A NOTE ON ABSOLUTE CONTINUITY IN FREE SEMIGROUP ALGEBRAS

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ABSTRACT. An absolutely continuous free semigroup algebra is weak-$*$ type L.

A free semigroup algebra is the WOT-closed (nonself-adjoint, unital) algebra $\mathfrak{S}$ generated by n isometries $S_1, \dots, S_n$ with pairwise orthogonal ranges. See [**4**] for an introduction. There is now a significant literature on these algebras [**1, 2, 9, 10, 11, 7, 6, 8, 5, 16, 15, 18, 17, 20**].

The prototype is the non-commutative Toeplitz algebra $\mathfrak{L}_n$, the algebra generated by the left regular representation λ of the free semigroup on n letters, $\mathbb{F}_n^+$. A free semigroup algebra is *type L* if it is isomorphic to $\mathfrak{L}_n$. Algebraic isomorphism implies the much stronger property that they are completely isometrically isomorphic and weak-$*$ homeomorphic.

An open problem of central importance is whether every type L representation has a wandering vector; i.e. a vector ξ such that the set $\{S_w\xi : w \in \mathbb{F}_n^+\}$ is orthonormal. Here we write $S_w = S_{i_k} \dots S_{i_1}$ for a word $w = i_k \dots i_1$ in $\mathbb{F}_n^+$.

The C*-algebra generated by n isometries is $*$-isomorphic to either the Cuntz algebra $\mathcal{O}_n$ if $\sum_{i=1}^n S_i S_i^* = I$ or the Cuntz-Toeplitz algebra $\mathcal{E}_n$ if $\sum_{i=1}^n S_i S_i^* < I$. The norm closed nonself-adjoint subalgebra generated by $S_1, \dots, S_n$ is denoted by $\mathfrak{A}_n$, the non-commutative analytic disk algebra introduced by Popescu [**16**]. The quotient map of $\mathcal{E}_n$ onto $\mathcal{O}_n$ is completely isometric on $\mathfrak{A}_n$. So it may also be considered as a subalgebra of $\mathcal{O}_n$, which is its C*-envelope (because $\mathcal{O}_n$ is simple). So $\mathfrak{A}_n$ sits isometrically inside $\sigma(\mathcal{E}_n)$ for every $*$-representation σ. Let the abstract generators of $\mathfrak{A}_n$ and $\mathcal{E}_n$ be denoted by $\mathfrak{s}_1, \dots, \mathfrak{s}_n$.

We consider $*$-representations of $\mathcal{E}_n$ and $\mathcal{O}_n$ as a natural way of describing n-tuples of isometries with orthogonal ranges. If σ is a representation of $\mathcal{E}_n$, then let $\mathfrak{S}_\sigma = \overline{\sigma(\mathfrak{A}_n)}^{\text{WOT}}$ denote the corresponding free semigroup algebra. Note that σ splits as $\sigma = \lambda^{(\alpha)} \oplus \tau$, where λ is the identity representation of $\mathcal{E}_n$ and τ factors through the quotient onto $\mathcal{O}_n$. This is the C*-algebra equivalent of the Wold decomposition.

In [**6**], a structure theorem was established which shows that there is a canonical projection P in $\mathfrak{S}$ which is coinvariant so that $\mathfrak{S}P = \mathfrak{W}P$, where $\mathfrak{W}$ is the von Neumann algebra generated by $\mathfrak{S}$, and $\mathfrak{S}|_{P^\perp\mathcal{H}}$ is type L. In [**8**], a notion of absolute continuity was introduced in order to further refine the analysis of free semigroup algebras, and of weaker type L representations in particular.

A linear functional φ on $\mathfrak{A}_n$ is *absolutely continuous* if it extends to a weak-$*$-continuous functional on $\mathfrak{L}_n$. In this case, there are vectors ζ, η in the Fock space $\ell^2(\mathbb{F}_n^+)$ so that $\varphi(A) = \langle \lambda(A)\zeta, \eta\rangle$. In particular, the WOT topology and the weak-$*$ topology coincide on $\mathfrak{L}_n$ [**9**]. A vector in $\mathcal{H}_\sigma$ is called absolutely continuous if the functional $\varphi(A) = \langle \sigma(A)x, x\rangle$ is

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absolutely continuous. The representation σ is called absolutely continuous if every vector in $\mathcal{H}_\sigma$ is absolutely continuous.

We eventually hope to establish that absolutely continuous representations are type L, meaning that $\mathfrak{S}_\sigma$ is type L. In [**8**], representations were divided into several classes. Say that σ is *von Neumann type* if $\mathfrak{S}_\sigma$ is a von Neumann algebra. This is possible, as an example of Read [**20**] shows (see also [**5**]). A representation was called *dilation type* if it had no summand of type L or of von Neumann type. This was justified by the fact that in this case, the range of the canonical projection was a cyclic subspace on which the compression of the isometries is completely non-isometric in a certain sense, and the representation is uniquely obtained by the minimal Frahzo–Bunce–Popescu dilation [**12, 3, 14, 7, 8**].

