THE CHOQUET BOUNDARY OF
AN OPERATOR SYSTEM

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ABSTRACT. We show that every operator system (and hence every unital operator algebra) has sufficiently many boundary representations to generate the C*-envelope.

We solve a 45 year old problem of William Arveson that is central to his approach to non-commutative dilation theory. We show that every operator system and every unital operator algebra has sufficiently many boundary representations to completely norm it. Thus the C*-algebra generated by the image of the direct sum of these maps is the C*-envelope. This was a central problem left open in Arveson's seminal work [2] on dilation theory for arbitrary operator algebras. In the intervening years, the existence of the C*-envelope was established, but a general argument producing boundary representations has not been available.

Arveson [2, 3] reformulated the classical dilation theory of Sz. Nagy [14] so that it made sense for an arbitrary unital closed subalgebra $\mathcal{A}$ of a C*-algebra. A central theme was the use of completely positive and completely bounded maps. He proposed the existence of a family of special representations of $\mathcal{A}$, called *boundary representations*, which have unique completely positive extensions to $C^*(\mathcal{A})$ that are *irreducible* $*$-representations. The set of boundary representations is a noncommutative analogue of the Choquet boundary of a function algebra, i.e. the set of points with unique representing measures. Arveson proposed that there should be sufficiently many boundary representations, so that their direct sum recovers the norm on $\mathcal{M}_n(\mathcal{A})$ for all $n \geq 1$. In this case, he showed that the C*-algebra generated by this direct sum enjoys an important universal property, and provides a realization of the *C*-envelope* of $\mathcal{A}$.

Arveson was not able to prove the existence of boundary representations in general, although in various concrete cases they can be exhibited. Consequently, he was also unable to prove the existence of the C*-envelope. However, a decade later, Hamana [10] established the existence of the C*-envelope using other methods. His proof, via the construction of a minimal injective operator system containing $\mathcal{A} + \mathcal{A}^*$, did little to answer questions about boundary representations. Nevertheless, it did lead to a variety of cases in which the C*-envelope can be explicitly described. (We will not review the extensive literature on this topic.)

Nearly 20 years later, Muhly and Solel [12] showed that boundary representations (and more generally, *-representations that factor through the C*-envelope) have homological properties that distinguish them from other representations. However, since their argument relied on Hamana's theorem, it did not lead to a new construction of the C*-envelope.

About a decade ago, Dritschel and McCullough [7] came up with an exciting new proof of the existence of the C*-envelope. It was a bona fide dilation argument, building on ideas of Agler [1], and introduced the idea of *maximal dilations*. This direct dilation theory approach had the following important consequence: if you begin with a completely isometric representation of $\mathcal{A}$, and find a maximal dilation, then the C*-algebra generated by the image of this dilation is the C*-envelope. Consequently, there has been considerable interest in maximal dilations.

Arveson [4] revisited the problem of the existence of boundary representations using the ideas of Dritschel and McCullough. Using the disintegration theory of representations of C*-algebras, he established that, in the separable case, sufficiently many boundary representations exist. He expressed regret at the time that these delicate measure-theoretic methods appeared to be necessary—but reminded the audience he had been looking for *any* way of doing it for nearly 40 years[1].

It is therefore of interest that our proof is a direct dilation-theoretic argument, building on ideas from Arveson's original 1969 paper, and the more recent work of Dritschel and McCullough. In particular, our arguments do not require any disintegration theory nor do they require separability.

Arveson observed in his original work that a completely contractive unital map of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ extends uniquely to a self-adjoint map on the

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[1]At the Fields Institute in Toronto, July, 2007, in response to a question from Richard Kadison.

operator system $\mathcal{S} = \overline{\mathcal{A} + \mathcal{A}^*}$ which is unital and completely positive. Consequently, he formulated much of his theory around dilations of completely positive maps of operator systems. We also work in this more general setting.

Arveson developed many other important ideas in his seminal paper. One example which is particularly relevant to our work is the notion of a *pure* completely positive map. He showed that a completely positive map defined on a C*-algebra is pure if and only if the minimal Stinespring dilation is irreducible. For completely positive maps on general operator systems, this is a necessary but not sufficient condition.

We begin our approach by showing that every pure unital completely positive map on an operator system $\mathcal{S}$ has a pure maximal dilation. This dilation has a unique extension to $\mathrm{C}^*(\mathcal{S})$ which is an irreducible *-representation that necessarily factors through the C*-envelope. Some results of Farenick [8, 9] are then used to show that there are sufficiently many finite dimensional pure u.c.p. maps (a.k.a *matrix states*) to completely norm $\mathcal{S}$. Dilating these matrix states to boundary representations then yields a sufficient family of boundary representations.

Craig Kleski [11] has some closely related results. In the separable case, he uses Arveson's measure theoretic approach to show that pure states have dilations to boundary representations. Also in connection with the second part of our paper, he shows that the pure states on $\mathcal{S}$ norm it, and in the separable case, the supremum is attained. He does not show that pure states completely norm $\mathcal{S}$, which we need.

