$ be an irreducible operator system, so that $\mathrm{C}^*(S) = \mathcal{K}(H)$, and let $n = \dim H$. Since $\mathcal{K}(H)$ is simple, $\mathrm{C}^*_{\min}(S) = \mathcal{K}(H)$. Let K denote the nc state space of S. For an extreme point $x \in (\partial K)_n$, the corresponding representation $\delta_x : \mathrm{C}(K) \to \mathcal{M}_n$ is an irreducible representation that factors through $\mathcal{K}(H)$. Since every irreducible representation of $\mathcal{K}(H)$ is equivalent to the identity representation $\mathrm{id} : \mathcal{K}(H) \to \mathcal{K}(H)$ and ∂K is closed under unitary equivalence, it follows that

$$\partial K = (\partial K)_n = \{\alpha \,\mathrm{id}\, \alpha^* : \alpha \in \mathrm{U}(H)\},$$

where $n = \dim H$.

Example 6.6.2. For $d \geq 2$, the Cuntz algebra $\mathcal{O}_d$ is the universal C*-algebra generated by d elements $s_1, \ldots, s_d$ satisfying the Cuntz relations

$$\sum_{i=1}^{d} s_i s_i^* = 1, \quad s_i^* s_j = \delta_{ij} 1.$$

The algebra $\mathcal{O}_d$ is simple and infinite dimensional.

Let $S = \operatorname{span}\{1, s_1, s_1^*, \ldots, s_d, s_d^*\}$. Then $\mathrm{C}^*(S) = \mathcal{O}_d$, so by the simplicity of $\mathcal{O}_d$, $\mathrm{C}^*_{\min}(S) = \mathcal{O}_d$. Let K denote the nc state space of S.

Every point $x \in K_n$, is completely determined by the row contraction $X = [x(s_1), \ldots, x(s_d)] \in \mathcal{M}_n^d$. We say that X is irreducible if it cannot be decomposed as a (non-trivial) direct sum. Note that X is irreducible if and only if the representation $\delta_x : \mathrm{C}(K) \to \mathcal{M}_n$ is irreducible. We say that X is a coisometry if $XX^* = \sum_{i=1}^d x(s_i)x(s_i)^* = 1_n$. If, in addition, $X^*X = \big[x(s_i)^* x(s_j)\big] = 1_d \otimes 1_n$, then we say that X is a row unitary. Note that X is a row unitary if and only if $x(s_1), \ldots, x(s_d)$ satisfy the Cuntz relations.

Suppose that the representation δ_x is the unique representing map for x. Proposition 5.2.3 and Proposition 5.2.4 imply that δ_x factors through $\mathrm{C}^*_{\min}(S) = \mathcal{O}_d$, so $x(s_1), \ldots, x(s_d)$ satisfy the Cuntz relations. Hence X is a row unitary. In particular, if x is extreme, then X is a row unitary.

Conversely, suppose that X is a row unitary and let $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ be a unital completely positive map with barycenter x. Then by the Kadison-Schwartz inequality,

$$1_n = \sum_{i=1}^{n} x(s_i)x(s_i)^* = \sum_{i=1}^{n} \mu(s_i)\mu(s_i)^* \leq \sum_{i=1}^{n} \mu(s_i s_i^*) = 1_n.$$

This implies that $\mu(s_i s_i^*) = x(s_i)x(s_i)^*$ for each i, so S belongs to the multiplicative domain of μ. Hence $\mu = \delta_x$, implying δ_x is the unique representing map for x.

It now follows from Theorem 6.1.9 that x is extreme if and only if X is an irreducible row unitary. Hence there is a correspondence between points in the extreme boundary ∂K and irreducible representations of $\mathrm{C}^*_{\min}(S) = \mathcal{O}_d$. In particular, K has no finite dimensional extreme points.

Example 6.6.3. Let $a_1, a_2 \in M$ be freely independent semicircular (self-adjoint) operators contained in a von Neumann algebra of type II_1. For example, letting s_1, s_2 denote the generators of the Cuntz algebra $\mathcal{O}_2$ as in Example 6.6.2, we can take $a_i = s_i + s_i^*$ for each i. Consider the operator system $S = \operatorname{span}\{1, a_1, a_2\}$ with nc state space K.

Let $A = \mathrm{C}^*(S)$. Then A is simple by [18], so $\mathrm{C}^*_{\min}(S) = A$. Furthermore, since A is separable, Voiculescu's theorem [47] (see [11, Corollary II.5.6]) implies that all representations of A are approximately unitarily equivalent. In other words, every separable representation of A is a point-norm limit of representations that are all unitarily equivalent to any other separable representation. For our purposes, the interesting thing about the operator system S is that not every irreducible representation of A restricts to an extreme point in ∂K. In particular, ∂K is not closed in the point-norm topology, and thus is not closed in the point-weak* topology.

We now consider specific representations of $\mathcal{O}_2$ belonging to the class of atomic representations classified in [14, Section 3]. Consider the Fock space $F_2 = l^2(\mathbb{F}_2^+)$, where $\mathbb{F}_2^+$ denotes the free semigroup on $\{1, 2\}$, with $\epsilon \in \mathbb{F}_2^+$ denoting the empty word. The canonical orthonormal basis for F_2 is $\{\delta_w : w \in \mathbb{F}_2^+\}$. Let $L_1, L_2 \in \mathcal{B}(F_2)$ denote the isometries defined by $L_i\delta_w = \delta_i w$ for $i = 1, 2$ and $w \in \mathbb{F}_2^+$. Let $\mathcal{H} = \mathbb{C} \oplus F_2$.

For $\lambda \in \mathbb{T}$, consider the representation $\pi_\lambda : \mathcal{O}_2 \to \mathcal{H}$ defined by

$$\pi_\lambda(s_1) = \begin{bmatrix} \lambda & 0 \\ 0 & L_1 \end{bmatrix}, \quad \pi_\lambda(s_2) = \begin{bmatrix} 0 & 0 \\ \delta_\epsilon & L_2 \end{bmatrix}.$$

Let $\sigma_\lambda = \pi_\lambda|_A$.

Identifying $H_{\aleph_0}$ with $\mathcal{H}$, we obtain $x_\lambda \in K_{\aleph_0}$ by setting $x_\lambda = \sigma_\lambda|_S$. The corresponding representation $\delta_{x_\lambda} : \mathrm{C}(K) \to \mathcal{M}_{\aleph_0}$ satisfies $\delta_{x_\lambda} = \sigma_\lambda \circ q$, where $q : \mathrm{C}(K) \to \mathrm{C}^*_{\min}(S) = A$ denotes the canonical quotient homomorphism. We will show that δ_{x_λ} is irreducible for all λ, but that x_λ is an extreme point if and only if $\lambda = \pm 1$.

Since π_λ is an atomic representation of $\mathcal{O}_2$ corresponding to the primitive word '1', it is irreducible by [14]. Write $\lambda = r + is$. Let p be a projection in $\sigma_\lambda(A)'$. Let $\eta = 1 \oplus 0 \in \mathcal{H}$. Replace p by $p^\perp$ if necessary to ensure that $p\eta \neq 0$. We will show that $p = 1_{\mathcal{H}}$.

First note that $\sigma_\lambda(a_1)\eta = 2r\eta$. Hence

$$\sigma_\lambda(a_1)p\eta = p\sigma_\lambda(a_1)\eta = 2rp\eta.$$

Since $L_1 + L_1^*$ does not have any eigenvectors, this implies $p\eta = \eta$. Thus $\sigma_\lambda(a_2)\eta = 0 \oplus \delta_\epsilon$ lies in $\mathrm{Ran}(p)$. It can now be shown by induction on word length that $0 \oplus \delta_w \in \mathrm{Ran}(p)$ for all words $w \in \mathbb{F}_2^+$. Hence $p = 1_{\mathcal{H}}$ and σ_λ is irreducible. It follows that δ_{x_λ} is irreducible.

Now suppose $\lambda = 1$. Since δ_{x_1} is irreducible, Theorem 6.1.9 implies that x_1 is an extreme point if and only if x_1 has a unique representing map. By Proposition 5.2.4, this is the case if and only if the only representing map of x_1 that factors through $C^*_{\min}(S) = \mathcal{O}_2$ is δ_{x_1}. Equivalently, x_1 is an extreme point if and only if whenever $\varphi : A \to \mathcal{B}(\mathcal{H})$ is a unital completely positive map satisfying $\varphi|_S = \sigma_1|_S$, then $\varphi = \sigma_1$.

Let $\varphi : A \to \mathcal{B}(\mathcal{H})$ be a unital completely positive map satisfying $\varphi|_S = \sigma_1|_S$. By Arveson's extension theorem we can extend φ to a unital completely positive map $\varphi_1 : \mathcal{O}_2 \to \mathcal{B}(\mathcal{H})$. Let $\rho_1 : \mathcal{O}_2 \to \mathcal{B}(\mathcal{K})$ be a minimal Stinespring representation for φ_1, so that $\varphi_1 = p_{\mathcal{H}}\rho_1|_{\mathcal{H}}$.

Each $\rho_1(s_i)$ is an isometry, and

$$\begin{aligned} 2\eta &= \sigma_1(a_1)\eta = \varphi_1(a_1)\eta \\ &= p_{\mathcal{H}}\rho_1(a_1)\eta = p_{\mathcal{H}}(\rho_1(s_1) + \rho_1(s_1)^*)\eta. \end{aligned}$$

It follows that $\rho_1(s_1)\eta = \eta$ and $\rho_1(s_1)^*\eta = \eta$. In particular, η lies in $\operatorname{Ran}(\rho_1(s_1)) = \operatorname{Ran}(\rho_1(s_2))^\perp$, so

$$\rho_1(a_2)\eta = (\rho_1(s_1) + \rho_1(s_1)^*)\eta = \rho_1(s_2)\eta.$$

This also shows that η is coinvariant for $\rho_1(s_1)$ and $\rho_1(s_2)$. Hence

$$\delta_\epsilon = \sigma_1(a_2)\eta = \varphi_1(a_2)\eta = p_{\mathcal{H}}\rho_1(a_2)\eta = p_{\mathcal{H}}\rho_1(s_2)\eta.$$

Therefore $\rho_1(s_2)\eta = \delta_\epsilon$.

It now follows that δ_ϵ is a wandering vector for the tuple $(\rho_1(s_1), \rho_1(s_2))$ in the sense of [14]. To see this, note that for $w \in \mathbb{F}_2^+$,

$$\langle \rho_1(s_w)\delta_\epsilon, \delta_\epsilon\rangle = \langle \delta_\epsilon, \rho_1(s_w)^*\delta_\epsilon\rangle.$$

If $w = 1v$ for $v \in \mathbb{F}_2^+$, then $\rho_1(s_w)^*\delta_\epsilon = \rho_1(s_v)^*\rho_1(s_1)^*\delta_\epsilon = 0$. Otherwise, if $w = 2v$, then $\rho_1(s_w)^*\delta_\epsilon = \rho_1(s_v)^*\rho_1(s_2)^*\delta_\epsilon = \rho_1(s_v)^*\eta \in \mathbb{C} \oplus 0$. Either way, the inner product vanishes.

An easy induction argument now shows that $\rho_1(s_i)\delta_w = \sigma_1(s_i)\delta_w$ and $\rho_1(s_i)^*\delta_w = \sigma_1(s_i)^*\delta_w$ for all $w \in \mathbb{F}_2^+$. This implies $\mathbb{C} \oplus F_2$ is invariant for $\rho_1(\mathcal{O}_2)$, and hence that $\rho_1 = \sigma_1 \oplus \rho_1'$ for some representation $\rho_1' : \mathcal{O}_2 \to \mathcal{B}(\mathcal{K} \ominus \mathcal{H})$. Therefore, $\varphi = \sigma_1$, and x_1 is an extreme point. A similar argument works when $\lambda = -1$, so x_{-1} is also an extreme point.

Now suppose $\lambda \neq \pm 1$. Let $\mathcal{L} = \mathbb{C} \oplus F_2 \oplus F_2$ and consider the representation $\tau_\lambda : \mathcal{O}_2 \to \mathcal{B}(\mathcal{L})$ defined by

$$\tau_\lambda(s_1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ s & r & 0 & 0 \\ 0 & 0 & L_1 & 0 \\ -r\delta_\epsilon & s\delta_\epsilon & 0 & L_1 \end{bmatrix}, \quad \tau_\lambda(s_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \delta_\epsilon & L_2 & 0 \\ 0 & 0 & 0 & L_2 \end{bmatrix}.$$

Define $\psi_\lambda : \mathcal{O}_2 \to \mathcal{B}(H)$ by $\psi_\lambda(a) = p_{\mathcal{H}}\tau_\lambda(a)|_{\mathcal{H}}$. Then $\psi_\lambda|_A = \sigma_\lambda|_A$, however

$$\psi_\lambda(a_1^2) - \sigma_\lambda(a_1^2) = s^2\eta\eta^*.$$

Hence in this case x_λ is not an extreme point.

7. Noncommutative convex functions

7.1. **Convex functions and convex envelopes.** The convex structure of a compact convex set C gives rise to the notion of convexity for a function on C. The Stone-Weierstrass theorem for lattices implies that the convex functions span a dense subset of the C*-algebra $\mathrm{C}(C)$ of continuous functions on C. We saw in Section 4.1 that $\mathrm{C}(C)$ is the maximal commutative C*-algebra generated by the function system $\mathrm{A}(C)$ of continuous affine functions on C in a certain precise sense. There is another important idea connecting $\mathrm{C}(C)$ to $\mathrm{A}(C)$ for which the convex structure of C is essential.

For $f \in \mathrm{C}(C)$, the convex envelope $\overline{f}$ of f is the best approximation of f from below by a convex lower semicontinuous function. It is defined by

$$\overline{f} = \sup\{a \in \mathrm{A}(C) : a \leq f\}.$$

The function f is convex if and only if $\bar{f} = f$.

There is also a geometric definition which is more readily generalized. Let $\operatorname{epi}(f) = \{(x,t) : x \in C,\ t \geq f(x)\}$. Then

$$\operatorname{epi}(\overline{f}) = \overline{\operatorname{conv}}(\operatorname{epi}(f))$$

is the closed convex hull of the epigraph of f.

One explanation for the importance of the convex envelope is that it encodes information about the set of representing measures of a point. Specifically, if C is a compact convex set and $f \in \mathrm{C}(C)$ is a continuous function with convex envelope $\overline{f}$, then for $x \in C$,

$$\overline{f}(x) = \inf_{\mu} \int_C f \, d\mu,$$

where the infimum is taken over all probability measures μ with barycenter x. This infimum is attained. Moreover, the measure μ is supported on the extreme boundary ∂C in an appropriate sense if and only if

$$\int_C f \, d\mu = \int_C \overline{f} \, d\mu$$

for every $f \in \mathrm{C}(C)$. We will revisit this characterization in Section 8.

7.2. **Noncommutative convex functions.** In this section we will introduce a notion of convexity for nc functions. We will need to consider matrices of bounded nc functions. For a compact nc convex set K and $f = (f_{ij}) \in \mathcal{M}_n(\mathrm{B}(K))$, we view f as a function $f : K \to \mathcal{M}_n(\mathcal{M})$ defined by $f(x) = (f_{ij}(x))$ for $x \in K$. Note that f is graded, respects direct sums and is unitarily equivariant in an appropriate sense, so we will refer to f as a nc function on K. We will say that f is *self-adjoint* if $f(x) \in \mathcal{M}_n(\mathcal{M}_k)_{sa}$ for all k and all $x \in K_k$.

Definition 7.2.1. Let K be a compact nc convex set and let $f \in \mathcal{M}_n(\mathrm{B}(K))$ be self-adjoint bounded nc function. The *epigraph* of f is the subset $\mathrm{Epi}(f) \subseteq \coprod_m K_m \times \mathcal{M}_n(\mathcal{M}_m)$ defined by

$$\mathrm{Epi}_m(f) = \{(x, \alpha) \in K_m \times \mathcal{M}_n(\mathcal{M}_m) : x \in K_m \text{ and } \alpha \geq f(x)\}.$$

We will say that f is *convex* if $\mathrm{Epi}(f)$ is an nc convex set, and that f is *lower semicontinuous* if $\mathrm{Epi}(f)$ is closed.

Remark 7.2.2. For a self-adjoint bounded nc function $f \in \mathcal{M}_n(\mathrm{B}(K))$, the fact that f is graded and respects direct sums implies that $\mathrm{Epi}(f)$ is a nc convex set if and only if

$$f(\alpha^* x \alpha) \leq (1_n \otimes \alpha^*) f(x) (1_n \otimes \alpha)$$

for every m, every $x \in K_m$ and every isometry $\alpha \in \mathcal{M}_{m,l}$.