1. BACKGROUND

We refer the reader to Paulsen's book [13] for the background needed for this paper. For a nice treatment of maximal dilations (a là Dritschel-McCullough), see section 2 of [4]. We briefly recall the central notions that we require.

An *operator system* $\mathcal{S}$ is a unital norm-closed self-adjoint subspace of a C*-algebra. We always view $\mathcal{S}$ as being contained in the C*-algebra that it generates, $\mathrm{C}^*(\mathcal{S})$. Sometimes these are called concrete operator systems. Choi and Effros [6] gave an abstract axiomatic definition of an operator system, and established a representation theorem showing that they can all be represented as concrete operator systems.

A *unital operator algebra* $\mathcal{A}$ is a closed unital subalgebra of a C*-algebra. Again, there is a definition of an abstract operator algebra, and a corresponding representation theorem due to Blecher, Ruan and Sinclair [5] showing that they can all be represented (completely isometrically) as subalgebras of C*-algebras. So our theory applies to

both abstract operator algebras and abstract operator systems. For our purposes, we will assume that $\mathcal{S}$ or $\mathcal{A}$ is already sitting in a C*-algebra.

A map φ from any subspace $\mathcal{M}$ of a C*-algebra $\mathfrak{A}$ into a C*-algebra $\mathfrak{B}$ determines a family of maps $\varphi_n : \mathcal{M}_n(\mathcal{M}) \to \mathcal{M}_n(\mathfrak{B})$ given by $\varphi_n([a_{ij}]) = [\varphi(a_{ij})]$. Say that φ is *completely bounded* if

$$\|\varphi\|_{cb} = \sup_{n \geq 1} \|\varphi_n\| < \infty.$$

Say that φ is *completely contractive* (c.c.) if $\|\varphi\|_{cb} \leq 1$. If the domain of φ is an operator system $\mathcal{S}$, say that φ is *completely positive* (c.p.) if φ_n is positive for all $n \geq 1$; and say that φ is *unital completely positive* (u.c.p.) if $\varphi(1) = 1$. Since $\|\varphi\|_{cb} = \|\varphi(1)\|$ for c.p. maps, we see that u.c.p. maps are always completely contractive.

As mentioned in the introduction, every *unital* completely contractive map φ of a *unital* operator space $\mathcal{M}$ into a C*-algebra has a unique self-adjoint extension to $\mathcal{S} = \overline{\mathcal{M} + \mathcal{M}^*}$ given by

$$\tilde{\varphi}(a + b^*) = \varphi(a) + \varphi(b)^*.$$

Moreover, this map $\tilde{\varphi}$ is completely positive.

A u.c.p. map $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ (or a c.c. representation of an operator algebra $\mathcal{A}$) has the *unique extension property* if it has a unique u.c.p. extension to $C^*(\mathcal{A})$ which is a $*$-representation. If, in addition, the $*$-representation is irreducible, it is called a *boundary representation*. When $\mathcal{A}$ is a function algebra contained in $C(X)$, the irreducible $*$-representations are just point evaluations. The restriction of a point evaluation to $\mathcal{A}$ has the unique extension property if it has a unique representing measure (namely, the point mass at the point itself).

A *dilation* of a c.c. unital representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of an operator algebra $\mathcal{A}$ is a representation $\sigma : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ where $\mathcal{K}$ is a Hilbert space containing $\mathcal{H}$ such that $P_{\mathcal{H}}\sigma(a)|_{\mathcal{H}} = \rho(a)$ for $a \in \mathcal{A}$. Similarly a *dilation* of a u.c.p. map $\varphi : S \to \mathcal{B}(\mathcal{H})$ of an operator system $\mathcal{S}$ is a u.c.p. map $\psi : \mathcal{S} \to \mathcal{B}(\mathcal{K})$ where $\mathcal{K}$ is a Hilbert space containing $\mathcal{H}$ such that $P_{\mathcal{H}}\psi(s)|_{\mathcal{H}} = \varphi(s)$ for $s \in \mathcal{S}$. We will write $\varphi \prec \psi$ or $\psi \succ \varphi$ to denote that ψ dilates φ. The map (ρ or φ) is called *maximal* if every dilation (of ρ or φ) is obtained by attaching a direct summand (i.e. $\psi \succ \varphi$ implies $\psi = \varphi \oplus \psi'$ for some ψ').

As noted above, a representation ρ of an operator algebra $\mathcal{A}$ extends to a unique u.c.p. map $\tilde{\rho}$ on the operator system $\mathcal{S} = \overline{\mathcal{A} + \mathcal{A}^*}$. It is easy to see that a dilation σ of ρ extends to a dilation $\tilde{\sigma}$ of $\tilde{\rho}$. However, this does not work in reverse. Indeed, a dilation of $\tilde{\rho}$ need not be multiplicative on $\mathcal{A}$, in which case it is not the extension of a representation.

Dritschel and McCullough [7] show that c.c. representations of an operator algebra $\mathcal{A}$ always have maximal dilations. Arveson [4] has a somewhat nicer proof, along similar lines, which is valid for u.c.p. maps on an operator system $\mathcal{S}$. Dritschel and McCullough show that maximal dilations extend to $*$-representations of $C^*(\mathcal{A})$. Arveson [4] shows that being a maximal dilation of a u.c.p. map on $\mathcal{S}$ is equivalent to having the unique extension property. Thus a maximal dilation of a u.c.p. map is multiplicative. This implies that if ρ is a c.c. representation of $\mathcal{A}$, and ψ is a maximal dilation of $\tilde{\rho}$, then $\psi|_{\mathcal{A}}$ is a maximal dilation of ρ. So establishing results for operator systems recovers the results for operator algebras at the same time.

The *C*-envelope* of an operator system $\mathcal{S}$ consists of a C*-algebra $\mathfrak{A} =: C^*_{\text{env}}(\mathcal{S})$ and a completely isometric unital imbedding $\iota : \mathcal{S} \to \mathfrak{A}$ such that $\mathfrak{A} = C^*(\iota(\mathcal{S}))$, with the following universal property: whenever $j : \mathcal{S} \to \mathfrak{B} = C^*(j(\mathcal{S}))$ is a unital completely isometric map, then there is a $*$-homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ such that $\iota = \pi j$. Hamana [10] proved that the C*-envelope always exists. Dritschel and McCullough [7] gave a new proof by showing that any maximal u.c.p. map on $\mathcal{S}$ extends to a $*$-representation of $C^*(\mathcal{S})$ which factors through $C^*_{\text{env}}(\mathcal{S})$. In particular, when the original map is completely isometric, the maximal dilation yields a $*$-representation onto the C*-envelope.

Arveson [2] calls a c.p. map φ *pure* if the only c.p. maps satisfying $0 \leq \psi \leq \varphi$ are scalar multiples of φ. When φ is defined on a C*-algebra $\mathfrak{A}$, it has a unique minimal Stinespring dilation $\varphi(a) = V^*\pi(a)V$, where π is a $*$-representation of $\mathfrak{A}$ on $\mathcal{K}$ and $V \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Arveson shows that the intermediate c.p. maps ψ are precisely those maps of the form $\psi(a) = V^*T\pi(a)V$, for $T \in \pi(\mathfrak{A})'$ with $0 \leq T \leq I$. Moreover, this is a bijective correspondence. Thus, a c.p. map on $\mathfrak{A}$ is pure if and only if the minimal Stinespring dilation is irreducible. For a c.p. map φ on an operator system $\mathcal{S}$, the minimal Stinespring dilation is not unique. However, φ is not pure if any minimal Stinespring representation is reducible.

We will observe that if φ is maximal and pure, then it extends to an irreducible $*$-representation of $C^*(\mathcal{S})$. Our goal will be to establish that every pure u.c.p. map from $\mathcal{S}$ into $\mathcal{B}(\mathcal{H})$ has a pure maximal dilation which is a boundary representation. This will be accomplished in Section 2. In Section 3, we gather the details needed to show that there are enough boundary representations to completely norm $\mathcal{S}$, so that their direct sum provides a completely isometric maximal representation of $\mathcal{S}$. This relies on results of Farenick [8, 9] on pure matrix states of operator systems, based on the Krein-Milman type theorem for matrix convex sets due to Webster and Winkler [15]. Altogether, our results

establish that there are sufficiently many boundary representations to construct the C*-envelope.

2. Extending pure maps

First a simple observation mentioned in the preceding section.

Lemma 2.1. *Every pure maximal u.c.p. map $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ extends to an irreducible $*$-representation of $\mathrm{C}^*(\mathcal{S})$, and hence is a boundary representation.*

Proof. Arveson [4] showed that maximal u.c.p. maps have the unique extension property. So φ extends uniquely to a $*$-representation π of $\mathrm{C}^*(\mathcal{S})$. It remains to show that π is irreducible. If π is not irreducible, then there is a proper projection P commuting with $\pi(\mathrm{C}^*(\mathcal{S}))$. Thus $\psi(s) = P\varphi(s)$ is a c.p. map such that $0 \le \psi \le \varphi$. However, $\psi(1) = P$ is not a scalar multiple of $I = \varphi(1)$. So φ is not pure, contrary to our hypothesis. Hence π is irreducible, and therefore is a boundary representation. ∎

The proof in [4] that maximal dilations exist uses the following concept. A u.c.p. map φ is *maximal at* (s_0, x_0) for $s_0 \in \mathcal{S}$ and $x_0 \in \mathcal{H}$ if whenever $\psi \succ \varphi$, we have $\psi(s_0)x_0 = \varphi(s_0)x_0$. It is clear that this is true precisely when $\|\psi(s_0)x_0\| = \|\varphi(s_0)x_0\|$ for all $\psi \succ \varphi$.