This next proposition shows that scalar convexity of an nc function implies nc convexity. This is a higher dimensional analogue of the Hansen-Pedersem result [27, Theorem 2.1] for an interval. See Example 7.2.4.

Proposition 7.2.3. *Let K be a compact nc convex set and let $f \in \mathcal{M}_n(\mathrm{B}(K))$ be a self-adjoint bounded nc function. Then f is convex if and only if*

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{7.2.1}$$

for all m, all $x, y \in K_m$ and all $\lambda \in [0,1]$.

Proof. Suppose f is convex. For $x, y \in K_m$ and $\lambda \in [0,1]$, the fact that (7.2.1) holds follows from Remark 7.2.2 and the factorization

$$\lambda x + (1-\lambda) y = \begin{bmatrix} \sqrt{\lambda} 1_m & \sqrt{1-\lambda} 1_m \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} \sqrt{\lambda} 1_m \\ \sqrt{1-\lambda} 1_m \end{bmatrix}.$$

Conversely, suppose f satisfies (7.2.1). By Remark 7.2.2, to show that f is convex, it suffices to show that for $y \in K_m$ and an isometry $\alpha \in \mathcal{M}_{m,l}$, $f(\alpha^* y \alpha) \leq (1_n \otimes \alpha^*) f(y) (1_n \otimes \alpha)$. Define a unitary $\beta \in \mathcal{M}_m$ by $\beta = \alpha\alpha^* - (1 - \alpha\alpha^*)$. Decompose $H_m = \mathrm{Ran}(\alpha\alpha^*) \oplus \mathrm{Ran}(1_m - \alpha\alpha^*)$ and identify $\mathrm{Ran}(\alpha\alpha^*)$ and $\mathrm{Ran}(1_m - \alpha\alpha^*)$ with H_l and H_{m-l} respectively. Then we can write

$$y = \begin{bmatrix} x & * \\ * & z \end{bmatrix}, \qquad \beta = \begin{bmatrix} 1_l & 0 \\ 0 & -1_{m-l} \end{bmatrix}, \quad \text{and} \quad \alpha = \begin{bmatrix} \gamma \\ 0 \end{bmatrix}$$

for some $x \in K_l$, $z \in K_{m-l}$, and γ is unitary. Observe that by unitary equivariance and equation (7.2.1),

$$\begin{aligned} f(\alpha^* y \alpha) = f(\gamma^* x \gamma) &= (1_n \otimes \gamma)^* f(x) (1_n \otimes \gamma) \\ &= (1_n \otimes \alpha)^* \big(f(x) \oplus f(z)\big) (1_n \otimes \alpha) \\ &= (1_n \otimes \alpha)^* f(x \oplus z) (1_n \otimes \alpha) \\ &= (1_n \otimes \alpha)^* f\big(\tfrac{1}{2}(y + \beta^* y \beta)\big) (1_n \otimes \alpha) \\ &\le (1_n \otimes \alpha)^* \tfrac{1}{2}\big(f(y) + f(\beta^* y \beta)\big) (1_n \otimes \alpha) \\ &= (1_n \otimes \alpha)^* \tfrac{1}{2}\big(f(y) + \beta^* f(y) \beta\big) (1_n \otimes \alpha) \\ &= (1_n \otimes \alpha)^* f(y) (1_n \otimes \alpha). \end{aligned}$$

Therefore f is nc convex. □

Example 7.2.4. Let K be a compact nc convex set. It is easy to check that every continuous self-adjoint nc affine function $a \in \mathcal{M}_n(\mathrm{A}(K))$ is convex and lower semicontinuous.

Example 7.2.5. Fix a compact interval $I \subseteq \mathbb{R}$. Recall that a continuous real-valued function $f \in \mathrm{C}(I)$ is operator convex if

$$f(\lambda \alpha + (1-\lambda)\beta) \le \lambda f(\alpha) + (1-\lambda) f(\beta)$$

for all n, all self-adjoint $\alpha, \beta \in (\mathcal{M}_n)_{sa}$ with spectrum in I and all $\lambda \in [0,1]$.

Define a nc convex set K by setting

$$K_n = \{\alpha \in (\mathcal{M}_n)_{sa} : \sigma(\alpha) \subseteq I\},$$

where $\sigma(\alpha)$ denotes the spectrum of α. Note that $K_1 = I$. A continuous real-valued function $f \in \mathrm{C}(I)$ determines a continuous self-adjoint nc function in $\mathrm{C}(K)$ by the continuous functional calculus, while on the other hand, a continuous self-adjoint nc function in $\mathrm{C}(K)$ restricts to a continuous real-valued function in $\mathrm{C}(I)$. It follows immediately from Proposition 7.2.3 that a self-adjoint nc function $f \in \mathrm{C}(K)$ is convex if and only if it restricts to an operator convex function in $\mathrm{C}(I)$.

This fact, that scalar operator convexity on an interval implies nc convexity in the sense of Definition 7.2.1, is essentially the Hansen-Pedersen-Jensen inequality [27, Theorem 2.1].

Example 7.2.6. Let K be a compact nc convex set. If $[c,d] \subseteq \mathbb{R}$ is a compact interval and $f \in \mathrm{C}([c,d])$ is an operator convex function on $[c,d]$, then for self-adjoint $a \in \mathrm{A}(K)$ satisfying $c1_{\mathrm{A}(K)} \le a \le d1_{\mathrm{A}(K)}$, then it follows as in Example 7.2.4 and Remark 7.2.2 that $f(a) \in \mathrm{C}(K)$ is convex.

Remark 7.2.7. There is also a natural notion of concave nc function in the noncommutative setting. However, a self-adjoint bounded nc function $f \in \mathcal{M}_n(\mathrm{B}(K))$ is convex if and only if $-f$ is concave, so there is no disadvantage to working only with convex functions.

For a compact nc convex set K, we do not know if convex nc functions in $\mathrm{C}(K)$ always span a dense subset of $\mathrm{C}(K)$. However, the next result will be sufficient our purposes.

Proposition 7.2.8. *Let K be a compact nc convex set and let $\mu, \nu : \mathrm{C}(K) \to \mathcal{M}_n$ be unital completely positive maps such that $\mu(f) = \nu(f)$ for every n and every convex nc function $f \in \mathcal{M}_n(\mathrm{C}(K))$. Then $\mu = \nu$.*

Proof. For $t \in [-1,1]$, the function $h_t(x) = x^2(1-tx)^{-1}$ is operator convex on the interval $(-1,1)$ [4]. The Taylor series expansion of h_t at $x = 0$ is $h_t(x) = \sum_{n \geq 0} t^n x^{n+2}$. Hence for self-adjoint $a \in \mathcal{M}_n(\mathrm{A}(K))$ with $\|a\| < 1$, Example 7.2.6 implies that the continuous nc function $h_t(a) \in \mathrm{C}(K)$ is convex. Hence by assumption,

$$0 = (\mu - \nu)(h_t(a)) = \sum_{n \geq 0} (\mu - \nu)(a^{n+2}) t^n \quad \text{for all} \quad t \in [-1,1].$$

It follows that the analytic function $k(z) = \sum_{n \geq 0} (\mu - \nu)(a^{n+2}) z^n$ is identically zero. Therefore $\mu(a^n) = \nu(a^n)$ for all $n \geq 2$. This also holds for $n = 0$ and $n = 1$ by hypothesis.

We now show that if $f = a_1 \cdots a_n$ for self-adjoint $a_1, \ldots, a_n \in \mathrm{A}(K)$, then $\mu(f) = \nu(f)$. To see this, define self-adjoint $b = (b_{ij}) \in \mathcal{M}_{n+1}(\mathrm{A}(K))$ by setting $b_{i,i+1} = b_{i+1,i} = a_i$ for $1 \leq i \leq n$ and setting all other entries to zero. It is easy to check that the $(1, n+1)$ entry of b^n is f. From above, $\mu(b^n) = \nu(b^n)$. Hence $\mu(f) = \nu(f)$. It follows that μ and ν agree on the C*-algebra generated by $\mathrm{A}(K)$, namely $\mathrm{C}(K)$, and we conclude that $\mu = \nu$. ☐

7.3. **Multivalued noncommutative functions.** A major difficulty in the noncommutative setting is the fact that the self-adjoint elements of a noncommutative von Neumann algebra do not form a lattice. Inspired by work of Wittstock [50] and Winkler [49], we will overcome this difficulty by working with multivalued functions.

Definition 7.3.1. Let K be an nc convex set and let $F : K \to \mathcal{M}_n(\mathcal{M})_{sa}$ be a multivalued self-adjoint function. We say that F is a *multivalued nc function* if it is non-degenerate, graded, unitarily equivariant and upwards directed, meaning that

(1) $F(x) \neq \emptyset$ for every $x \in K$,
(2) $F(K_m) \subseteq \mathcal{M}_n(\mathcal{M}_m)$ for all m,

(3) $F(\beta x\beta^*) = (1_n \otimes \beta)F(x)(1_n \otimes \beta^*)$ for every $x \in K_m$ and every unitary $\beta \in \mathcal{M}_m$,

(4) $F(x) = F(x) + \mathcal{M}_n(\mathcal{M}_m)_+$ for every m and every $x \in K_m$.

We say that F is *bounded* if there is a constant $\lambda > 0$ such that for every $\beta \in F(x)$ there is $\alpha \in F(x)$ with $\alpha \leq \beta$ such that $\|\alpha\| \leq \lambda$. If F is bounded, then we let $\|F\|$ denote the infimum of all λ as above. Otherwise we write $\|F\| = \infty$. If $G : K \to \mathcal{M}_n(\mathcal{M})$ is another multivalued nc function, then we will write $F \leq G$ if $F(x) \supseteq G(x)$ for every $x \in K$.

Definition 7.3.2. Let K be a compact nc convex set and let $F : K \to \mathcal{M}_n(\mathcal{M})$ be a bounded multivalued nc function. The *graph* of F is the subset $\operatorname{Graph}(F) \subseteq \coprod_m K_m \times \mathcal{M}_n(\mathcal{M}_m)$ defined by

$$\operatorname{Graph}_m(F) = \{(x, \alpha) \in K_m \times \mathcal{M}_n(\mathcal{M}_m) : x \in K_m \text{ and } \alpha \in F(x)\}.$$

We say that F is *convex* if $\operatorname{Graph}(F)$ is an nc convex set, and that F is *lower semicontinuous* if $\operatorname{Graph} F$ is closed.

Example 7.3.3. Let K be a compact nc convex set and let $f \in \mathcal{M}_n(\mathrm{B}(K))$ be self-adjoint. There is a bounded multivalued nc function $F : K \to \mathcal{M}_n(\mathcal{M})$ naturally associated to f defined by $F(x) = [f(x), +\infty)$ for $x \in K$. Note that F is the unique multivalued nc function with $\operatorname{Graph}(F) = \operatorname{Epi}(f)$.

Recall that if K is a compact nc convex set and $\mu : \mathrm{C}(K) \to \mathcal{M}_k$ is a unital completely positive map, then μ can be extended to a unital completely positive map on the C*-algebra $\mathrm{B}(K)$ of bounded single-valued nc functions. We will extend μ further and make sense of the expression $\mu(F)$ when $F : K \to \mathcal{M}_n(\mathcal{M})$ is a bounded multivalued nc function.

Definition 7.3.4. Let K be a compact nc convex set and let $\mu : \mathrm{C}(K) \to \mathcal{M}_k$ be a unital completely positive map. Let $(x, \alpha) \in K_m \times \mathcal{M}_{m,k}$ be a minimal representation for μ. For a bounded multivalued nc function $F : K \to \mathcal{M}_n(\mathcal{M})$, we define

$$\mu(F) = (1_n \otimes \alpha^*)F(x)(1_n \otimes \alpha).$$

Remark 7.3.5. The fact that this extension of μ is well defined follows from unitary equivariance of multivalued nc functions. The argument is similar to the argument for bounded single-valued nc functions from Section 5.

Let $f \in \mathcal{M}_n(\mathrm{B}(K))$ be self-adjoint and let $F : K \to \mathcal{M}_n(\mathcal{M})$ denote the corresponding bounded multivalued nc function defined as in Example 7.3.3. If $G : K \to \mathcal{M}_n(\mathcal{M})$ is a multivalued nc function, then we will write $f = G$, $f \leq G$ and $f \geq G$ if $F = G$, $F \leq G$ or $F \geq G$ respectively.

7.4. **Noncommutative convex envelopes.** In this section we will introduce a notion of convex envelope for continuous nc functions that will play a similarly important role in the noncommutative setting. We will need to work with multivalued nc functions, and this introduces some technical difficulties. However, the results in this section will also apply to single-valued functions via the correspondence in Example 7.3.3.

We will define the convex envelope of a function geometrically in terms of the graph of the function. The non-trivial fact that this definition of the convex envelope is equivalent to an appropriate approximation from below by continuous nc affine functions will be the main result in this section.

Let K be a compact nc convex set. For cardinals m and n, we will view $f \in \mathcal{M}_m(\mathcal{M}_n(\mathrm{B}(K)))$ as a function $f : K \to \mathcal{M}_m(\mathcal{M}_n(\mathcal{M}))$ in the obvious way. For another function $g \in \mathcal{M}_m(\mathcal{M}_n(\mathrm{B}(K)))$, we will write $f \leq g$ if f and g are self-adjoint and $f(x) \leq g(x)$ for all $x \in K$. For a multivalued nc function $F : K \to \mathcal{M}_n(\mathcal{M})$, we define a multivalued function $1_m \otimes F : K \to \mathcal{M}_m(\mathcal{M}_n(\mathcal{M}))$ by

$$(1_m \otimes F)(x) = \{1_m \otimes \alpha : \alpha \in F(x)\}, \quad x \in K.$$

Note that $1_m \otimes F$ is not an nc function since $1_m \otimes \alpha \leq \beta$ does not imply that $\beta = 1_m \otimes \beta'$.

Definition 7.4.1. Let K be a compact nc convex set. The *convex envelope* of a bounded multivalued function $F : K \to \mathcal{M}_n(\mathcal{M})$ is the multivalued nc function $\overline{F} : K \to \mathcal{M}_n(\mathcal{M})$ determined by the property

$$\operatorname{Graph} \overline{F} = \overline{\operatorname{ncconv}}(\operatorname{Graph}(F)).$$

That is, the graph of $\overline{F}$ is the closed nc convex hull of the graph of F.

Proposition 7.4.2. *Let K be an nc convex set and let $F : K \to \mathcal{M}_n(\mathcal{M})$ be a bounded multivalued nc function with convex envelope $\overline{F}$. Then*

1. *$\overline{F}$ is a lower semicontinuous convex multivalued nc function,*
2. *$\overline{F} \leq F$,*
3. *if F is nc convex and lower semicontinuous, then $\overline{F} = F$,*
4. *if F is bounded by λ, then so is $\overline{F}$, and*
5. *if G is a convex nc function such that $G \leq F$, then $G \leq \overline{F}$.*

Proof. Since the graph of $\overline{F}$ is defined to be nc convex and closed, (1) is immediate. Also, evidently $\operatorname{Graph} \overline{F} \supset \operatorname{Graph}(F)$, so $\overline{F} \le F$. If $\operatorname{Graph}(F)$ is already closed and nc convex, then clearly $\overline{F} = F$.

Suppose that F is bounded by λ. Then $-\lambda I_n \otimes I_k \le F(x)$ for all $x \in K_k$, and this persists for $\overline{F}$. Suppose that (x,β) belongs to the (algebraic) nc convex hull of $\operatorname{Graph}(F)$; say $(x,\beta) = \sum \alpha_i^*(x_i,\beta_i)\alpha_i$ where $\sum \alpha_i^*\alpha_i = I_k$. Then since F is bounded by λ, there exist $\gamma_i \in F(x_i)$ with $\gamma_i \le \beta_i$ and $\|\gamma_i\| \le \lambda$. It follows that $(x,\gamma) \in \overline{F}(x)$ where $\gamma = \sum \alpha_i^*\gamma_i\alpha_i \le \beta$ and $\|\gamma\| \le \lambda$. In general, if (x,β) is a limit of a net of such points (x_j,β_j), find (x_j,γ_j) with $\gamma_j \le \beta_j$ and $\|\gamma_j\| \le \lambda$. Extract a convergent cofinal subnet with limit (x,γ). Then $\gamma \le \beta$ and $\|\gamma\| \le \lambda$. So $\overline{F}$ is bounded by λ.

Finally it is clear from the definition that $\overline{F}$ is the largest convex nc function smaller than F. ∎

The next result is a noncommutative analogue of the classical fact that the convex envelope of a function is obtained as the supremum of the continuous affine functions dominated by the function.

Theorem 7.4.3. *Let K be a compact nc convex set and let $F : K \to \mathcal{M}_n(\mathcal{M})$ be a bounded multivalued nc function. Then for $x \in K_p$,*

$$\overline{F}(x) = \bigcap_m \bigcap_{a \le 1_m \otimes F} \{\alpha \in (\mathcal{M}_n(\mathcal{M}_p))_{sa} : 1_m \otimes \alpha \ge a(x)\},$$

where the intersection is taken over all m and all self-adjoint nc affine functions $a \in \mathcal{M}_m(\mathcal{M}_n(\mathrm{A}(K)))_{sa}$ satisfying $a \le 1_m \otimes F$.

Proof. Let $\tilde{F}(x) := \bigcap_m \bigcap_{a \le 1_m \otimes F} \{\alpha \in (\mathcal{M}_n(\mathcal{M}_p))_{sa} : 1_m \otimes \alpha \ge a(x)\}$ for $x \in K_p$. It is easy to see that

$$\operatorname{Graph}(\tilde{F}) = \bigcap_m \bigcap_{a \le 1_m \otimes F} \big\{(x,\alpha) \in K \times (\mathcal{M}_n(\mathcal{M}))_{sa} : (x, 1_m \otimes \alpha) \in \operatorname{Epi}(a)\big\},$$

where the intersection is taken over all m and all self-adjoint nc affine functions a in $\mathcal{M}_m(\mathcal{M}_n(\mathrm{A}(K)))_{sa}$ satisfying $a \le 1_m \otimes F$. This is an intersection of closed nc convex sets, so $\tilde{F}$ is a lower semicontinuous convex function with $\tilde{F} \le F$. Thus by definition of the convex envelope, we have that $\tilde{F} \le \overline{F}$. We will prove that $\operatorname{Graph} \tilde{F} = \operatorname{Graph} \overline{F}$. It remains to show that $\overline{F} \le \tilde{F}$.

Since F is bounded, $\overline{F}$ is bounded by Proposition 7.4.2. By replacing F by $F(x) + (\|F\|+1)$, we may assume that $\overline{F}(x) \subseteq [1_n \otimes 1_k, +\infty)$ for every k and every $x \in K_k$. Then for $(x,\alpha) \in \operatorname{Graph}_k(\overline{F})$, $\alpha \ge 1_n \otimes 1_k$.

Fix $x_0 \in K_k$ and self-adjoint $\alpha_0 \in \mathcal{M}_n(\mathcal{M}_k)$ such that $(x_0,\alpha_0) \notin \operatorname{Graph}_k(\overline{F})$. To show that $(x_0,\alpha_0) \notin \operatorname{Graph}_k(\tilde{F})$, we must show there is a cardinal m and an nc affine function $a \in (\mathcal{M}_m(\mathcal{M}_n(\mathrm{A}(K))))_{sa}$ such that $a \leq 1_m \otimes F$, in the sense that $[a(x), +\infty) \supseteq 1_m \otimes F(x)$ for all $x \in K$, but $a(x_0) \not\leq 1_m \otimes \alpha_0$. Since $\mathrm{Graph}(\tilde{F})$ and $\mathrm{Graph}(\overline{F})$ are both closed and nc convex, Proposition 2.2.9 and Section 2.3 show that we can assume that k is finite.

Let E be an operator system containing K. Since $\mathrm{Graph}(\overline{F})$ is closed and nc convex, it follows from Corollary 2.4.2 that there is a normal completely bounded self-adjoint map $\theta : E \oplus \mathcal{M}_n \to \mathcal{M}_k$ and self-adjoint $\gamma \in \mathcal{M}_k$ such that $\theta_l((x, \alpha)) \leq \gamma \otimes 1_l$ for every l and every $(x, \alpha) \in \mathrm{Graph}_l(\overline{F})$, but $\theta_k((x_0, \alpha_0)) \not\leq \gamma \otimes 1_k$. Here we write θ_l for the amplification $\theta \otimes \mathrm{id}_l$.

Define normal completely bounded maps $\varphi : E \to \mathcal{M}_k$ and $\psi : \mathcal{M}_n \to \mathcal{M}_k$ by $\varphi(x) = \theta(x, 0_n)$ for $x \in E$ and $\psi(\alpha) = -\theta(0_E, \alpha)$ for $\alpha \in \mathcal{M}_n$. As above, we write $\varphi_l = \varphi \otimes \mathrm{id}_l$ and $\psi_l = \psi \otimes \mathrm{id}_l$. Then for every l and every $(x, \alpha) \in \mathrm{Graph}_l(\overline{F})$,

$$\varphi_l(x) - \psi_l(\alpha) = \theta_l((x, \alpha)) \leq \gamma \otimes 1_l.$$

Rearranging gives

$$\varphi_l(x) - \gamma \otimes 1_l \leq \psi_l(\alpha). \tag{7.4.1}$$

We first claim that ψ is completely positive. To see this, note that for l and $(x, \alpha) \in \mathrm{Graph}_l(\overline{F})$, the normalization ensures that $\alpha \geq 1_n \otimes 1_l$; and the fact that $\overline{F}$ is upwards directed implies $(x, \lambda\alpha) \in \mathrm{Graph}_l(\overline{F})$ for every $\lambda \geq 1$. Hence by (7.4.1),

$$\varphi_l(x) - \gamma \otimes 1_l \leq \psi_l(\lambda\alpha) = \lambda\psi_l(\alpha). \tag{7.4.2}$$

Dividing both sides by λ and taking $\lambda \to \infty$ yields $\psi_l(\alpha) \geq 0$.

Now for positive $\beta \in \mathcal{M}_n(\mathcal{M}_l)$ and $\epsilon > 0$, there is some $\lambda > 0$ such that $\lambda(\beta + \epsilon 1_n \otimes 1_l) \geq \alpha$. Then since $\overline{F}$ is upwards directed,

$$(x, \lambda(\beta + \epsilon 1_n \otimes 1_l)) \in \mathrm{Graph}_l(\overline{F}).$$

Hence by (7.4.2),

$$\lambda\psi_l(\beta + \epsilon 1_n \otimes 1_l) = \psi_l(\lambda(\beta + \epsilon 1_n \otimes 1_l)) \geq 0.$$

Dividing by λ implies $\psi_l(\beta + \epsilon 1_n \otimes 1_l) \geq 0$, and taking $\epsilon \to 0$ gives $\psi_l(\beta) \geq 0$. Hence ψ is completely positive.

We claim that θ can be chosen to ensure that $\psi(\alpha)$ is invertible for all $\alpha \in \mathcal{M}_n$ with $\alpha \neq 0$ and $\alpha \geq 0$. To see this, choose a faithful state $\tau : \mathcal{M}_n \to \mathbb{C}$ and $\epsilon > 0$ such that

$$\theta_k(x_0, \alpha_0) - \epsilon\tau_k(\alpha_0) 1_k \otimes 1_k \not\leq \gamma \otimes 1_n.$$

Then the completely bounded self-adjoint map $\theta' : E \oplus \mathcal{M}_n \to \mathcal{M}_k$ defined by $\theta'(x, \alpha) = \theta(x, \alpha) - \epsilon\tau(\alpha) 1_k$ for $(x, \alpha) \in E \oplus \mathcal{M}_n$ satisfies

$\theta_l'((x,\alpha)) \leq \gamma \otimes 1_l$ for every l and every $(x,\alpha) \in \mathrm{Graph}_l(\overline{F})$, but $\theta_k'((x_0,\alpha_0)) \not\leq \gamma \otimes 1_k$. Furthermore, the map $\psi' : \mathcal{M}_n \to \mathcal{M}_k$ defined by $\psi'(\alpha) = -\theta'(x, 0_n)$ for $\alpha \in \mathcal{M}_n$ satisfies

$$\psi'(\alpha) = -\theta(0_E, \alpha) + \epsilon\tau(\alpha)1_k = \psi(x, 0_n) + \epsilon\tau(\alpha)1_k.$$

In particular, the positivity of ψ and the faithfulness of τ implies that $\psi'(\alpha)$ is invertible for all $\alpha \in \mathcal{M}_n$ with $\alpha \neq 0$ and $\alpha \geq 0$. By replacing θ by θ', we can therefore assume that ψ has this property.

Since ψ is completely positive and k and n are finite, Stinespring's theorem provides finite m and an operator $\beta : H_k \to H_n^m$ such that

$$\psi(\alpha) = \beta^*(1_m \otimes \alpha)\beta \quad \text{for} \quad \alpha \in \mathcal{M}_n.$$

Then

$$\psi_l(\alpha) = (\beta^* \otimes 1_l)(1_m \otimes \alpha)(\beta \otimes 1_l) \quad \text{for} \quad \alpha \in \mathcal{M}_n(\mathcal{M}_l).$$

Write $\beta = \nu|\beta|$, where $\nu : H_k \to H_n^m$ is a partial isometry with initial space $(\ker\beta)^\perp$. It follows from above that $|\beta|$ is invertible and hence $\nu^*\nu = 1_k$. Let $q = \nu\nu^*$.

Since $|\beta|$ is invertible, there is an element $\beta' = |\beta|^{-1} \in \mathcal{M}_k$. Then $\nu = \beta\beta'$, whence $\nu\beta'\beta^* = \nu\nu^* = q$. Thus for $\alpha \in \mathcal{M}_n(\mathcal{M}_l)$,

$$(\nu\beta' \otimes 1_l)\psi_l(\alpha)(\beta'\nu^* \otimes 1_l) = (q \otimes 1_l)(1_m \otimes \alpha)(q \otimes 1_l).$$

Hence decomposing $1_m \otimes \alpha \in \mathcal{M}_m(\mathcal{M}_n(\mathcal{M}_l)))$ as a block matrix with respect to the projection $q \otimes 1_l$ as

$$1_m \otimes \alpha = \begin{bmatrix} (1_m \otimes \alpha)_{11} & (1_m \otimes \alpha)_{12} \\ (1_m \otimes \alpha)_{21} & (1_m \otimes \alpha)_{22} \end{bmatrix},$$

we obtain

$$(1_m \otimes \alpha)_{11} = (\nu\beta' \otimes 1_l)\psi_l(\alpha)(\beta'\nu^* \otimes 1_l). \tag{7.4.3}$$

For $\epsilon > 0$, define a self-adjoint affine function $a_\epsilon \in \mathcal{M}_m(\mathcal{M}_n(\mathrm{A}(K)))$ by writing it in block matrix form with respect to the projection q as

$$a_\epsilon = \begin{bmatrix} a_{\epsilon,11} & 0 \\ 0 & a_{\epsilon,22} \end{bmatrix},$$

where

$$a_{\epsilon,11}(x) = (\nu\beta' \otimes 1_l)(\varphi_l(x) - \gamma \otimes 1_l - \epsilon 1_k \otimes 1_l)(\beta'\nu^* \otimes 1_l) \tag{7.4.4}$$

and

$$a_{\epsilon,22}(x) = -\lambda_\epsilon q^\perp$$

for $x \in K_l$, where $\lambda_\epsilon > 0$ is chosen to satisfy

$$\lambda_\epsilon > \epsilon^{-1}\|\beta\|^2\|\overline{F}\|^2 + \|\alpha\|.$$

We claim that $a_\epsilon \leq 1_m \otimes \overline{F}$ in the sense that $[a_\epsilon(x), +\infty) \supseteq 1_m \otimes \overline{F}(x)$ for all $x \in K$. The boundedness of $\overline{F}$ implies that for every l and every $(x, \alpha') \in \operatorname{Graph}_l(\overline{F})$, there is $\alpha \in \overline{F}(x)$ such that $\alpha \leq \alpha'$ and $\|\alpha\| \leq \|\overline{F}\|$. Therefore, in order to show that $a_\epsilon \leq 1_m \otimes \overline{F}$, it suffices to show that $a_\epsilon(x) \leq 1_m \otimes \alpha$ for every l and every $(x, \alpha) \in \operatorname{Graph}_l(\overline{F})$ with $\|\alpha\| \leq \|\overline{F}\|$. Taking the Schur complement of the block matrix of $1_m \otimes \alpha - a_\epsilon(x)$ with respect to the projection $q \otimes 1_l$ implies that this condition is equivalent to the inequalities

$$(1_m \otimes \alpha)_{22} - a_{\epsilon,22}(x) \geq 0$$

and

$$\begin{aligned}(1_m \otimes \alpha)_{11} - a_{\epsilon,11}(x)& \\ &\geq (1_m \otimes \alpha)_{12}\big((1_m \otimes \alpha)_{22} - a_{\epsilon,22}(x)\big)^{-1}(1_m \otimes \alpha)_{21}.\end{aligned}$$

The first inequality follows immediately from the choice of λ_ϵ, since

$$\begin{aligned}(1_m \otimes \alpha)_{22} - a_{\epsilon,22}(x) &= (1_m \otimes \alpha)_{22} + \lambda_\epsilon q^\perp \otimes 1_l \\ &\geq \epsilon^{-1}\|\beta\|^2\|\overline{F}\|^2 q^\perp \otimes 1_l.\end{aligned} \tag{7.4.5}$$

For the second inequality, observe that (7.4.3) and (7.4.4) imply

$$\begin{aligned}(1_m&\otimes\alpha)_{11} - a_{\epsilon,11}(x) \\ &= (\nu\beta' \otimes 1_l)\big((\psi_l(\alpha) - \varphi_l(x) + \gamma \otimes 1_l) + \epsilon 1_k \otimes 1_l\big)(\beta'\nu^* \otimes 1_l) \\ &\geq \epsilon\nu(\beta')^2\nu^* \otimes 1_l \\ &\geq \frac{\epsilon}{\|\beta\|^2} q \otimes 1_l.\end{aligned}$$

Then since $\|\alpha\| \leq \|\overline{F}\|$, (7.4.5) implies that

$$\begin{aligned}(1_m \otimes \alpha)_{12}\big((1_m &\otimes \alpha)_{22} - a_{\epsilon,22}(x)\big)^{-1}(1_m \otimes \alpha)_{21} \\ &= (1_m \otimes \alpha)_{12}((1_m \otimes \alpha)_{22} + \lambda_\epsilon q^\perp \otimes 1_l)^{-1}(1_m \otimes \alpha)_{21} \\ &\leq \frac{\epsilon}{\|\beta\|^2} q \otimes 1_l.\end{aligned}$$

Hence the second inequality is also satisfied. Therefore, $a_\epsilon \leq 1_m \otimes \overline{F}$.

Finally, we claim there is an $\epsilon > 0$ such that $a_\epsilon(x_0) \not\leq 1_m \otimes \alpha_0$. To see this, suppose for the sake of contradiction that $a_\epsilon(x_0) \leq 1_m \otimes \alpha_0$ for all $\epsilon > 0$. Then in particular, looking at the top left corner of the block matrix of $1_m \otimes \alpha_0 - a_\epsilon(x)$ with respect to the projection $q \otimes 1_k$

and applying (7.4.4) implies

$$\begin{aligned}0 &\le (1_m \otimes \alpha_0)_{11} - a_{\epsilon,11}(x_0)\\ &= (\nu\beta' \otimes 1_k)(\psi_l(\alpha_0) - \varphi_k(x_0) + \gamma \otimes 1_k + \epsilon 1_k \otimes 1_k)(\beta'\nu^* \otimes 1_k)\\ &= (\nu\beta' \otimes 1_k)(-\theta((x_0,\alpha_0)) + \gamma \otimes 1_k + \epsilon 1_k \otimes 1_k)(\beta'\nu^* \otimes 1_k)\end{aligned}$$

Then multiplying on the left by $\beta^* \otimes 1_k$ and on the right by $\beta \otimes 1_k$ and taking $\epsilon \to 0$ implies $\theta_k((x_0,\alpha_0)) \le \gamma \otimes 1_k$, contradicting our original separation of (x_0,α_0) from the graph of $\overline{F}$. We conclude that for some $\epsilon > 0$, the nc affine function a_ε achieves the desired separation. □

We will need a useful fact regarding the convex envelope and multiplicity.