The BW topology on $\mathcal{B}(\mathcal{S}, \mathcal{B}(\mathcal{H}))$ is the point-weak-$*$ topology. An easy application of the Banach-Alaoglu Theorem shows that the unit ball is compact since, in the BW topology, it embeds as a closed subset of the product of closed balls of $\mathcal{B}(\mathcal{H})$ with the weak-$*$ topology. In fact, $\mathcal{B}(\mathcal{S}, \mathcal{B}(\mathcal{H}))$ is a dual space, with the BW topology coinciding with the weak-$*$ topology on bounded sets [13, Lemma 7.1]; but we do not need this fact. The c.p. and u.c.p. maps are closed in this topology [2]; and thus the set of u.c.p. maps is BW-compact.

Lemma 2.2. *Let $\mathcal{S}$ be an operator system, and let $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be a u.c.p. map. Given $s_0 \in \mathcal{S}$ and $x_0 \in \mathcal{H}$, there is a u.c.p. dilation of φ to a map $\psi : \mathcal{S} \to \mathcal{B}(\mathcal{H} \oplus \mathbb{C})$ which is maximal at (s_0, x_0), i.e.*

$$\|\psi(s_0)x_0\| = \sup\{\|\rho(s_0)x_0\| : \rho \succ \varphi\}.$$

Proof. First note that if $\rho \succ \varphi$, then the compression of ρ to the Hilbert space $\mathrm{span}\{\mathcal{H}, \rho(s)x\}$ yields a u.c.p. dilation ρ' of φ into $\mathcal{H} \oplus \mathbb{C}$ with $\|\rho'(s_0)x_0\| = \|\rho(s_0)x_0\|$. So the supremum is the same if we consider only u.c.p. maps into $\mathcal{B}(\mathcal{H} \oplus \mathbb{C})$. The set of all such maps is compact in the BW topology. Hence a routine compactness argument yields the desired map ψ. ∎

This next lemma is motivated by Farenick's result [8, Theorem B] which states that a matrix state is pure if and only if it is a matrix extreme point. However our arguments will work in Hilbert spaces of arbitrary dimension. The goal is to construct a one dimensional dilation of a pure u.c.p. map to a u.c.p. map which is maximal at (s_0, x_0) while conserving purity.

Lemma 2.3. *Let $\mathcal{S}$ be an operator system, and let $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be a pure u.c.p. map. Given $s_0 \in \mathcal{S}$ and $x_0 \in \mathcal{H}$ at which φ is not maximal, there is a pure u.c.p. dilation $\psi : \mathcal{S} \to \mathcal{B}(\mathcal{H} \oplus \mathbb{C})$ which is maximal at (s_0, x_0).*

Proof. Let

$$L = \sup\{\|\rho(s_0)x_0\| : \rho \succ \varphi\} \quad \text{and} \quad \eta = (L^2 - \|\varphi(s_0)x_0\|^2)^{1/2}.$$

Let

$$X = \{\psi \in \mathrm{UCP}(\mathcal{S}, \mathcal{B}(\mathcal{H} \oplus \mathbb{C})) : \psi \succ \varphi \text{ and } \psi(s_0)x_0 = \varphi(s_0)x_0 \oplus \eta\}.$$

Conjugating the map obtained in the previous lemma by a unitary of the form $I \oplus \zeta$ implies that this is a non-empty BW-compact convex set. Let ψ_0 be an extreme point of X. Note that X is a face of

$$Y = \{\psi \in \mathrm{UCP}(\mathcal{S}, \mathcal{B}(\mathcal{H} \oplus \mathbb{C})) : \psi \succ \varphi\}.$$

Hence ψ_0 is also an extreme point of Y.

We claim that ψ_0 is pure. To this end, suppose that ψ_1 is a c.p. map into $\mathcal{H} \oplus \mathbb{C}$ such that $0 \le \psi_1 \le \psi_0$. Set $\psi_2 = \psi_0 - \psi_1$. To avoid the possibility that $\psi_i(1)$ may not be invertible, take a small $\varepsilon > 0$ and use

$$\psi_i' = (1 - 2\varepsilon)\psi_i + \varepsilon\psi_0 \quad \text{for } i = 1, 2.$$

Then $\psi_0 = \psi_1' + \psi_2'$ and $\psi_i'(1) =: Q_i \ge \varepsilon I$. Thus Q_i is invertible, and $Q_1 + Q_2 = \psi_0(1) = I$. If we show that ψ_1' is a scalar multiple of ψ_0, then the same follows for ψ_1.

Observe that $P_{\mathcal{H}}\psi_i'(\cdot)|_{\mathcal{H}} \le \varphi$. By purity of φ, there are positive scalars λ_i so that $P_{\mathcal{H}}\psi_i'(\cdot)|_{\mathcal{H}} = \lambda_i\varphi$. Clearly $\lambda_1 + \lambda_2 = 1$ and $\lambda_i \ge \varepsilon$. Thus writing Q_i as a matrix with respect to the decomposition $\mathcal{H} \oplus \mathbb{C}$, there is a vector $x_i \in \mathcal{H}$ and scalar α_i so that

$$Q_i = \begin{bmatrix} \lambda_i & \lambda_i^{1/2}x_i \\ \lambda_i^{1/2}x_i^* & \alpha_i \end{bmatrix} = \begin{bmatrix} \lambda_i^{1/2} & 0 \\ x_i^* & \beta_i \end{bmatrix} \begin{bmatrix} \lambda_i^{1/2} & x_i \\ 0 & \beta_i \end{bmatrix}.$$