Corollary 7.4.4. *Let K be an nc convex set and let $F : K \to \mathcal{M}_n(\mathcal{M})$ be a self-adjoint bounded multivalued nc function with convex envelope $\overline{F}$. Then $\overline{1_l \otimes F} \le 1_l \otimes \overline{F}$.*

Proof. Note that

$$\begin{aligned}\operatorname{Graph}(\overline{1_l \otimes F}) &= \overline{\operatorname{ncconv}}\big(\operatorname{Graph}(1_l \otimes F)\big)\\ &= \overline{\operatorname{ncconv}}\left\{\coprod_{x\in K}(x, 1_l \otimes F(x))\right\}.\end{aligned}$$

The nc convex combinations include all points obtained using points x_i and contractions of the form $1_l \otimes \alpha_i$. Therefore

$$\begin{aligned}\operatorname{Graph}(\overline{1_l \otimes F}) &\supset \coprod_{x\in K}(x, 1_l \otimes \overline{F}(x))\\ &= \operatorname{Graph}(1_l \otimes \overline{F}).\end{aligned}$$ □

The next result is a noncommutative analogue of a result of Mokobodzki (see e.g. [1, Proposition I.5.1]).

Proposition 7.4.5. *Let K be a compact nc convex set and let $F : K \to \mathcal{M}_n(\mathcal{M})$ be a self-adjoint bounded multivalued bounded nc function with convex envelope $\overline{F}$. Then for $x \in K_p$,*

$$\overline{F}(x) = \bigcap_{g \le 1_m \otimes F} \{\alpha \in \mathcal{M}_n(\mathcal{M}_p) : 1_m \otimes \alpha \ge g(x)\},$$

where the intersection is taken over all m and all convex nc functions $g \in \mathcal{M}_m(\mathcal{M}_n(\mathrm{C}(K)))$ satisfying $g \le 1_m \otimes F$.

Proof. By Theorem 7.4.3, $\overline{F}(x)$ is the intersection over such sets with respect to continuous affine nc functions. Since every affine nc function is a convex nc function, the intersection over all m and all convex nc functions $g \in \mathcal{M}_m(\mathcal{M}_n(\mathrm{C}(K)))$ satisfying $g \leq 1_m \otimes F$ is smaller. On the other hand, by Proposition 7.4.2, $\overline{1_m \otimes F}$ is the largest lower semicontinuous convex multivalued nc function dominated by $1_m \otimes F$, meaning that for all such g, $g \leq \overline{1_m \otimes F}$. Hence by Corollary 7.4.4, $g \leq 1_m \otimes \overline{F}$. Therefore, the intersection is precisely $\mathrm{Graph}(\overline{F})$. ☐

7.5. **Completely positive maps.** The next result shows that, as in the classical setting, the noncommutative convex envelope encodes information about the set of representing maps of a point.

Theorem 7.5.1. *Let K be a compact nc convex set and let $f : K \to \mathcal{M}_n(\mathcal{M})$ be a self-adjoint lower semicontinuous bounded nc function with convex envelope $\overline{f}$. Then for $x \in K_m$,*

$$\overline{f}(x) = \bigcup_{\mu} [\mu(f), +\infty),$$

where the union is taken over all unital completely positive maps $\mu : \mathrm{C}(K) \to \mathcal{M}_m$ with barycenter x.

Proof. Define $F : K \to \mathcal{M}$ by $F(x) = \cup_\mu [\mu(f), +\infty)$ for $x \in K_m$, where the union is taken over all unital completely positive maps $\mu : \mathrm{C}(K) \to \mathcal{M}_m$ with barycenter x. Then F is a self-adjoint bounded multivalued nc function since it is clearly graded, unitarily equivariant and upward directed.

We claim that F is lower semicontinuous. Let (x_i, α_i) be a net in $\mathrm{Graph}_m(F)$ converging to $(x, \alpha) \in K_m \times \mathcal{M}_n(M_m)$. Then there are unital completely positive maps $\mu_i : \mathrm{C}(K) \to \mathcal{M}_m$ such that μ_i has barycenter x_i and $\mu_i(f) \leq \alpha_i$. Let $\mu : \mathrm{C}(K) \to \mathcal{M}_m$ be a cluster point of the net (μ_i). Then μ has barycenter x and $\mu(f) \leq \alpha$, so $(x, \alpha) \in \mathrm{Graph}_m(F)$.

Next we show that F is convex. Suppose that $(x_i, \alpha_i) \in \mathrm{Graph}(F)$, where $x_i \in K_{n_i}$ and μ_i is a unital completely positive map $\mu_i : \mathrm{C}(K) \to \mathcal{M}_m$ with barycenter x_i such that $\mu_i(f) \leq \alpha_i$. If $\beta_i \in \mathcal{M}_{n,n_i}$ so that $\sum \beta_i^* \beta_i = 1_n$, let

$$x := \sum_i \beta_i^* x_i \beta_i \in K_n \quad \text{and} \quad \alpha := \sum_i (1_m \otimes \beta_i^*) \alpha_i (1_m \otimes \beta_i).$$

We need to verify that $(x, \alpha) \in \mathrm{Graph}(F)$. Observe that

$$\mu := \sum (1_m \otimes \beta_i^*) \mu_i (1_m \otimes \beta_i)$$

is a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_m$ with barycenter x. In addition,

$$\begin{aligned}\mu(f) &= \sum (1_m \otimes \beta_i^*)\mu_i(f)(1_m \otimes \beta_i) \\ &\le \sum (1_m \otimes \beta_i^*)\alpha_i(1_m \otimes \beta_i) = \alpha.\end{aligned}$$

Therefore $(x, \alpha) \in \mathrm{Graph}(F)$.

It now suffices to show that $\overline{f} = F$. We will accomplish using Theorem 7.4.3 by showing that if $a \in \mathcal{M}_m(\mathcal{M}_n(\mathrm{A}(K)))$, then $a \le 1_m \otimes f$ if and only if $a \le 1_m \otimes F$ for every $x \in K_m$.

If $a \le 1_m \otimes F$, then for $x \in K_m$, then $a(x) \le 1_m \otimes \mu(f)$ for every unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_m$ with barycenter x. In particular, taking $\mu = \delta_x$ implies $a(x) \le 1_m \otimes f(x)$. Hence $a \le 1_m \otimes f$.

On the other hand, if $a \le 1_m \otimes f$, then for $x \in K_m$ and every unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_m$ with barycenter x,

$$a(x) = \mu(a) \le \mu(1_m \otimes f) = 1_m \otimes \mu(f).$$

Hence $a \le 1_m \otimes F$. $\square$

The next result extends Proposition 7.4.5.

Corollary 7.5.2. *Let K be a compact nc convex set and let $F : K \to \mathcal{M}_n(\mathcal{M})$ be a self-adjoint multivalued bounded nc function. Then for every unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_p$,*

$$\mu(\overline{F}) = \bigcap_{g \le 1_m \otimes F} \{\alpha \in \mathcal{M}_n(\mathcal{M}_p) : 1_m \otimes \alpha \ge \mu(g)\},$$

where the intersection is taken over all m and all convex nc functions $g \in \mathcal{M}_m(\mathcal{M}_n(\mathrm{C}(K)))$ satisfying $g \le 1_m \otimes F$.

Proof. Fix a minimal representation $(x, \nu) \in K_q \times \mathcal{M}_{q,p}$ for μ so that $\mu(f) = \nu^* f(x)\nu$ for $f \in \mathrm{C}(K)$. Then by Proposition 7.4.5,

$$\begin{aligned}\mu(\overline{F}) &= (1_n \otimes \nu^*)\overline{F}(x)(1_n \otimes \nu) \\ &= (1_n \otimes \nu^*)\Big(\bigcap_{g \le 1_m \otimes F} \{\alpha \in \mathcal{M}_n(\mathcal{M}_q) : 1_m \otimes \alpha \ge g(x)\}\Big)(1_n \otimes \nu) \\ &= \bigcap_{g \le 1_m \otimes F} \big\{(1_n \otimes \nu^*)\alpha(1_n \otimes \nu) \in \mathcal{M}_n(\mathcal{M}_q) : 1_m \otimes \alpha \ge g(x)\big\},\end{aligned}$$

where the intersection is taken over all m and all convex nc functions $g \in \mathcal{M}_m(\mathcal{M}_n(\mathrm{C}(K)))$ satisfying $g \le 1_m \otimes F$.

Thus if $1_m \otimes \alpha \ge g(x)$, then setting $\beta = (1_n \otimes \nu^*)\alpha(1_n \otimes \nu)$, it follows that

$$1_m \otimes \beta \ge (1_m \otimes 1_n \otimes \nu^*)g(x)(1_m \otimes 1_n \otimes \nu) = \mu(g).$$

Conversely, if $\beta \in \mathcal{M}_n(\mathcal{M}_p)$ satisfies $1_m \otimes \beta \geq \mu(g)$, then for $\varepsilon > 0$, define $\alpha_\varepsilon \in \mathcal{M}_n(\mathcal{M}_q)$ by

$$\begin{aligned}\alpha_\varepsilon &= (1_n \otimes \nu)(\beta + \varepsilon 1_p)(1_n \otimes \nu^*) + (1_n \otimes (1_q - \nu\nu^*))(\varepsilon^{-1}\|g\|^2 + \|g\|) \\ &\simeq \begin{bmatrix} \beta + \varepsilon & 0 \\ 0 & \varepsilon^{-1}\|g\|^2 + \|g\| \end{bmatrix},\end{aligned}$$

where the decomposition is taken with respect to the range of $1_n \otimes \nu$. If we also decompose $g(x)$ with respect to the range of $1_m \otimes 1_n \otimes \nu$, it has the form

$$g(x) \simeq \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

and by hypothesis $g_{11} \simeq \mu(g) \leq 1_m \otimes \beta$. Thus

$$\begin{aligned}1_m \otimes \alpha_\varepsilon - g(x) &\simeq \begin{bmatrix} 1_m \otimes \beta - g_{11} + \varepsilon(1_m \otimes 1_p) & -g_{12} \\ -g_{21} & \varepsilon^{-1}\|g\|^2 + (\|g\| - g_{22}) \end{bmatrix} \\ &\geq \begin{bmatrix} \varepsilon & -g_{12} \\ -g_{21} & \varepsilon^{-1}\|g\|^2 \end{bmatrix} \\ &\geq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Since $\mu(\overline{F})$ is a closed set, we deduce that

$$\begin{aligned}\mu(\overline{F}) &= \bigcap_{g \leq 1_m \otimes F} \left\{ (1_n \otimes \nu^*)\alpha(1_n \otimes \nu) \in \mathcal{M}_n(\mathcal{M}_q) : 1_m \otimes \alpha \geq g(x) \right\} \\ &= \bigcap_{g \leq 1_m \otimes F} \left\{ \beta \in \mathcal{M}_n(\mathcal{M}_p) : 1_m \otimes \beta \geq \mu(g) \right\},\end{aligned}$$

where the intersection is taken over all m and all convex nc functions $g \in \mathcal{M}_m(\mathcal{M}_n(\mathrm{C}(K)))$ satisfying $g \leq 1_m \otimes F$. □

7.6. **Noncommutative Jensen inequality.** The next result is a natural noncommutative analogue of the classical Jensen inequality.

Theorem 7.6.1 (Noncommutative Jensen inequality)**.** *Let K be a compact nc convex set and let $f \in \mathrm{B}(K)$ be a self-adjoint lower semicontinuous convex nc function. Then for any completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ with barycenter $x \in K_n$, $f(x) \leq \mu(f)$.*

Proof. For $x \in K_n$, Theorem 7.4.3 and Theorem 7.5.1 imply that

$$[f(x), +\infty) = \overline{f}(x) = \bigcup_{\mu} [\mu(f), +\infty),$$

where the union is taken over all unital completely positive maps $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ with barycenter x. In particular, $f(x) \leq \mu(f)$ for all such μ. □

8. Orders on Completely Positive Maps

8.1. **Classical Choquet order.** The classical Choquet order is a generalization of the even more classical notion of majorization. Let C be a compact convex set. For probability measures μ and ν on $\mathcal{C}$, ν is said to dominate μ in the Choquet order, written $\mu \prec_c \nu$, if $\mu(f) \leq \nu(f)$ for every convex function $f \in \mathrm{C}(C)$. The Choquet order is a partial order on the space of probability measures on C.

Heuristically, the Choquet order measures the how far the support of a probability measure is from the extreme boundary ∂C, in the sense that if $\mu \prec_c \nu$, then the support of ν is closer to ∂C than the support of μ. In fact, if ν is maximal in the Choquet order and C is metrizable, then ν is actually supported on ∂C. If C is non-metrizable, then ∂C is not necessarily Borel, but the maximality of μ still implies that it is supported on ∂C in an appropriate sense.

8.2. **Noncommutative Choquet order.** In this section we will introduce a noncommutative analogue of the Choquet order for unital completely positive maps. The comparison will be with respect to convex continuous nc functions in the sense of Section 7.2. Eventually, we will see that this order measures how far the support of a unital completely positive map is from the extreme boundary in an appropriate sense.

Definition 8.2.1. Let K be a compact nc convex set and let $\mu, \nu : \mathrm{C}(K) \to \mathcal{M}_m$ be unital completely positive maps. We say that μ is dominated by ν in the *nc Choquet order* and write $\mu \prec_c \nu$ if $\mu(f) \leq \nu(f)$ for every n and every convex nc function $f \in \mathcal{M}_n(\mathrm{C}(K))$.

Lemma 8.2.2. *Let K be a compact nc convex set and let $\mu, \nu : \mathrm{C}(K) \to \mathcal{M}_p$ be unital completely positive maps with $\mu \prec_c \nu$. Then μ and ν have the same barycenter.*

Proof. Suppose $\mu \prec_c \nu$. For $a \in \mathrm{A}(K)$, both a and $-a$ are convex, so $\mu(a) \leq \nu(a)$ and $\mu(-a) \leq \nu(-a)$, implying $\mu(a) = \nu(a)$. Hence $\mu|_{\mathrm{A}(K)} = \nu|_{\mathrm{A}(K)}$. □

By Proposition 7.4.2, if $f \in \mathrm{C}(K)$ is convex, then $\overline{f} = f$. The next result follows immediately from this fact.

Proposition 8.2.3. *Let K be a compact nc convex set and let $\mu, \nu : \mathrm{C}(K) \to \mathcal{M}_p$ be unital completely positive maps. Then $\mu \prec_c \nu$ if and only if $\mu(\overline{f}) \leq \nu(\overline{f})$ for every n and every self-adjoint nc function $f \in \mathcal{M}_n(\mathrm{C}(K))$.*

Proposition 8.2.4. *Let K be a compact nc convex set and let L denote the nc state space of* $\mathrm{C}(K)$*. Then for each n, the nc Choquet order is a partial order on L_n.*

Proof. It is easy to see that the nc Choquet order is reflexive and transitive. Antisymmetry follows from Proposition 7.2.8. □

8.3. **Dilation order.** There is another natural order for unital completely positive maps relating to the dilation theory of completely positive maps.

Definition 8.3.1. Let K be a compact nc convex set and let $\mu, \nu : \mathrm{C}(K) \to \mathcal{M}_m$ be unital completely positive maps. We say that μ is dominated by ν in the *dilation order* and write $\mu \prec_d \nu$ if there are representations $(x, \alpha) \in K_n \times \mathcal{M}_{n,m}$ for μ and $(y, \beta) \in K_p \times \mathcal{M}_{p,m}$ for ν along with an isometry $\gamma \in \mathcal{M}_{p,n}$ such that $\beta = \gamma\alpha$ and $x = \gamma^* y \gamma$.

Remark 8.3.2. Note that α and β are isometries satisfying $\mu = \alpha^* \delta_x \alpha$ and $\nu = \beta^* \delta_y \beta$. The condition $x = \gamma^* y \gamma$ implies that y dilates x.

Lemma 8.3.3. *Let K be a compact nc convex set and let $\mu, \nu : \mathrm{C}(K) \to \mathcal{M}_m$ be unital completely positive maps with $\mu \prec_d \nu$. Then μ and ν have the same barycenter.*

Proof. Let $(x, \alpha) \in K_n \times \mathcal{M}_{n,m}$, $(y, \beta) \in K_p \times \mathcal{M}_{p,m}$ and $\gamma \in \mathcal{M}_{p,n}$ be as in Definition 8.3.1. Then for $a \in \mathrm{A}(K)$,

$$\mu(a) = \alpha^* a(x) \alpha = \alpha^* a(\gamma^* y \gamma)\alpha = \alpha^* \gamma^* a(y) \gamma \alpha = \beta^* a(y) \beta = \nu(a). \quad \square$$

The next result provides a useful reformulation of the dilation order that we will use frequently.

Proposition 8.3.4. *Let K be a compact nc convex set and let $\mu, \nu : \mathrm{C}(K) \to \mathcal{M}_m$ be unital completely positive maps. Then $\mu \prec_d \nu$ if and only if there is a representation $(x, \alpha) \in K_n \times \mathcal{M}_{n,m}$ for μ and a unital completely positive map $\tau : \mathrm{C}(K) \to \mathcal{M}_n$ with barycenter x satisfying $\nu = \alpha^* \tau \alpha$.*

Proof. Suppose $\mu \prec_d \nu$ and let $(x, \alpha) \in K_n \times \mathcal{M}_{n,m}$, $(y, \beta) \in K_p \times \mathcal{M}_{p,m}$ and $\gamma \in \mathcal{M}_{p,n}$ be as in Definition 8.3.1. Let $\tau = \gamma^* \delta_y \gamma$. Then $\tau|_{\mathrm{A}(K)} = \gamma^* y \gamma = x$ and $\nu = \alpha^* \tau \alpha$.

Conversely, suppose there is a representation $(x, \alpha) \in K_n \times \mathcal{M}_{n,m}$ for μ and a unital completely positive map $\tau : \mathrm{C}(K) \to \mathcal{M}_n$ with barycenter x satisfying $\nu = \alpha^* \tau \alpha$. Choose a representation $(y, \gamma) \in K_p \times \mathcal{M}_{p,m}$ for τ. Then letting $\beta = \gamma\alpha$, (y, β) is a representation for ν and $\gamma^* y \gamma = \tau|_{\mathrm{A}(K)} = x$. □

Remark 8.3.5. Let K be a compact nc convex set and let $\mu, \nu : \mathrm{C}(K) \to \mathcal{M}_m$ be unital completely positive maps such that $\mu \prec_d \nu$. If $(x, \alpha) \in K_n \times \mathcal{M}_{m,n}$ is any representation of μ, then it follows as in the proof of Proposition 8.3.4 that there is representation $(y, \beta) \in K_p \times \mathcal{M}_{p,m}$ for ν and an isometry $\gamma \in \mathcal{M}_{p,n}$ such that $\beta = \gamma\alpha$ and $x = \gamma^* y \gamma$. In particular we can always assume that the representation (x, α) of μ is minimal. Note that it is not necessarily true that (y, β) will be a minimal representation of ν.

Proposition 8.3.6. *Let K be a compact nc convex set. For $x \in K$, the corresponding representation δ_x is the unique minimal element among the family of representing maps of x with respect to both the nc Choquet order and the dilation order.*

Proof. For $x \in K_m$, note that $(x, 1_m)$ is a minimal representation of δ_x. Let $\mu : K_m \to \mathcal{M}_m$ be a unital completely positive map with barycenter x and let $(y, \alpha) \in K_n \times \mathcal{M}_{n,m}$ be a minimal representation of μ. Then $x = \alpha^* y \alpha$ and $\mu = \alpha^* \delta_y \alpha$. So $\delta_x \prec_d \mu$.

Let $f \in \mathcal{M}_n(\mathrm{C}(K))$ be a convex nc function. Then by Remark 7.2.2,

$$f(x) = f(\alpha^* y \alpha) \leq \alpha^* f(y) \alpha = \mu(f).$$

Hence $\delta_x \prec_c \mu$. ☐

Theorem 8.3.7. *Let K be a compact nc convex set and let $\mu : \mathrm{C}(K) \to \mathcal{M}_m$ be a unital completely positive map with representation $(x, \alpha) \in K_n \times \mathcal{M}_{n,m}$. If μ is maximal in the dilation order and the representation (x, α) is minimal, then x is a maximal point. Conversely, if x is a maximal point then μ is maximal in the dilation order.*

Proof. Suppose μ is maximal in the dilation order and the representation (x, α) is minimal. By Theorem 5.1.3, there is maximal point $y \in K_p$ that dilates x. The maximality of y implies that it has a unique representing map, namely δ_y. Let $\beta \in \mathcal{M}_{p,n}$ be an isometry such that $\beta^* y \beta = x$ and define a unital completely positive map $\nu : \mathrm{C}(K) \to \mathcal{M}_m$ by $\nu = \alpha^* \beta^* \delta_y \beta \alpha$. Then by Proposition 8.3.4, $\mu \prec_d \nu$. Hence by the maximality of μ, $\mu = \nu$. Thus the pair $(y, \beta\alpha)$ is a representation of μ. By the minimality of the representation (x, α) and the uniqueness of minimal representations, $y \cong x \oplus z$ for some $z \in K$. Since y is a maximal point, it follows that x is a maximal point.

Conversely, suppose x is a maximal point. Let $\nu : \mathrm{C}(K) \to \mathcal{M}_m$ be a unital completely positive map such that $\mu \prec_d \nu$. Then by Remark 8.3.5 there is a completely positive map $\tau : \mathrm{C}(K) \to \mathcal{M}_n$ with barycenter x such that $\nu = \alpha^* \tau \alpha$. The map τ has barycenter x. Since x is maximal, it has a unique representing map, and hence $\tau = \delta_x$. Hence $\nu = \alpha^* \delta_x \alpha = \mu$ and we conclude that μ is maximal in the dilation order. ☐

Corollary 8.3.8. *Let K be a compact nc convex set and let $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ be a unital completely positive map. Then there is a unital completely positive map $\nu : \mathrm{C}(K) \to \mathcal{M}_n$ such that $\mu \prec_d \nu$ and ν is maximal in the dilation order.*

Proof. Choose a representation $(x, \alpha) \in K_n \times \mathcal{M}_{n,m}$ for μ. Following the proof of Theorem 8.3.7, apply Theorem 5.1.3 to obtain maximal $y \in K_p$ that dilates x and has a unique representing map, namely δ_y. Let $\beta \in \mathcal{M}_{p,n}$ be an isometry such that $\beta^* y \beta = x$. Define a unital completely positive map $\nu : \mathrm{C}(K) \to \mathcal{M}_m$ by $\nu = \alpha^* \beta^* \delta_y \beta \alpha$. Then by Proposition 8.3.4, $\mu \prec_d \nu$. There is a summand y_0 of y so that (y_0, α) is a minimal representation of ν. Every summand of y is maximal. Therefore, Theorem 8.3.7 implies that ν is maximal in the dilation order. □

8.4. **Dilation order and convex envelopes.** In this section we will make a connection between convex envelopes and the dilation order. This will be the key fact used in the next section to show that the two orders coincide. In view of Proposition 8.3.6, this generalizes Theorem 7.5.1.

Theorem 8.4.1. *Let K be a compact nc convex set. Let $f \in \mathcal{M}_n(\mathrm{B}(K))$ be a self-adjoint bounded nc function with convex envelope $\overline{f}$. Then for a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_k$,*

$$\mu(\overline{f}) = \bigcup_{\mu \prec_d \nu} [\nu(f), +\infty),$$

where the union is taken over all unital completely positive maps $\nu : \mathrm{C}(K) \to \mathcal{M}_k$ with $\mu \prec_d \nu$.

Proof. Let $(x, \alpha) \in K_m \times \mathcal{M}_{m,k}$ be a minimal representation of μ. Then by Theorem 7.5.1,

$$\mu(\overline{f}) = \alpha^* \overline{f}(x) \alpha = \bigcup_{\tau} \alpha^* [\tau(f), +\infty) \alpha = \bigcup_{\tau} [\alpha^* \tau(f) \alpha, +\infty),$$

where the union is taken over all unital completely positive maps $\tau : \mathrm{C}(K) \to \mathcal{M}_m$ with barycenter x.

The result now follows from Proposition 8.3.4 which says that a unital completely positive map $\nu : \mathrm{C}(K) \to \mathcal{M}_k$ satisfies $\mu \prec_d \nu$ if and only if there is a unital completely positive map $\tau : \mathrm{C}(K) \to \mathcal{M}_m$ with barycenter x such that $\nu = \alpha^* \tau \alpha$. □

Corollary 8.4.2. *Let K be a compact nc convex set. A unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_k$ is maximal in the dilation order if and only if $[\mu(f), +\infty) = \mu(\overline{f})$ for all $f \in \mathrm{C}(K)$.*

Proof. If μ is maximal in the dilation order, then Theorem 8.4.1 immediately implies $[\mu(f), +\infty) = \mu(\overline{f})$ for all $f \in \mathrm{C}(K)$.

Conversely, suppose that $[\mu(f), +\infty) = \mu(\overline{f})$ for all $f \in \mathrm{C}(K)$. Let $\nu : \mathrm{C}(K) \to \mathcal{M}_k$ be a unital completely positive map with $\mu \prec_d \nu$. For $f \in \mathrm{C}(K)$, Theorem 8.4.1 implies that

$$[\mu(f), +\infty) = \mu(\overline{f}) \supseteq [\mu(f), +\infty) \cup [\nu(f), +\infty)$$

and

$$[\mu(-f), +\infty) = \mu(\overline{-f}) \supseteq [\mu(-f), +\infty) \cup [\nu(-f), +\infty).$$

Therefore $\mu(f) \leq \nu(f)$ and $\mu(-f) \leq \nu(-f)$, implying $\mu(f) = \nu(f)$. Hence $\mu = \nu$ and therefore μ is maximal. □

8.5. **Equivalence of orders.** In this section we will show that the noncommutative Choquet order and the dilation order coincide.

Theorem 8.5.1. *Let K be a compact nc convex set and let $\mu, \nu : \mathrm{C}(K) \to \mathcal{M}_n$ be unital completely positive maps. Then $\mu \prec_c \nu$ if and only if $\mu \prec_d \nu$.*

Proof. If $\mu \prec_d \nu$, then Theorem 8.4.1 implies that for every self-adjoint continuous convex nc function $f \in \mathcal{M}_n(\mathrm{C}(K))$,

$$\mu(f) = \mu(\overline{f}) = \bigcup_{\mu \prec_d \lambda} [\lambda(f), +\infty) \supseteq \bigcup_{\nu \prec_d \lambda} [\lambda(f), +\infty) = \nu(\overline{f}) = \nu(f).$$

Therefore $\mu(f) \leq \nu(f)$. That is, $\mu \prec_c \nu$.

Conversely, if $\mu \prec_c \nu$, then Proposition 8.2.3 implies $\mu(\overline{f}) \leq \nu(\overline{f})$ for every n and every self-adjoint nc function $f \in \mathcal{M}_n(\mathrm{C}(K))$. Let $\mathcal{F}$ be a dense family of functions in $\mathrm{C}(K)_{sa}$ such that $\mathcal{F} = -\mathcal{F}$ and define $h \in \mathcal{M}_m(\mathrm{C}(K))$ by $h = \oplus_{f \in \mathcal{F}} f$. Then from above, $\mu(\overline{h}) \leq \nu(\overline{h}) \leq \nu(h)$. Hence by Theorem 8.4.1, there is a unital completely positive map $\lambda : \mathrm{C}(K) \to \mathcal{M}_p$ such that $\mu \prec_d \lambda$ and $\lambda(h) \leq \nu(h)$. But then $\lambda(f) \leq \nu(f)$ and $\lambda(-f) \leq \nu(-f)$ for all $f \in \mathcal{F}$, implying $\lambda(f) = \nu(f)$ for all $f \in \mathcal{F}$. Since $\mathcal{F}$ is dense in $\mathrm{C}(K)$, it follows that $\lambda = \nu$. Hence $\mu \prec_d \nu$. □

8.6. **Scalar convex envelope.** In this section we will show that the maximality of a unital completely positive map on the C*-algebra of continuous nc functions can be detected by looking at scalar-valued functions. As usual we will let L denote the nc state space of $\mathrm{C}(K)$. Thus L_1 denotes the space of (scalar) states of $\mathrm{C}(K)$. Recall that ∂L_1 denotes the scalar extreme points of the convex set L_1.

The next two results follow immediately from the definition of the nc Choquet order.

Lemma 8.6.1. *Let K be a compact nc convex set and let $\mu_1, \ldots, \mu_k$ and $\nu_1, \ldots, \nu_k$ be states on $\mathrm{C}(K)$ such that $\mu_i \prec_c \nu_i$ for each i. Then for scalars $\lambda_1, \ldots, \lambda_k \geq 0$ with $\lambda_1 + \cdots + \lambda_k = 1$, $\sum \lambda_i \mu_i \prec_c \sum \lambda_i \nu_i$.*

Lemma 8.6.2. *Let K be a compact nc convex set and let $\{\mu_i\}$ and $\{\nu_i\}$ be nets of states on $\mathrm{C}(K)$ converging in the point-weak* topology to states μ and ν on $\mathrm{C}(K)$ respectively. If $\mu_i \prec_c \nu_i$ for each i, then $\mu \prec_c \nu$.*

Definition 8.6.3. Let K be a compact nc convex set and let L_1 denote the state space of $\mathrm{C}(K)$. For $f \in \mathrm{C}(K)$ with convex envelope $\overline{f}$, let $\hat{f}$ and $\breve{f}$ denote the scalar-valued functions on L_1 defined by

$$\hat{f}(\mu) = \mu(f) \quad \text{and} \quad \breve{f}(\mu) = \inf \mu(\overline{f}) \quad \text{for} \quad \mu \in L_1.$$

Remark 8.6.4. Kadison's representation theorem [29] implies that as a function system, $\mathrm{C}(K)$ is order isomorphic to the function system $\mathrm{A}(L_1)$ of continuous affine functions on L_1. For $f \in \mathrm{C}(K)$, the function $\hat{f}$ is precisely the image of f under the corresponding order isomorphism.

Proposition 8.6.5. *Let K be a compact nc convex set and let L_1 denote the state space of $\mathrm{C}(K)$. Let $f \in \mathrm{C}(K)$ be a self-adjoint continuous nc function. Then the corresponding functions $\hat{f}, \breve{f} : L_1 \to \mathbb{C}$ satisfy*

(1) $\breve{f} \leq \hat{f}$,
(2) $\breve{f}$ *is lower semicontinuous,*
(3) $\breve{f}$ *is convex.*

Proof. (1) This follows immediately from Proposition 7.4.2.

(2) Let $\{\mu_i\}$ be a net in L_1 converging to $\mu \in L_1$. We must show that $\liminf \breve{f}(\mu_i) \geq \breve{f}(\mu)$. For $\epsilon > 0$, Theorem 8.4.1 implies that for each i there is $\nu_i \in L_1$ such that $\mu_i \prec_d \nu_i$ and $\nu_i(f) < \inf \mu_i(\overline{f}) + \epsilon$. If $\{\nu_j\}$ is a subnet converging in the weak* topology to $\nu \in L_1$, then Lemma 8.6.2 implies $\mu \prec_c \nu$. So by Theorem 8.5.1, $\mu \prec_d \nu$. Thus applying Theorem 8.4.1 again implies $[\nu(f), +\infty) \subseteq \mu(\overline{f})$. Hence

$$\begin{aligned}\breve{f}(\mu) = \inf \mu(\overline{f}) &\leq \nu(f) = \lim \nu_j(f) \\ &\leq \liminf(\inf \mu_j(\overline{f})) + \epsilon = \liminf \breve{f}(\mu_j) + \epsilon.\end{aligned}$$

Taking $\epsilon \to 0$ gives the desired result.

(3) For $\mu_1, \mu_2 \in L_1$ and $t \in (0,1)$, Theorem 8.4.1 implies

$$\begin{aligned}t\mu_1(\overline{f}) + (1-t)\mu_2(\overline{f}) &= t \bigcup_{\mu_1 \prec_d \nu_1} [\nu_1(f), +\infty) + (1-t) \bigcup_{\mu_2 \prec_d \nu_2} [\nu_2(f), +\infty) \\ &= \bigcup_{\substack{\mu_1 \prec_d \nu_1 \\ \mu_2 \prec_d \nu_2}} [t\nu_1(f) + (1-t)\nu_2(f), +\infty),\end{aligned}$$

where the unions are taken over all states $\nu_1, \nu_2 \in L_1$ with $\mu_1 \prec_d \nu_1$ and $\mu_2 \prec_d \nu_2$. For such ν_1, ν_2, Lemma 8.6.1 and Theorem 8.5.1 imply that

$$t\mu_1 + (1-t)\mu_2 \prec_d t\nu_1 + (1-t)\nu_2.$$

Therefore

$$t\mu_1(\overline{f}) + (1-t)\mu_2(\overline{f}) \subseteq \bigcup_{\lambda} [\lambda(f), +\infty) = (t\mu_1 + (1-t)\mu_2)(\overline{f}),$$

where the union is taken over all $\lambda \in L_1$ with $t\mu_1 + (1-t)\mu_2 \prec_d \lambda$. Hence

$$\begin{aligned} t\breve{f}(\mu_1) + (1-t)\breve{f}(\mu_2) &= t \inf \mu_1(\overline{f}) + (1-t) \inf \mu_2(\overline{f}) \\ &\geq \inf (t\mu_1 + (1-t)\mu_2)(\overline{f}) = \breve{f}(t\mu_1 + (1-t)\mu_2), \end{aligned}$$

and we conclude that $\breve{f}$ is convex. □

We obtain the following characterization of maximal elements in K in terms of the scalar-valued functions in Definition 8.6.3.