The factorization is possible by the Cholesky algorithm, where the positivity and invertibility of Q_i guarantee that

$$\beta_i = (\alpha_i - \|x_i\|^2)^{1/2} > 0.$$

Since $Q_1 + Q_2 = I$, we obtain

$$(1) \qquad \lambda_1^{1/2} x_1 + \lambda_2^{1/2} x_2 = 0$$

and

$$(2) \qquad \|x_1\|^2 + \beta_1^2 + \|x_2\|^2 + \beta_2^2 = \alpha_1 + \alpha_2 = 1.$$

Let

$$\gamma_i = \begin{bmatrix} \lambda_i^{1/2} & x_i \\ 0 & \beta_i \end{bmatrix}; \quad \text{then} \quad \gamma_i^{-1} = \begin{bmatrix} \lambda_i^{-1/2} & -\lambda_i^{-1/2} \beta_i^{-1} x_i \\ 0 & \beta_i^{-1} \end{bmatrix}.$$

Also,

$$\gamma_1^* \gamma_1 + \gamma_2^* \gamma_2 = Q_1 + Q_2 = I.$$

Define u.c.p. maps

$$\tau_i(\cdot) = \gamma_i^{-1*} \psi_i'(\cdot) \gamma_i^{-1}.$$

Then

$$\psi_i'(\cdot) = \gamma_i^* \tau_i(\cdot) \gamma_i,$$

and from above,

$$P_{\mathcal{H}} \tau_i(\cdot)|_{\mathcal{H}} = \lambda_i^{-1} P_{\mathcal{H}} \psi_i'(\cdot)|_{\mathcal{H}} = \varphi.$$

Hence τ_i is a dilation of φ.

Since τ_i is unital and positive, there is a state f_i on $\mathcal{S}$ and a linear map T_i in $\mathcal{B}(\mathcal{S}, \mathcal{H})$ so that

$$\tau_i(s) = \begin{bmatrix} \varphi(s) & T_i(s) \\ T_i(s^*)^* & f_i(s) \end{bmatrix}.$$

Observe that $T_i(1) = 0$; and since τ_i is a dilation of φ, we have, by the definition of η,

$$|T_i(s_0^*)^* x_0| \leq \eta.$$

Therefore,

$$\psi_i'(s) = \gamma_i^* \tau_i(\cdot) \gamma_i$$
$$= \begin{bmatrix} \lambda_i \varphi(s) & \lambda_i^{1/2}\big(\varphi(s)x_i + \beta_i T_i(s)\big) \\ \lambda_i^{1/2}\big(\varphi(s^*)x_i + \beta_i T_i(s^*)\big)^* & x_i \varphi(s) x_i + 2\beta_i \operatorname{Re}(x_i^* T_i(s)) + \beta_i^2 f_i(s) \end{bmatrix}.$$

Consideration of $\eta = P_{\mathbb{C}}\psi_0(s_0)x_0$ yields

$$
\begin{aligned}
\eta^2 &= \|P_{\mathbb{C}}\psi_0(s_0)x_0\|^2 \\
&= \left\|P_{\mathbb{C}}\big(\psi_1'(s_0)x_0 + \psi_2'(s_0)x_0\big)\right\|^2 \\
&= \left|\big(\varphi(s_0)(\lambda_1^{1/2}x_1 + \lambda_2^{1/2}x_2)\big)^*x_0 + \lambda_1^{1/2}\beta_1 T_1(s_0^*)^*x_0 + \lambda_2^{1/2}\beta_2 T_2(s_0^*)^*x_0\right|^2 \\
&= \left|\lambda_1^{1/2}\beta_1 T_1(s_0^*)^*x_0 + \lambda_2^{1/2}\beta_2 T_2(s_0^*)^*x_0\right|^2 \\
&\leq (\lambda_1 + \lambda_2)\big(\beta_1^2 |T_1(s_0^*)^*x_0|^2 + \beta_2^2 |T_2(s_0^*)^*x_0|^2\big) \\
&\leq (\beta_1^2 + \beta_2^2)\eta^2 \\
&\leq (\alpha_1 + \alpha_2)\eta^2 \\
&= \eta^2,
\end{aligned}
$$

where we have used (1) and (2), and the Cauchy-Schwarz inequality.