Proposition 8.6.6. *Let K be a compact nc convex set and let L_1 denote the state space of $\mathrm{C}(K)$. A state $\mu \in L_1$ is maximal in the dilation order if and only if $\hat{f}(\mu) = \breve{f}(\mu)$ for all self-adjoint $f \in \mathrm{C}(K)$.*

Proof. For self-adjoint $f \in \mathrm{C}(K)$, Proposition 7.4.2 implies that $\mu(\overline{f}) \supseteq [\mu(f), +\infty)$. Hence if $\inf \mu(\overline{f}) = \breve{f}(\mu) = \overline{f}(\mu) = \mu(f)$, then $\mu(\overline{f}) = [\mu(f), +\infty)$. The result now follows from Corollary 8.4.2. □

8.7. **Extreme points revisited.** In this section we will apply the equivalence between the nc Choquet order and the dilation order to give short proofs of Theorem 5.1.3 about the existence of maximal dilations and Theorem 6.2.2 about the existence of extreme points.

The next result is Theorem 5.1.3. It was proved by Dritschel and McCullough [17, Theorem 1.2].

Theorem 5.1.3. *Let K be a compact nc convex set. Then every point in K has a maximal dilation.*

Proof. Fix $x \in K_m$. By an easy Zorn's lemma argument there is a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ with barycenter x that is maximal in the dilation order. Let $(y, \alpha) \in K_n \times \mathcal{M}_{n,m}$ be a minimal representation of μ. Then y dilates x and by Theorem 8.3.7, y is maximal. □

The next result is Theorem 6.2.2. It was proved in [12, Theorem 2.4]. Note that the proof is completely independent of Theorem 5.1.3.

Theorem 6.2.2. *Let K be a compact nc convex set. Then every pure point in K has an extreme dilation.*

Proof. Our primary goal is to find a pure dilation-maximal representing map μ for x on $\mathrm{C}(K)$. Let L denote the nc state space of $\mathrm{C}(K)$. Let $F = \{\mu \in L_n : \mu \text{ has barycenter } x\}$. Then F is a closed face since if $(1/2)(\nu_1 + \nu_2) \in F$, then by the pureness of x, both ν_1 and ν_2 have barycenter x. Furthermore, F is hereditary with respect to the dilation order, meaning that if $\mu \in F$ and $\mu \prec_d \nu$, then $\nu \in F$ because the barycenters of μ and ν agree by Lemma 8.3.3.

Say that a face F' is hereditary if $\mu \in F'$ and $\mu \prec_d \nu$ implies that $\nu \in F'$. Apply Zorn's lemma to the family of all closed hereditary faces contained in F to get a minimal closed hereditary face F_0. We claim that F_0 is a single point. Suppose otherwise that there are $\mu, \nu \in F_0$ with $\mu \neq \nu$. By Proposition 7.2.8, there is a convex nc function $f \in \mathrm{C}(K)$ such that $\mu(f) \neq \nu(f)$. The set $\{\mu(f) : \mu \in F_0\}$ is a compact convex subset of $(\mathcal{M}_n)_{sa}$. Therefore there is a maximal element A of this set in the usual order on self-adjoint matrices. Let $F_1 = \{\mu \in F_0 : \mu(f) = A\}$. The maximality of A shows that this is a proper closed face of F_0. Moreover since f is convex, this set is hereditary. This contradicts the minimality of F_0 as a closed hereditary face. Thus $F_0 = \{\mu_0\}$ is a singleton. Thus μ_0 is an extreme point of F. Since F_0 is a face, μ_0 is pure. Since F_0 is hereditary, μ_0 must be maximal in the dilation order.

Let (y, α) be a minimal representation of μ_0. By [2, Corollary I.4.3], δ_y is irreducible. By Theorem 8.3.7, y has a unique representing map. Thus y is an nc extreme point of K by Theorem 6.1.9. By Corollary 6.2.1, δ_y is a boundary representation. □

The proof of Theorem 6.1.9 implies the next result.

Corollary 8.7.1. *If $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ is a pure dilation maximal unital completely positive map with minimal representation (y, α) for $y \in K_p$ and an isometry $\alpha \in \mathcal{M}_{p,n}$, then y is an nc extreme point of K.*

9. Noncommutative Choquet-Bishop-de Leeuw theorem

9.1. **Classical Choquet-Bishop-de Leeuw theorem.** The classical Choquet-Bishop-de Leeuw theorem asserts that for a compact convex set C, every point $x \in C$ can be represented by a probability measure μ supported on the extreme boundary ∂C of C. The result was proved for metrizable C by Choquet [10], and for non-metrizable C by Bishop and de Leeuw [8] (see [1, Section I.4]).

The set C is metrizable if and only if the corresponding function system $\mathrm{A}(C)$ is separable. In this case, ∂C is G_δ, and as usual, μ is said to be supported on ∂C if $\mu(C \setminus \partial C) = 0$. Otherwise, ∂C is not necessarily even Borel, and in this case μ is said to be supported on ∂C if $\mu(X) = 0$ for every *Baire* set $X \subseteq C$ that is disjoint from ∂C. Equivalently, $\int_C f\, d\mu = 0$ for every bounded Baire function f on C with support in $C \setminus \partial C$.

9.2. **Noncommutative Choquet-Bishop-de Leeuw theorem.** In this section, we will establish a noncommutative generalization of the Choquet-Bishop-de Leeuw theorem. This result will not require any assumptions about separability. However, as in the classical theory, technical difficulties arise in the non-separable setting. In order to handle these difficulties, and in order to define an appropriate notion of support for a representing map, we will require an appropriate notion of bounded Baire nc function. Before stating the definition, we first recall the definition of the Baire-Pedersen envelope of a C*-algebra, introduced by Pedersen under a different name (see [39, Section 4.5]).

Let A be a C*-algebra. The *Baire-Pedersen envelope* $\mathfrak{B}(A)$ of A is a C*-subalgebra of the bidual A^{**} that contains A. It is constructed as the monotone sequential closure of A in its universal representation. If A is commutative, say $A = \mathrm{C}(X)$ for a compact Hausdorff space X, then $\mathfrak{B}(\mathrm{C}(X))$ is isomorphic to the C*-algebra of bounded Baire functions on X.

Definition 9.2.1. For a compact nc convex set K, we let $\mathrm{B}^\infty(K)$ denote the Baire-Pedersen envelope $\mathfrak{B}(\mathrm{C}(K))$ of $\mathrm{C}(K)$ and refer to the elements in $\mathrm{B}^\infty(K)$ as the *bounded Baire nc functions* on K. We say that a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ is *supported* on the extreme boundary ∂K if $\mu(f) = 0$ for every bounded Baire nc function $f \in \mathrm{B}^\infty(K)$ satisfying $f(x) = 0$ for all $x \in \partial K$.

Remark 9.2.2. Note that since $\mathrm{B}^\infty(K) \subseteq \mathrm{C}(K)^{**} = \mathrm{B}(K)$, the elements in $\mathrm{B}^\infty(K)$ are bounded nc functions. For a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$, the restriction to $\mathrm{B}^\infty(K)$ of the unique normal extension of μ to $\mathrm{B}(K)$ coincides with the unique sequentially normal extension of μ to $\mathrm{B}^\infty(K)$ (see [39, Theorem 4.5.9]).

The next result is a noncommutative analogue of the Choquet-Bishop-de Leeuw theorem. Note that the result does not place any restrictions on K. In particular, $\mathrm{A}(K)$ is not required to be separable.

Theorem 9.2.3 (Noncommutative Choquet-Bishop-de Leeuw theorem)**.** *Let K be a compact nc convex set. For $x \in K_n$ there is a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ that represents x and is supported on the extreme boundary ∂K.*

Remark 9.2.4. Since the restriction to $\mathrm{A}(K)$ of the unique surjective homomorphism from $\mathrm{C}(K)$ onto $\mathrm{C}^*_{\min}(\mathrm{A}(K))$ is a unital complete order embedding, Arveson's extension theorem implies that for $x \in K_n$ there is a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ that represents x and factors through $\mathrm{C}^*_{\min}(\mathrm{A}(K))$. Hence we can always choose a representing map for x that is supported on the irreducible representations of $\mathrm{C}^*_{\min}(\mathrm{A}(K))$, i.e. on the Shilov boundary of $\mathrm{A}(K)$.

The discussion in Section 6.5 shows that ∂K corresponds to an (often proper) subset of the irreducible representations of $\mathrm{C}^*_{\min}(\mathrm{A}(K))$. Therefore, the the assertion in Theorem 9.2.3 is much stronger. It says that we can always choose a representing map for x that is supported on ∂K, i.e. on the Choquet boundary of $\mathrm{A}(K)$.

In order to prove Theorem 9.2.3, we will require some preliminary results about the separable case.

Proposition 9.2.5. *Let K be a compact nc convex set such that $\mathrm{A}(K)$ is separable and let L_1 denote the state space of $\mathrm{C}(K)$. Then the set*

$$Z = \{\nu \in L_1 : \nu \textit{ is pure and dilation maximal}\}$$

is G_δ. If $\mu \in L_1$ is dilation maximal, then there is a regular Borel probability measure ρ on L_1 supported on Z with barycenter μ, meaning that $\rho(Z) = 1$ and

$$\mu(f) = \int_Z \nu(f)\, d\rho(\nu) \quad \textit{for} \quad f \in \mathrm{C}(K).$$

Moreover any regular Borel probability measure ρ on L_1 with barycenter μ is supported on Z in the above sense.

Proof. Let $\{f_k\}$ be a dense sequence in $\mathrm{A}(K)_{sa}$. For $k, m \in \mathbb{N}$, let

$$X_{km} = \{\nu \in L_1 : (\hat{f}_k - \breve{f}_k)(\nu) \geq \tfrac{1}{m}\},$$

where $\hat{f}$ and $\breve{f}$ are defined as in Section 8.6. By Proposition 8.6.5, $\breve{f}$ is convex and lower semicontinuous. Since $\hat{f}$ is continuous and affine, $\hat{f} - \breve{f}$ is concave and upper semicontinuous. Therefore X_{km} is closed.

By Proposition 8.6.6, the G_δ set

$$Y = K \setminus \big(\bigcup_{k,m} X_{km}\big)$$

is precisely the set of dilation maximal states on $\mathrm{C}(K)$. Since $\mathrm{A}(K)$ is separable, $\mathrm{C}(K)$ is also separable. Hence L_1 is metrizable, so ∂L_1 is G_δ (see e.g. [1, Corollary I.4.4]). It follows that $Z = \partial L_1 \cap Y$ is G_δ.

By Choquet's integral representation theorem, there is a Borel measure ρ on C supported on ∂C that represents μ, i.e. such that

$$\mu(f) = \int_{\partial C} \nu(f)\, d\rho(\nu) \quad \text{for} \quad f \in \mathrm{C}(K).$$

It remains to show that ρ is supported on Z, or equivalently that $\rho(X_{km}) = 0$ for $k, m \in \mathbb{N}$.

Suppose for the sake of contradiction that $\rho(X_{km}) > 0$ for some $k, m \in \mathbb{N}$. Define probability measures τ and η on L_1 by

$$\sigma = \rho(X_{km})^{-1} \rho|_{X_{km}} \quad \text{and} \quad \tau = \rho(L_1 \setminus X_{km})^{-1} \rho|_{L_1 \setminus X_{km}}.$$

Let $\xi \in L_1$ denote the barycenter of σ and let $\eta \in L_1$ denote the barycenter of τ. Note that $\mu = \rho(X_{km})\xi + \rho(L_1 \setminus X_{km})\eta$.

Since σ is supported on X_{km}, there is a sequence $\{\sigma_i\}$ of finitely supported probability measures on X_{km} such that $\lim \sigma_i = \sigma$ in the weak* topology. Each σ_i can be written as a finite convex combination $\sigma_i = \sum c_{ij} \delta_{\nu_{ij}}$ of states $\nu_{ij} \in X_{km}$. Let $\xi_i \in L_1$ denote the barycenter of σ_i. Then by the continuity of the barycenter map, $\lim \xi_i = \xi$ in the weak* topology. Hence

$$\begin{aligned}
\xi(f_k) = \lim_i \xi_i(f_k) &= \lim_i \sum_j c_{ij} \nu_{ij}(f_k) \\
&\geq \frac{1}{m} + \liminf_i \sum_j c_{ij} \breve{f}_k(\nu_{ij}) \\
&\geq \frac{1}{m} + \liminf_i \breve{f}_k(\xi_i) \\
&\geq \frac{1}{m} + \breve{f}_k(\xi),
\end{aligned}$$

where we have used the convexity and lower semicontinuity of $\breve{f}_k$ from Proposition 8.6.5. Another application of the convexity of $\breve{f}_k$ yields

$$\begin{aligned}
\mu(f_k) &= \rho(X_{km})\xi(f_k) + \rho(L_1 \setminus X_{km})\eta(f_k) \\
&\geq \frac{\rho(X_{km})}{m} + \rho(X_{km})\breve{f}_k(\xi) + \rho(L_1 \setminus X_{km})\breve{f}_k(\eta) \\
&\geq \frac{\rho(X_{km})}{m} + \breve{f}_k(\mu).
\end{aligned}$$

In particular, $\hat{f}_k(\mu) \neq \breve{f}_k(\mu)$. Therefore, by Proposition 8.6.6, μ is not maximal in the dilation order, providing a contradiction. $\square$

The next result will provide the connection between the separable and the non-separable case.

Proposition 9.2.6. *Let K be a compact nc convex set and let $S \subseteq \mathrm{A}(K)$ be an operator system with nc state space K_0. Identify S with $\mathrm{A}(K_0)$ and identify $\mathrm{C}(K_0)$ with the C*-subalgebra of $\mathrm{C}(K)$ generated by S. Then every dilation maximal pure state on $\mathrm{C}(K_0)$ extends to a dilation maximal pure state on $\mathrm{C}(K)$.*

Proof. Let μ_0 be a dilation maximal pure state on $C(K_0)$ and let (x_0, α_0) be a minimal representation of μ_0 for $x_0 \in (K_0)_n$ and an isometry $\alpha_0 \in \mathcal{M}_{n,1}$. Note that by Theorem 8.3.7 and Theorem 6.1.9, x_0 is an extreme point of K_0.

Let $F = \{x \in K_n : x|_{\mathrm{A}(K_0)} = x_0\}$. Then F is a closed face of K_n. By the (classical) Krein-Milman theorem, the set of extreme points of F is non-empty. Let $x \in F$ be an extreme point. Then x is pure in K. By Theorem 6.2.2, we can dilate x to an extreme point $y \in K_p$. In particular, the representation δ_y is irreducible and dilation maximal. Let $\beta \in \mathcal{M}_{p,n}$ be an isometry such that $x = \beta^* y \beta$.

Define a state μ on $\mathrm{C}(K)$ by $\mu = \alpha_0^* \beta^* \delta_y \beta \alpha_0$. Then $(y, \beta\alpha_0)$ is a representation of μ. Since δ_y is irreducible, $(y, \beta\alpha_0)$ is minimal. As y is an extreme point, it follows that μ is pure and maximal in the dilation order. Furthermore, since x_0 is an extreme point in K_0, the restriction $y|_{\mathrm{A}(K_0)}$ must be a trivial dilation of x_0. Hence $\mu|_{\mathrm{C}(K_0)} = \mu_0$. □

Proposition 9.2.7. *Let K be a compact nc convex set. Every dilation maximal state on $\mathrm{C}(K)$ is supported on the extreme boundary ∂K.*

Proof. Let μ be a dilation maximal state on $\mathrm{C}(K)$ and fix $f \in \mathrm{B}^\infty(K)$ such that $f(x) = 0$ for $x \in \partial K$. We must show that $\mu(f) = 0$.

By [39, Lemma 4.5.3], there is a separable operator system $S \subseteq \mathrm{A}(K)$ with nc state space K_0 such that if we identify S with $\mathrm{A}(K_0)$ and identify $\mathrm{C}(K_0)$ with the subalgebra of $\mathrm{C}(K)$ generated by S, then $f \in \mathrm{B}^\infty(K_0) \subseteq \mathrm{B}^\infty(K)$. The key point is that f is a Baire nc function on the nc state space of a separable operator subsystem.