Since this is an equality, the last inequality yields that $\beta_i^2 = \alpha_i$, and hence $x_i = 0$ for $i = 1, 2$. The second inequality yields that $|T_i(s_0^*)^*x_0|^2 = \eta^2$, so that we can write $T_i(s_0^*)^*x_0 = \mu_i\eta$, where $|\mu_i| = 1$. Moreover, since we have equality in the use of the Cauchy-Schwarz inequality,

$$
\begin{aligned}
(\lambda_1^{1/2}, \lambda_2^{1/2}) &= \nu(\beta_1 T_1(s_0^*)^*x_0, \beta_2 T_2(s_0^*)^*x_0) \\
&= \nu(\beta_1\mu_1\eta, \beta_2\mu_2\eta)
\end{aligned}
$$

for some ν. Since both λ_i and both β_i are positive, this implies $\mu_1 = \mu_2$, which gives

$$
\begin{aligned}
\eta &= P_{\mathbb{C}}\psi_0(s_0)x_0 \\
&= \lambda_1 T_1(s_0^*)^*x_0 + \lambda_2 T_2(s_0^*)^*x_0 \\
&= (\lambda_1 + \lambda_2)\mu_1\eta \\
&= \mu_1\eta.
\end{aligned}
$$

Thus $\mu_1 = \mu_2 = 1$, giving $T_i(s_0^*)^*x_0 = \eta$. Hence both τ_i belong to X.

The last two inequalities above also yield $\beta_1^2 + \beta_2^2 = 1$. Writing $\lambda_i^{1/2} = \nu\eta\beta_i$, this implies

$$
1 = \lambda_1 + \lambda_2 = \nu^2\eta^2(\beta_1^2 + \beta_2^2) = \nu^2\eta^2.
$$

Since λ_i, η, and β_i are all positive, ν is also positive. Hence $\nu = \eta^{-1}$, and $\lambda_i^{1/2} = \beta_i$. Therefore, $\gamma_i = \lambda_i^{1/2}I$ is a scalar matrix.

From above,

$$
\psi_i'(\cdot) = \gamma_i^*\tau_i(\cdot)\gamma_i = \lambda_i\tau_i(\cdot).
$$

Therefore $\psi_0 = \lambda_1 \tau_1 + \lambda_2 \tau_2$ is a convex combination of the τ_i. Since ψ_0 is an extreme point of X, we obtain that $\tau_i = \psi_0$. Thus $\psi_1' = \lambda_1 \psi_0$. So ψ_0 is pure. $\blacksquare$

It is easy to see that φ is maximal if and only if it is maximal at every (s,x) for $s \in \mathcal{S}$ and $x \in \mathcal{H}$ [4]. We will establish the existence of pure maximal dilations by a transfinite induction. In the separable case, a simple induction is possible.

Theorem 2.4. *Let $\mathcal{S}$ be an operator system, and let $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be a pure u.c.p. map. Then φ has a pure maximal dilation ψ. Therefore ψ extends to a $*$-representation of $\mathrm{C}^*(\mathcal{S})$ which is a boundary representation of $\mathcal{S}$.*

Proof. Enumerate a dense subset of $b_1(\mathcal{S}) \times b_1(\mathcal{H})$, the product of unit balls of $\mathcal{S}$ and $\mathcal{H}$, using an ordinal Λ, as

$$\{(s_\lambda, x_\lambda) : \lambda < \Lambda, \text{ such that } \lambda \text{ is a successor ordinal}\}.$$

We will use transfinite induction to construct a pure u.c.p. dilation of φ which is maximal at each (s_λ, x_λ).

Start with $\varphi_0 := \varphi$. At each successor ordinal $1+\lambda$ for $\lambda \geq 0$, we have a pure u.c.p. dilation φ_λ of φ into a Hilbert space $\mathcal{H}_\lambda$ which is maximal at (s_α, x_α) for all $\alpha \leq \lambda$. If φ_λ is already maximal at $(s_{1+\lambda}, x_{1+\lambda})$, set $\varphi_{1+\lambda} = \varphi_\lambda$. Otherwise, use Lemma 2.3 to obtain a 1-dimensional dilation of φ_λ to a pure u.c.p. map $\varphi_{1+\lambda}$ into a Hilbert space $\mathcal{H}_{\lambda+1}$ which is maximal at $(s_{1+\lambda}, x_{1+\lambda})$.

At each limit ordinal μ, for each $\alpha < \mu$, we have a Hilbert space $\mathcal{H}_\alpha$ and pure u.c.p. dilation φ_α which is maximal at (s_λ, x_λ) for each successor ordinal $\lambda \leq \alpha$. Moreover if $\lambda < \alpha$, then $\mathcal{H}_\lambda \subset \mathcal{H}_\alpha$ and $\varphi_\lambda \prec \varphi_\alpha$. Let $\mathcal{H}_\mu$ be the direct limit of the Hilbert spaces $\mathcal{H}_\alpha$, which we can consider as the completion of the union $\bigcup_{\alpha<\mu} \mathcal{H}_\alpha$. Then we define φ_μ so that the compression of φ_μ to $\mathcal{H}_\alpha$ is φ_α for all $\alpha < \mu$. Clearly φ_μ is a u.c.p. map which is a dilation of φ_α for each $\alpha < \mu$. To see that φ_μ is pure, suppose that $0 \leq \tau \leq \varphi_\mu$. The compression of τ to $\mathcal{H}_\alpha$ satisfies

$$0 \leq P_{\mathcal{H}_\alpha} \tau(\cdot)|_{\mathcal{H}_\alpha} \leq \varphi_\alpha.$$

By purity, there is a scalar t so that $P_{\mathcal{H}_\alpha} \tau(\cdot)|_{\mathcal{H}_\alpha} = t\varphi_\alpha$. Moreover $tI = P_{\mathcal{H}_\alpha} \tau(1)|_{\mathcal{H}_\alpha}$; so t is independent of α. By continuity, $\tau = t\varphi_\mu$ and hence φ_μ is pure.

The result at the end of this induction is a pure u.c.p. dilation ψ_1 of φ acting on a Hilbert space $\mathcal{K}_1$, which by continuity is maximal at (s,x) for every $s \in \mathcal{S}$ and $x \in \mathcal{H}$. Repeat this procedure recursively to obtain a sequence of pure u.c.p. dilations ψ_k acting on $\mathcal{K}_k$ which are

maximal at (s, x) for every $s \in \mathcal{S}$ and $x \in \mathcal{K}_{k-1}$. The direct limit of this sequence is a pure u.c.p. dilation ψ_∞ acting on $\mathcal{K}_\infty$ which is maximal at (s, x) for every $s \in \mathcal{S}$ and $x \in \mathcal{K}_\infty$; and thus is maximal. Arguing as in the limit ordinal case above, ψ_∞ is pure. Finally, by Lemma 2.1, ψ_∞ extends to an irreducible $*$-representation of $\mathrm{C}^*(\mathcal{S})$ which is a boundary representation of $\mathcal{S}$. ∎

Remark 2.5. If $\mathcal{H}$ is finite dimensional and $\mathcal{S}$ is separable, the intermediate dilations of the previous proof can be kept finite dimensional, so that only the final limit dilation is infinite dimensional. This is accomplished by doing the dilations at the k-th stage only for a finite set of pairs (s_i, x_j^k), where $1 \le i \le N_k$ and $\{x_j^k\}$ forms a finite ε_k-net in the unit sphere of $\mathcal{H}_k$. Here, N_k and ε_k are chosen such that $\lim_k N_k = \infty$ and $\lim_{k \to \infty} \varepsilon_k = 0$. In the limit, one still obtains a maximal dilation.

3. Sufficiency of boundary representations

Now that we have a method for constructing boundary representations, we show that there are enough of them to yield the C*-envelope. A *matrix state* is a u.c.p. map of $\mathcal{S}$ into the $k \times k$ matrices $\mathcal{M}_k$. Let $S_k(\mathcal{S}) = \mathrm{UCP}(\mathcal{S}, \mathcal{M}_k)$ be the set of all u.c.p. maps from $\mathcal{S}$ into $\mathcal{M}_k$. The set of all matrix states is $S(\mathcal{S}) = (S_k(\mathcal{S}))_{k \ge 1}$. We begin with an easy observation.

Lemma 3.1. *The set of all matrix states completely norms $\mathcal{S}$; i.e. for every $S \in \mathcal{M}_n(\mathcal{S})$,*

$$\|S\| = \sup\{\|\varphi_n(S)\| : \varphi \in S(\mathcal{S})\}.$$

Proof. Let π be a faithful $*$-representation of $\mathrm{C}^*(\mathcal{S})$ on $\mathcal{H}$. Then π is completely isometric. Hence the set of compressions of π to all finite dimensional subspaces of $\mathcal{H}$ also completely norms $\mathcal{S}$. ∎

Now the issue is to replace the set of all matrix states with the set of pure matrix states. For this, we need the notions of matrix convexity and matrix extreme points.

A *matrix convex set* in a vector space V is a collection $K = (K_k)$ of subsets $K_k \subset \mathcal{M}_k(V)$ such that K_k contains all elements of the form

$$\sum_{i=1}^p \gamma_i^* v_i \gamma_i \quad \text{for all } v_i \in K_{k_i}, \ \gamma_i \in \mathcal{M}_{k_i, k}, \text{ such that } \sum_{i=1}^p \gamma_i^* \gamma_i = I_k.$$

If $S = (S_k)$ is a collection of subsets of $\mathcal{M}_k(V)$, then there is a smallest closed matrix convex set generated by S called $\overline{\mathrm{conv}}(S)$.

A matrix convex combination $v = \sum_{i=1}^p \gamma_i^* v_i \gamma_i$ is *proper* if each γ_i has a right inverse belonging to $\mathcal{M}_{k, k_i}$, i.e., if γ_i is surjective. In particular,

we must have that $k \geq k_i$. A point $v \in K_k$ is a *matrix extreme point* if whenever v is a proper matrix convex combination of $v_i \in K_{k_i}$ for $1 \leq i \leq p$, then each $k_i = k$ and $v_i = u_i v u_i^*$ for some unitary $u_i \in \mathcal{M}_k$. In particular, at level $k = 1$, matrix extreme points are just extreme points. Webster and Winkler [15, Theorem 4.3] prove a Krein-Milman Theorem for matrix convex sets stating that a compact matrix convex set is the closed matrix convex hull of its matrix extreme points.