Let $(L_0)_1$ denote the (scalar) state space of $\mathrm{C}(K_0)$. Then by Proposition 9.2.5, the set

$$Z = \{\nu \in (L_0)_1 : \nu \text{ is pure and dilation maximal}\}$$

is G_δ, and there is a regular Borel probability measure ρ on $(L_0)_1$ supported on Z such that

$$\mu(g) = \int_Z \nu(g)\, d\rho(\nu), \quad \text{for} \quad g \in \mathrm{C}(K_0) \tag{9.2.1}$$

Since $f(x) = 0$ for $x \in \partial K$, Proposition 9.2.6 implies that $f(x) = 0$ for $x \in \partial K_0$. Hence by Corollary 8.7.1, $\nu(f) = 0$ for every $\nu \in Z$.

By [39, Corollary 4.5.13], f is universally measurable, so the barycenter formula (9.2.1) also holds for f (see e.g. [6, Section 5]). Hence $\mu(f) = 0$. □

Putting all of these ingredients together yields a proof of our noncommutative Bishop-de Leeuw Theorem.

Proof of Theorem 9.2.3. Let $y \in K_p$ be a maximal dilation of x and let $\alpha \in \mathcal{M}_{p,n}$ be an isometry such that $x = \alpha^* y \alpha$. Then the corresponding representation δ_y is dilation maximal, and hence by Proposition 9.2.7, δ_y is supported on ∂K. Define a unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ by $\mu = \alpha^* \delta_y \alpha$. Then μ represents x and is supported on ∂K. □

10. Noncommutative integral representation theorem

10.1. **Motivation.** In this section we will restrict our attention to the separable setting and prove a noncommutative analogue of Choquet's integral representation theorem using the results from Section 9. We suspect that these ideas may work in greater generality, however we will utilize results about direct integral decompositions of representations of separable C*-algebras.

Let K be a compact nc convex set. For $x \in K_n$, the corresponding representation $\delta_x : \mathrm{C}(K) \to \mathcal{M}_n$ should be viewed as a noncommutative Dirac measure on K supported at the point x. More generally, for a finite set of points $\{x_i \in K_{n_i}\}$ and operators $\alpha_i \in \mathcal{M}_{n_i,n}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$, the unital completely positive map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ defined by

$$\mu = \sum \alpha_i^* \delta_{x_i} \alpha_i$$

should be viewed as a finitely supported nc probability measure on K.

For a continuous nc function $f \in \mathrm{C}(K)$, the expression

$$\mu(f) = \sum_{i=1}^{k} \alpha_i^* f(x_i) \alpha_i,$$

should be viewed as the integral of f against the nc measure μ. Note that if $x \in K_n$ denotes the barycenter of μ, then in particular, for a continuous nc affine function $a \in \mathrm{A}(K)$,

$$a(x) = \mu(a) = \sum_{i=1}^{k} \alpha_i^* a(x_i) \alpha_i.$$

In the next section we will consider a basic theory of noncommutative integration.

10.2. **Noncommutative integration.** In this section we will outline a basic theory of integration against measures taking values in spaces of completely positive maps following an approach originally due to Fujimoto [25].

Let M and N be separable von Neumann algebras and let $\mathrm{CP}_{\mathrm{nor}}(M,N)$ denote the space of all normal completely positive maps from M to N. Let Z be a topological space and let $\mathrm{Bor}(Z)$ denote the σ-algebra of Borel subsets of Z.

A $\mathrm{CP}_{\mathrm{nor}}(M,N)$-valued Borel measure on Z is a countably additive map $\lambda : \mathrm{Bor}(Z) \to \mathrm{CP}_{\mathrm{nor}}(M,N)$, meaning that if (E_k) is a disjoint sequence in $\mathrm{Bor}(Z)$ and $E = \bigcup_{k=1}^\infty E_k$, then $\lambda(E) = \sum_{k=1}^\infty \lambda(E_k)$, where the right hand side converges with respect to the point-weak* topology.

Following Fujimoto [25, Definition 3.6], we will require that λ satisfies an absolute continuity-type condition with respect to a scalar-valued Borel measure.

Let ν be a scalar-valued Borel measure on Z. For each $a \in M$ and $\rho \in N_*$, we obtain a scalar-valued Borel measure $\lambda_{a,\rho}$ on Z defined by

$$\lambda_{a,\rho}(E) = \langle \lambda(E)(a), \rho \rangle, \quad \text{for} \quad E \in \mathrm{Bor}(Z).$$

We say that λ is absolutely continuous with respect to ν if each $\lambda_{a,\rho}$ is absolutely continuous with respect to ν for each $\rho \in N_*$. In this case, the Radon-Nikodym theorem implies that for each $\lambda_{a,\rho}$, there is unique $r_{a,\rho} \in L^1(Z,\nu)$ satisfying

$$\lambda_{a,\rho}(E) = \langle \lambda(E)(a), \rho \rangle = \int_E r_{a,\rho}(z)\, d\nu(z), \quad \text{for} \quad E \in \mathrm{Bor}(Z).$$

We say that a function $f : Z \to M$ is *bounded* if

$$\sup\{\|f(z)\| : z \in Z\} < \infty.$$

We say that f is *measurable* if its range $f(Z)$ is separable and $f^{-1}(E)$ is Borel for every weak*-Borel set $E \subseteq M$. We will let $\mathrm{BM}(Z,M)$ denote the space of all bounded measurable functions from Z to M. Since M is separable, the relative weak* topology on bounded subsets of M is metrizable. We equip bounded subsets of $\mathrm{BM}(Z,M)$ with the corresponding topology of uniform convergence with respect to this metric, which we refer to as the *weak*-uniform convergence topology*.

We say that $f \in \mathrm{BM}(Z,M)$ is *simple* if there are sequences $(E_k)_{k=1}^\infty$ in $\mathrm{Bor}(Z)$ and $(a_k)_{k=1}^\infty$ in M such that

$$f = \sum_{k=1}^\infty \chi_{E_k} a_k.$$

Fujimoto showed [25, Lemma 3.3] that the space of bounded measurable simple functions is dense in $\mathrm{BM}(Z, M)$ with respect to the weak*-uniform convergence topology.

For a $\mathrm{CP}_{\mathrm{nor}}(M, N)$-valued Borel measure λ on Z and a simple function $f \in \mathrm{BM}(Z, M)$ expressed as above, the integral of f with respect to λ is defined by

$$\int_X f\, d\lambda = \sum_{i=1}^{\infty} \lambda(E_i)(a_i),$$

where the right hand side converges in the weak* topology on N (see [25, Lemma 3.1]). As usual, this definition does not depend on any particular expression of f.

Viewing integration against λ as a linear map on the space of bounded measurable simple functions, Fujimoto showed [25, Definition 3.8] that if λ is absolutely continuous with respect to a scalar measure ν on Z, then there is a unique linear extension to $\mathrm{BM}(Z, M)$ that is continuous with respect to the weak*-uniform convergence topology. For f in $\mathrm{B}(Z, M)$, we will let

$$\int_Z f\, d\lambda.$$

denote the value of this extension at f, and refer to it as the *integral of f against λ*. We will say that f is *λ-integrable*. For $\rho \in N_*$, it follows from above that

$$\Big\langle \int_Z f\, d\lambda, \rho \Big\rangle = \int_Z r_{f(z),\rho}(z)\, d\nu(z).$$

10.3. **Noncommutative integral representation theorem.** In this section we will introduce a definition of nc measure along with a corresponding notion of integration for nc functions. We will then apply these ideas to establish our noncommutative integral representation theorem.

Lemma 10.3.1. *Let K be a compact nc convex set such that $\mathrm{A}(K)$ is metrizable. Then for each n, $(\partial K)_n := \partial K \cap K_n$ is a Borel set.*

Proof. In [3, Theorem 2.5], Arveson shows that $x \in K_n$ is maximal if and only if for every $a \in \mathrm{A}(K)$ and unit vector $\xi \in H_n$, for any dilation y of x on a larger space, one has $\|y(a)\xi\| = \|x(a)\xi\|$. And as noted in [12], it suffices to consider dilations y on a Hilbert space $H_n' = H_n \oplus \mathbb{C}$. Let L_n denote the space of unital completely positive maps from $\mathrm{A}(K)$ into H_n'. The compression map from H_n' onto H_n determines a surjective continuous map $\rho : L_n \to K_n$. Note that $y \in L_n$ is a dilation of x precisely when $\rho(y) = x$.

Fix a countable dense subset $\{a_i\} \subset \mathrm{A}(K)$ and a countable dense subset $\{\xi_j\}$ of the unit sphere of H_n. Observe that

$$F_{ij}(x) = \|x(a_i)\xi_j\| = \sup_k |\langle x(a_i)\xi_j, \xi_k\rangle| \quad \text{for} \quad x \in K_n$$

is the supremum of continuous functions and thus is lower semicontinuous, and in particular is Borel. Consider the function

$$G_{ij}(x) = \sup_{y \in \rho^{-1}(x)} \|y(a_i)\xi_j\| = \sup_k \sup_{y \in \rho^{-1}(x)} |\langle y(a_i)\xi_j, \eta_k\rangle|,$$

where $\{\eta_k\}$ is a dense subset of the unit sphere of H_n'. This is the supremum of the functions

$$g_{ijk}(x) = \sup_{y \in \rho^{-1}(x)} |\langle y(a_i)\xi_j, \eta_k\rangle|.$$

This function is upper semicontinuous because if x_m converges to x, pick $y_m \in \rho^{-1}(x_m)$ attaining the value $g_{ijk}(x_m)$. Dropping to a subsequence, we may suppose that $|\langle y_m(a_i)\xi_j, \eta_k\rangle|$ approaches $\limsup_m g_{ijk}(x_m)$ and the y_m converge to $y \in \rho^{-1}(x)$. Thus

$$g_{ijk}(x) \geq |\langle y(a_i)\xi_j, \eta_k\rangle| = \limsup_m g_{ijk}(x_m).$$

Hence g_{ijk} is Borel and so G_{ij} is also Borel.

It follows that $H_{ij}(x) = G_{ij}(x) - F_{ij}(x)$ is Borel. By the discussion in the first paragraph, x is maximal if and only if $x \in \bigcap_{i,j} H_{ij}^{-1}(\{0\})$. Therefore the set of maximal points in K_n is Borel. Since K_n is metrizable, the set of pure points ∂K_n is G_δ, and hence Borel. Since $(\partial K)_n$ is the intersection of ∂K_n and the set of maximal elements by Proposition 6.1.4, it is Borel. □

Definition 10.3.2. Let K be a compact nc convex set such that $\mathrm{A}(K)$ is separable. For $n \leq \aleph_0$, a *$\mathcal{M}_n$-valued finite nc measure* on K is a sequence $\lambda = (\lambda_m)_{m \leq \aleph_0}$ such that each λ_m is a $\mathrm{CP}_{\mathrm{nor}}(\mathcal{M}_m, \mathcal{M}_n)$-valued Borel measure and the sum

$$\sum_{m \leq \aleph_0} \lambda_m(K_m)(1_m) \in \mathcal{M}_n$$

is weak* convergent. For $E \in \mathrm{Bor}(K)$, we define $\lambda(E)$ by

$$\lambda(E) = \sum_{m \leq \aleph_0} \lambda_m(E_m).$$

We will say that λ is *supported on the extreme boundary* ∂K if

$$\lambda_m(K_m \setminus \partial K) = 0_{\mathcal{M}_m, \mathcal{M}_n} \quad \text{for all} \quad m \leq \aleph_0.$$

We will say that λ is a *$\mathcal{M}_n$-valued nc probability measure* on K if the above sum is equal to 1_n. Finally, we will say that λ is *admissible* if each

λ_m is absolutely continuous with respect to a scalar-valued measure on K_m.

Remark 10.3.3. Let λ be an admissible $\mathcal{M}_n$-valued finite nc measure on K as above. For a bounded Baire nc function $f \in \mathrm{B}^\infty(K)$ and $m \leq \aleph_0$, the restriction $f|_{K_m} : K_m \to \mathcal{M}_m$ is a bounded and measurable $\mathcal{M}_m$-valued function on K_m, i.e. $f|_{K_m} \in \mathrm{BM}(K_m, \mathcal{M}_m)$. Hence by the discussion in Section 10.2, $f|_{K_m}$ is λ_m-integrable.

Example 10.3.4. Let K be a compact nc convex set such that $\mathrm{A}(K)$ is separable. For $x \in K_n$, define a $\mathcal{M}_n$-valued finite nc measure $\lambda_x = (\lambda_{x,m})_{m \leq \aleph_0}$ on K by letting $\lambda_{x,m} = 0_{\mathcal{M}_m, \mathcal{M}_n}$ for $m \neq n$ and

$$\lambda_{x,n}(E) = \begin{cases} \mathrm{id}_n & x \in E, \\ 0_n & x \notin E \end{cases}$$

for $E \subseteq \mathrm{Bor}(K_n)$. Then λ_x is the noncommutative analogue of a point mass. Note that λ_x is absolutely continuous with respect to the scalar-valued point mass δ_x on K_n. Hence λ_x is admissible.

Example 10.3.5. Let K be a compact nc convex set such that $\mathrm{A}(K)$ is separable and let $\lambda = (\lambda_m)_{m \leq \aleph_0}$ be a $\mathcal{M}_n$-valued finite nc measure on K. For $\varphi \in \mathrm{CP}_{\mathrm{nor}}(\mathcal{M}_n, \mathcal{M}_p)$, the composition $\varphi \circ \lambda := (\varphi \circ \lambda_m)_{m \leq \aleph_0}$ is a $\mathcal{M}_p$-valued finite nc measure on K. If λ is a nc probability measure, and φ is unital, then $\varphi \circ \lambda$ is a nc probability measure. In this setting, scalars are replaced by normal completely positive maps, so $\varphi \circ \lambda$ is the noncommutative analogue of a scaling of λ.

If λ_m is absolutely continuous with respect to a scalar-valued probability measure ν_m on K_m, then $\varphi \circ \lambda_m$ is also absolutely continuous with respect to ν_m. Hence if λ is admissible, then so is $\varphi \circ \lambda$.

In particular, for $\alpha \in \mathcal{M}_{n,p}$, $\alpha^* \lambda \alpha := (\alpha^* \lambda_m \alpha)_{m \leq \aleph_0}$ is a $\mathcal{M}_p$-valued finite nc measure on K. If $\alpha^* \alpha = 1_p$, then $\alpha^* \lambda \alpha$ is an nc probability measure. More generally, this shows that the set of (admissible) finite nc measures on K and the set of (admissible) finite nc probability measures on K each form an nc convex set.

Definition 10.3.6. Let K be a compact nc convex set such that $\mathrm{A}(K)$ is separable and let λ be an admissible $\mathcal{M}_n$-valued finite nc measure on K. For $f \in \mathrm{B}^\infty(K)$, we define the *integral of f with respect to λ* by

$$\int_K f \, d\lambda := \sum_{m \leq \aleph_0} \int_{K_m} f \, d\lambda_m.$$

We say that λ *represents* $x \in K_n$ if $\int_K a \, d\lambda = a(x)$ for all $a \in \mathrm{A}(K)$.

Remark 10.3.7. For an admissible nc probability measure λ as above, it follows from the discussion in Section 10.2 that map $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ defined by $\mu(f) = \int_K f\, d\lambda$ is unital and completely positive.

Example 10.3.8. Let K be a compact nc convex set. Fix a finite set of points $\{x_i \in K_{n_i}\}$ and a corresponding finite family $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$. For each i, let $\lambda_{x_i} = (\lambda_{x_i,m})_{m \le \aleph_0}$ denote the nc probability measure corresponding to x_i as in Example 10.3.4. Define $\lambda = (\lambda_m)_{m \le \aleph_0}$ by $\lambda = \sum \alpha_i^* \lambda_x \alpha_i$. Then λ is an admissible finite nc probability measure on K. For a function $f \in \mathrm{B}^\infty(K)$, the integral of f with respect to λ is

$$\int_K f\, d\lambda = \sum_{m \le \aleph_0} \int_{K_m} f\, d\lambda_m = \sum_i \alpha_i^* f(x_i) \alpha_i.$$

For the the proof of the next result, we will utilize the theory of direct integral decompositions of representations of separable C*-algebras as presented in Takesaki's book [45, Sections IV.6 and IV.8].

Let A be a separable C*-algebra with state space L_1. Then L_1 is a compact convex subset of the dual of $\mathrm{C}(L_1)$ with respect to the weak* topology. The extreme boundary ∂L_1 is precisely the set of pure states of A. For a state $\nu \in L_1$, let $\pi_\nu : A \to \mathcal{B}(H_\nu)$ denote the GNS representation of ν.