The matrix state space $S(\mathcal{S}) = (S_k(\mathcal{S}))_{k \geq 1}$ of an operator system $\mathcal{S}$ forms a BW-compact matrix convex set. A result of Farenick [8, Theorem B] shows that a matrix state is pure if and only if it is a matrix extreme point of $S(\mathcal{S})$. Thus every matrix state is in the BW-closure of the matrix convex combinations of the pure matrix states. In [9], Farenick provides a simpler proof of the Webster-Winkler Theorem in the context of matrix states of an operator system. (Another result of Webster and Winkler shows that this is actually the general situation.)

Lemma 3.2. *The set of all pure matrix states completely norms $\mathcal{S}$; i.e. for every $S \in \mathcal{M}_n(\mathcal{S})$,*

$$\|S\| = \sup\{\|\varphi_n(S)\| : \varphi \in S(\mathcal{S}), \ \varphi \ pure\}.$$

Proof. It suffices to show that the supremum over all matrix convex combinations of pure matrix states is no larger than the supremum over pure matrix states. This inequality will then extend to the BW-closure by continuity. Thus by the remarks preceding the lemma, this will be the supremum over all matrix states. Hence the result follows from Lemma 3.1.

Suppose that $\varphi \in S_k(\mathcal{S})$ is a matrix convex combination of pure states $\varphi_i \in S_{k_i}(\mathcal{S})$. So there are linear maps $\gamma_i \in \mathcal{M}_{k_i,k}$ such that

$$\varphi = \sum_{i=1}^{p} \gamma_i^* \varphi_i \gamma_i \quad \text{and} \quad \sum_{i=1}^{p} \gamma_i^* \gamma_i = I_k.$$

Then $\psi := \varphi_1 \oplus \cdots \oplus \varphi_p$ belongs to $S_K(\mathcal{S})$ where $K = \sum_{i=1}^{p} k_i$. We can factor φ as

$$\varphi = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_p \end{bmatrix}^* \begin{bmatrix} \varphi_1 & 0 & \dots & 0 \\ 0 & \varphi_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \varphi_p \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_p \end{bmatrix} = \gamma^* \psi \gamma,$$

where $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_p)^T$. Observe that

$$\gamma^* \gamma = \sum_{i=1}^{p} \gamma_i^* \gamma_i = I_k.$$

Hence γ is an isometry.

Let $S \in \mathcal{M}_n(\mathcal{S})$. Then

$$\|\varphi_n(S)\| = \|(\gamma \otimes I_n)^* \psi_n(S)(\gamma \otimes I_n)\|$$

$$\leq \left\| \begin{bmatrix} (\varphi_1)_n(S) & 0 & \ldots & 0 \\ 0 & (\varphi_2)_n(S) & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & (\varphi_p)_n(S) \end{bmatrix} \right\|$$

$$= \max_{1 \leq i \leq p} \|(\varphi_i)_n(S)\|.$$

The right hand side is a maximum over pure states, as desired. ∎

We can now combine all of the ingredients to obtain the main result.

Theorem 3.3. *Let $\mathcal{S}$ be an operator system. Then $\mathcal{S}$ is completely normed by its boundary representations. Hence the direct sum of all boundary representations yields a completely isometric map $\iota : \mathcal{S} \to \mathcal{B}(\mathcal{K})$, so that $(\iota, \mathrm{C}^*(\iota(\mathcal{S})))$ is the C*-envelope of $\mathcal{S}$. (Here, the direct sum is taken over a set of fixed Hilbert spaces of dimensions ranging from 1 up to $\aleph_0 \dim \mathcal{S}$.)*

Proof. By Lemma 3.2, the pure matrix states completely norm $\mathcal{S}$. By Theorem 2.4, each of these pure matrix states can be dilated to a boundary representation of $\mathcal{S}$. Clearly this implies that the collection of all boundary representations completely norms $\mathcal{S}$. To get a set, we need to take the precaution to fix a set of Hilbert spaces of the proper dimensions to accomodate irreducible representations of $\mathrm{C}^*(\mathcal{S})$. This dimension is bounded above by $\aleph_0 \dim \mathcal{S}$. The direct sum π of this set of boundary representations is then completely isometric on $\mathcal{S}$. Each boundary representation is maximal, and thus any dilation of π must leave each boundary representation as a direct summand. Hence π is a direct summand of its dilation, and therefore is a maximal u.c.p. map. By the arguments of Dritschel and McCullough [7] or Arveson [4], the C*-envelope of $\mathcal{S}$ is the C*-algebra generated by this representation. ∎

Earlier remarks yield the corresponding result for operator algebras.

Corollary 3.4. *Let $\mathcal{A}$ be a unital operator algebra. Then $\mathcal{A}$ is completely normed by its boundary representations. Hence the direct sum of all boundary representations yields a completely isometric map $\iota : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, so that $(\iota, \mathrm{C}^*(\iota(\mathcal{A})))$ is the C*-envelope of $\mathcal{A}$. (Here, the direct sum is taken over a set of fixed Hilbert spaces of dimensions ranging from 1 up to $\aleph_0 \dim \mathcal{S}$.)*

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