For $\mu \in L_1$, Choquet's theorem implies there is a probability measure ρ on L_1 supported on ∂L_1 with barycenter μ. By [45, Proposition IV.6.23], we can in addition choose ρ to be orthogonal, which is equivalent to the commutative von Neumann algebra $L^\infty(L_1, \mu)$ being isomorphic to a subalgebra of the commutant $\pi_\mu(A)'$. By [45, Corollary IV.8.31], the orthogonality of ρ implies that π_μ is unitarily equivalent to the direct integral

$$\pi_\mu \simeq \int_{L_1}^{\oplus} \pi_\nu \, d\rho(\nu),$$

where π_ν denotes the GNS representation of ν. If there is a cardinal number n such that every state in the support of μ is pure and the corresponding GNS representation acts on a Hilbert space of dimension n, then [45, Corollary IV.8.30] implies $\pi_\mu(A)$ is isomorphic to a subalgebra of $L^\infty(L_1, \rho, \mathcal{B}(H)) := L^\infty(L_1, \rho) \overline{\otimes} \mathcal{B}(H)$, where H is a fixed Hilbert space of dimension n. In this case, the map $L_1 \to \mathrm{Rep}(A, H) : \nu \to \pi_\nu$ is ρ-measurable with respect to the point-weak* topology on $\mathrm{Rep}(A, H)$.

Proposition 10.3.9. *Let K be a compact nc convex set such that $\mathrm{A}(K)$ is separable. For $x \in K$ there is a nc probability measure λ on K such* *that*

$$\int_\lambda f\,d\lambda = \delta_x(f), \quad \textit{for} \quad f \in \mathrm{B}^\infty(K).$$

In particular, λ represents x. Moreover, if x is maximal then λ can be chosen so that it is supported on the extreme boundary ∂K.

Proof. We may assume that δ_x is cyclic. Let L_1 denote the (scalar) state space of $\mathrm{C}(K)$ and let $\mu \in L_1$ be a state with GNS representation $\pi_\mu = \delta_x$. Choose a maximal orthogonal measure ρ on L_1 with barycenter μ, so that in particular ρ is supported on ∂L_1. Then by the discussion preceding the proof, δ_x is unitarily equivalent to the direct integral

$$\delta_x \cong \int_{L_1}^\oplus \delta_{y_\nu}\,d\rho(\nu),$$

where for each $\nu \in L_1$, y_ν is a minimal representation for ν.

If x is maximal, then by Proposition 9.2.5, ν is pure and dilation maximal for ρ-almost every $\nu \in L_1$. In this case, Corollary 8.7.1 implies that δ_{y_ν} is an extreme point for ρ-almost every $\nu \in L_1$.

For $m \le \aleph_0$, let $C_m = \{\nu \in L_1 : y_\nu \in K_m\}$. Then by [15, Lemma 1, page 139], C_m is Borel. Let $\rho_m = \rho|_{C_m}$. Then from above, ρ_m is supported on $\partial L_1 \cap C_m$. Let

$$\pi_m = \int_{C_m}^\oplus \delta_{y_\nu}\,d\rho_m(\nu).$$

We can identify the range of π_m with a subalgebra of $L^\infty(C_m, \rho_m, \mathcal{M}_m)$ and by [42, Theorem 1.22.13], $L^\infty(C_m, \rho_m, \mathcal{M}_m)_* = L^1(C_m, \rho_m, (\mathcal{M}_m)_*)$.

Define $\iota_m : C_m \to K_m$ by $\iota_m(\nu) = y_\nu$. Then ι_m is the composition of the ρ_m-measurable map $C_m \to \mathrm{Rep}(\mathrm{C}(K), H_m) : \nu \to \delta_{y_\nu}$ with restriction to $\mathrm{A}(K)$. Since the latter map is continuous, ι_m is measurable.

Define a $\mathrm{CP}_{\mathrm{nor}}(\mathcal{M}_m, L^\infty(C_m, \rho_m, \mathcal{M}_m))$-valued Borel measure λ_m on K_m by

$$\lambda_m(E) = \int_{\iota_m^{-1}(E)}^\oplus \mathrm{id}_{\mathcal{M}_m}\,d\rho_m(\zeta), \quad E \in \mathrm{Bor}(K_m).$$

Then for $\alpha \in \mathcal{M}_m$ and $\tau \in L^1(C_m, \rho_m, (\mathcal{M}_m)_*)$,

$$(\lambda_m)_{\alpha,\tau}(E) = \langle \lambda_m(E)(\alpha), \tau \rangle = \int_{\iota_m^{-1}(E)} \langle \alpha, \tau(\nu) \rangle\,d\rho_m(\nu)$$

for $E \in \mathrm{Bor}(K_m)$. Hence $|\langle \lambda_m(E)(\alpha), \tau \rangle| \le \|\alpha\| \|\tau\|$, so we can define $r_{\alpha,\tau} \in L^1(C_m, \rho_m)$ by

$$r_{\alpha,\tau}(\nu) = \langle \alpha, \tau(\nu) \rangle, \quad \nu \in C_m.$$

Then

$$(\lambda_m)_{\alpha,\tau}(E) = \int_{\iota_m^{-1}(E)} r_{\alpha,\tau}(\nu)\, d\rho_m(\nu)$$

for $E \in \mathrm{Bor}(K_m)$. In particular, $(\lambda_m)_{\alpha,\tau}$ is absolutely continuous with respect to the scalar pushforward measure $\rho_m \circ \iota^{-1}$. Hence λ_m is absolutely continuous with respect to $\rho_m \circ \iota_m^{-1}$. Thus by Section 10.2, the integration map against λ_m has a unique extension to $\mathrm{BM}(K_m, \mathcal{M}_m)$ that is continuous with respect to the weak*-uniform convergence topology.

For $f \in \mathrm{B}^\infty(K)$ and $\tau \in L^1(C_m, \rho_m, (\mathcal{M}_m)_*)$,

$$\begin{aligned} \left\langle \int_{K_m} f\, d\lambda_m, \tau \right\rangle &= \int_{C_m} r_{f(y_\nu),\tau}(\nu)\, d\rho_m(\nu) \\ &= \int_{C_m} \langle f(y_\nu), \tau(\nu)\rangle\, d\rho_m(\nu) \\ &= \langle \pi_m(f), \tau\rangle. \end{aligned}$$

Hence

$$\int_{K_m} f\, d\lambda_m = \pi_m(f), \quad f \in \mathrm{B}^\infty(K).$$

Let $\lambda = (\lambda_m)_{m \le \aleph_0}$. Then λ is an admissible nc probability measure on K. Since $\delta_x \simeq \oplus \pi_m$, it follows from above that for $f \in \mathrm{B}^\infty(K)$,

$$\int_K f\, d\lambda = \bigoplus_{m \le \aleph_0} \int_{K_m} f\, d\lambda_m = \bigoplus_{m \le \aleph_0} \pi_m(f) \cong \delta_x(f).$$

If x is maximal, then Proposition 9.2.5 implies that ν is pure and dilation maximal for ρ-almost every $\nu \in L_1$. In this case, δ_{y_ν} is an extreme point for ρ-almost every $\nu \in L_1$. $\square$

The next result can be viewed as a kind of Riesz-Markov-Kakutani representation theorem for unital completely positive maps on the C*-algebra of nc continuous functions.

Theorem 10.3.10. *Let K be a compact nc convex set such that $\mathrm{A}(K)$ is separable and let $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ be a unital completely positive map. Then there is an admissible nc probability measure λ on K such that*

$$\mu(f) = \int_K f d\lambda, \quad \textit{for} \quad f \in \mathrm{B}^\infty(K).$$

Moreover, if μ is dilation maximal, then λ can be chosen so that it is supported on the extreme boundary ∂K.

Proof. Let $(x, \alpha) \in K_p \times \mathcal{M}_{p,n}$ be a minimal representation for μ. Applying Proposition 10.3.9, we obtain an nc probability measure $\sigma = (\sigma_m)_{m \leq \aleph_0}$ on K such that $\delta_x(f) = \int_K f \, d\sigma$ for all $f \in \mathrm{B}^\infty(K)$. If μ is dilation maximal, then x is maximal, in which case σ is supported on ∂K.

Define a nc probability measure $\lambda = (\lambda_m)_{m \leq \aleph_0}$ by $\lambda_m = \alpha^* \sigma_m \alpha$. Then

$$\int_K f \, d\lambda = \alpha^* \left(\int_K f \, d\lambda' \right) \alpha = \alpha^* f(x) \alpha = \mu(f)$$

for $f \in \mathrm{B}^\infty(K)$. If μ is dilation maximal, then from above σ is supported on ∂K in which case λ is supported on ∂K. □

Theorem 10.3.11 (Noncommutative integral representation theorem)**.** *Let K be a compact nc convex set such that $\mathrm{A}(K)$ is separable. Then for $x \in K$ there is an admissible nc probability measure λ on K that represents x and is supported on ∂K, i.e. such that*

$$a(x) = \int_K a \, d\lambda, \quad \textit{for} \quad a \in \mathrm{A}(K).$$

Proof. Suppose $x \in K_n$. Let $y \in K_p$ be a maximal dilation of x and let $\alpha \in \mathcal{M}_{p,n}$ be an isometry such that $x = \alpha^* y \alpha$. By Proposition 10.3.9, there is an admissible nc probability measure σ on K that represents y and is supported on ∂K. Define a nc probability measure $\lambda = (\lambda_m)_{m \leq \aleph_0}$ on K by $\lambda_m = \alpha^* \sigma_m \alpha$. Then λ represents x and is also supported on ∂K. □

References

[1] E. Alfsen, *Compact convex sets and boundary integrals*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. **57**, Springer-Verlag, New York-Heidelberg, 1971.

[2] W. Arveson, *Subalgebras of C*-algebras*, Acta Mathematica **123** (1969), 141–224.

[3] ———, *The noncommutative Choquet boundary*, Journal of the American Mathematical Society **21** (2008), 1065–1084.

[4] J. Bendat and S. Sherman, *Monotone and convex operator functions*, Transactions of the American Mathematical Society **79** (1955), 58–71.

[5] K. Bichteler, *A generalization to the non-separable case of Takesaki's duality theorem for C*-algebras*, Inventiones Mathematicae **9** (1969/1970), 89–98.

[6] L. Brown, *MASA's and certain type I closed faces of C*-algebras*, Group representations, ergodic theory, and mathematical physics: a tribute to George W. Mackey, 2008, pp. 69–98.

[7] N. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.

[8] E. Bishop and K. de Leeuw, *The representations of linear functionals by measures on sets of extreme points*, Annales de l'Institut Fourier (Grenoble) **9** (1959), 305–331.

[9] M.D. Choi and E. Effros, *Injectivity and operator spaces*, Journal of Functional Analysis **24** (1977), 156–209.

[10] G. Choquet, *Existence et unicité des représentations intégrales au moyen des points extrémaux dans les cônes convexes*, Séminaire Bourbaki, Vol. 4, 1995, pp. Exp. No. 139, 33–47.

[11] K.R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996.

[12] K.R. Davidson and M. Kennedy, *The Choquet boundary of an operator system*, Duke Mathematical Journal **164** (2015), 2989–3004.

[13] K. Davidson and M. Kennedy, *Choquet order and hyperrigidity for function systems*, Advances in Mathematics **385** (2021), Paper No. 107774, 30.

[14] K.R. Davidson and D. Pitts, *Invariant subspaces and hyper-reflexivity for free semigroup algebras*, Proceedings of the London Mathematical Society **78** (1999), 401–430.

[15] J. Dixmier, *C^*-algebras*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15.

[16] ———, *von Neumann algebras*, North-Holland Mathematical Library, vol. 27, North-Holland Publishing Co., Amsterdam-New York, 1981. With a preface by E. C. Lance, Translated from the second French edition by F. Jellett.

[17] M. Dritschel and S. McCullough, *Boundary representations for families of representations of operator algebras and spaces*, Journal of Operator Theory **53** (2005), 159–167.

[18] K.J. Dykema, *Simplicity and the stable rank of some free product C^*-algebras*, Transactions of the American Mathematical Society **351** (1999), 1–40.

[19] E. Effros and Z.J. Ruan, *Operator spacess*, Oxford University Press, 2000.

[20] ———, *Operator space tensor products and Hopf convolution algebras*, Journal of Operator Theory **50** (2003), 131–156.

[21] E. Effros and S.Winkler, *Matrix convexity: operator analogues of the bipolar and Hahn-Banach theorems*, Journal of Functional Analysis **144** (1997), 117–152.

[22] E. Evert, J.W. Helton, I. Klep, and S. McCullough, *Extreme points of matrix convex sets, free spectrahedra, and dilation theory*, Journal of Geometric Analysis **28** (2018), 1373–1408.

[23] D. Farenick, *Pure matrix states on operator systems*, Linear Algebra and its Applications **393** (2004), 149–173.

[24] D. Farenick and P. Morenz, *C^*-extreme points in the generalised state spaces of a C^*-algebra*, Transactions of the American Mathematical Society **349** (1997), 1725–1748.

[25] I. Fujimoto, *Decomposition of completely positive maps*, Journal of Operator Theory **32** (1994), 273–297.

[26] M. Hamana, *Injective envelopes of operator systems*, Publications of the Research Institute for Mathematical Sciences **15** (1979), 773–785.

[27] F. Hansen and G. Pedersen, *Jensen's operator inequality*, Bulletin of the London Mathematical Society **35** (2003), 553–564.

[28] A. Hopenwasser, R. Moore, and V. Paulsen, *C^*-extreme points*, Transactions of the American Mathematical Society **266** (1981), 291–307.

[29] R. Kadison, *A representation theory for commutative topological algebra*, Memoirs of the American Mathematical Society **No. 7** (1951).

[30] D. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, *Foundations of free noncommutative function theory*, Mathematical Surveys and Monographs, vol. 199, American Mathematical Society, Providence, RI, 2014.

[31] E. Kirchberg and S. Wassermann, *C*-algebras generated by operator systems*, Journal of Functional Analysis **155** (1998), 324–351.

[32] C. Kleski, *Korovkin-type properties for completely positive maps*, Illinois Journal of Mathematics **58** (2014), 1107–1116.

[33] T. Kriel, *An introduction to matrix convex sets and free spectrahedra*, arXiv preprint arXiv:1611.03103 (2018).

[34] J. Lukeš, J. Malý, Ivan I. Netuka, and J. Spurný, *Integral representation theory*, De Gruyter Studies in Mathematics, vol. 35, Walter de Gruyter & Co., Berlin, 2010. Applications to convexity, Banach spaces and potential theory.

[35] Bojan Magajna, *C*-convex sets and completely positive maps*, Integral Equations and Operator Theory **85** (2016), 37–62.

[36] P. Morenz, *The structure of C^*-convex sets*, Canadian Journal of Mathematics **46** (1994), 1007–1026.

[37] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002.

[38] V. Paulsen and M. Tomforde, *Vector spaces with an order unit*, Indiana University Mathematics Journal **58** (2009), 1319-1359.

[39] G. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs, vol. 14, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979.

[40] R. Phelps, *Lecture's on Choquet's Theorem,* 2nd edition, 2nd, Lecture Notes in Mathematics, vol. 1757, Springer-Verlag, Berlin, 2001.

[41] G. Pisier, *Introduction to operator space theory*, Vol. 294, Cambridge University Press, 2003.

[42] S. Sakai, *C*-algebras and W*-algebras*, Springer-Verlag, 2012.

[43] R. Smith, *Completely bounded maps between C^*-algebras*, Journal of the London Mathematical Society (2) **27** (1983), 157–166.

[44] M. Takesaki, *A duality in the representation theory of C*-algebras*, Annals of Mathematics (2) **85** (1967), 370–382.

[45] ______, *Theory of operator algebras I*, Springer-Verlag, 2003.

[46] J.L. Taylor, *A general framework for a multi-operator functional calculus*, Advances in Mathematics **9** (1972), 183–252.

[47] D. Voiculescu, *A non-commutative Weyl-von Neumann theorem*, Revue Roumaine de Mathématiques Pures et Appliquées **21** (1976), 97–113.

[48] C. Webster and S. Winkler, *The Krein-Milman theorem in operator convexity*, Transactions of the American Mathematical Society **351** (1999), 307–322.

[49] S. Winkler, *The non-commutative Legendre-Fenchel transform*, Mathematica Scandinavica **85** (1999), 30–48.

[50] G. Wittstock, *Ein operatorwertiger Hahn-Banach Satz*, Journal of Functional Analysis **40** (1981), 127–150.

[51] ———, *On matrix order and convexity*, Functional Analysis, Survey and Recent Results III, Math. Studies **90** (1984), 175-188.

Department of Pure Mathematics, University of Waterloo, Waterloo, ON, N2L 3G1, Canada

Email address: krdavids@uwaterloo.ca

Department of Pure Mathematics, University of Waterloo, Waterloo, ON, N2L 3G1, Canada

Email address: matt.kennedy@uwaterloo.ca