It is easy to establish that dilation type representations always have wandering vectors; while von Neumann type representations do not. No type L representation is known to exist without wandering vectors, and such an algebra would be an explicit example of a nonself-adjoint WOT-closed reductive algebra. Such an example would provide a counterexample to an important variant of the invariant subspace problem. However there is always a finite ampliation $\sigma^{(p)}$ of any type L representation which is spanned by its wandering vectors [**6**].

In [**8**], further refinements were introduced to try to identify 'type L behaviour' of a weaker type. Say that σ is *weak type L* if $\sigma \oplus \lambda$ is type L. The addition of a copy of λ introduces a wandering vector, and allows one to show that the wandering vectors must now span the whole space. Also we say that σ is *weak-$*$ type L* if the infinite ampliation $\sigma^{(\infty)}$ is type L. This ampliation is then spanned by its wandering vectors. The weak-$*$ closed algebra $\mathfrak{T}_\sigma = \overline{\sigma(\mathfrak{A}_n)}^{\text{w-}*}$ is isometrically isomorphic and weak-$*$ homeomorphic to the free semigroup algebra $\mathfrak{S}_{\sigma^{(\infty)}}$.

A weak-$*$ type L representation which is not type L would have to be of von Neumann type. So the weak-$*$ closure of $\sigma(\mathfrak{A})$ would be isomorphic to $\mathfrak{L}_n$ but the WOT-closure would be self-adjoint. Again no example of this strange phenonemon is known. However there are other analytic operator algebras exhibiting this behaviour [**13**].

The hierarchy of weak type L representations is that type L implies weak-$*$ type L implies weak type L implies absolutely continuous. For any of these, the existence of a single wandering vector implies that the representation is type L and is spanned by its wandering vectors [**8**]. It was also shown that absolutely continuous representations are weak type L, and indeed $\sigma \oplus \tau$ is type L for any type L representation τ.

In this note, we show that absolutely continuous representations are weak-$*$ type L.

1. Absolutely Continuous Representations

The canonical basis of $\ell^2(\mathbb{F}_n^+)$ is $\{\xi_w : w \in \mathbb{F}_n^+\}$. The generators of $\mathfrak{L}_n$ are denoted by $L_1, \dots, L_n$. Every element of $\mathfrak{L}_n$ and, a fortiori $\mathfrak{A}_n$, has a Fourier series determined by

$$\lambda(A)\xi_\varnothing = \sum_{w \in \mathbb{F}_n^+} a_w \xi_w.$$

We write $A \sim \sum_{w \in \mathbb{F}_n^+} a_w \mathfrak{s}_w$, and this makes sense using Cesaro summation [**9**]. The functional which reads off the constant coefficient $a_\varnothing$ is denoted $\varphi_0(A) = \langle \lambda(A)\xi_\varnothing, \xi_\varnothing \rangle$.

The starting point is a key observation in [**6**] that σ is von Neumann type if and only if φ_0 is not WOT-continuous on $\mathfrak{S}_\sigma$. This is equivalent to the fact that the WOT-closed ideal $\mathfrak{S}_{\sigma 0}$ generated by $\sigma(\mathfrak{s}_1), \dots, \sigma(\mathfrak{s}_n)$ equals all of $\mathfrak{S}_\sigma$. In this case, the canonical projection is $P = I$. In general, the intersection $\mathfrak{J} = \bigcap_{k \geq 0} \mathfrak{S}_{\sigma 0}^k$ is a right ideal of the enveloping von Neumann algebra $\mathfrak{W}_\sigma$, and so has the form $\mathfrak{W}_\sigma P$.

Similarly, σ is weak-$*$ type L if and only if φ_0 is weak-$*$ continuous on $\mathfrak{S}_\sigma$. If φ_0 is weak-$*$ continuous but not WOT-continuous, then σ would be weak-$*$ type L and von Neumann type. In order to show that absolutely continuous representations are weak-$*$ type L, it suffices to show that φ_0 is weak-$*$ continuous.

Theorem 1.1. *Every absolutely continuous free semigroup algebra is weak-$*$ type L.*

Proof. Let σ be an absolutely continuous representation of $\mathcal{E}_n$. Let $\mathfrak{M}_\sigma$ denote the predual of $\mathfrak{T}_\sigma$. Equivalently, this is the space of weak-$*$ continuous functionals on $\sigma(\mathfrak{A}_n)$. By definition of absolute continuity, this forms a closed subspace of $\mathfrak{M}_\lambda$, the predual of $\mathfrak{L}_n$.

As noted above, if φ_0 belongs to $\mathfrak{M}_\sigma$, then σ is weak-$*$ type L. If it is not, then by the Hahn–Banach Theorem, there is an element $L_0 \in \mathfrak{L}_n$ so that $\varphi_0(L_0) = 1$ and $\psi(L_0) = 0$ for $\psi \in \mathfrak{M}_\sigma$. We will show that this is impossible.

By [**8**], $\sigma \oplus \lambda$ is type L. The basic idea is that each vector x in $\mathcal{H}_\sigma$, being absolutely continuous, is in the range of a continuous map V in $\mathcal{B}(\ell^2(\mathbb{F}_n^+), \mathcal{H}_\sigma)$ which intertwines λ and σ in the sense: $\sigma(A)V = V\lambda(A)$ for $A \in \mathfrak{L}_n$. Thus $\begin{bmatrix} V & \pm I \end{bmatrix}^t$ are intertwiners between $\sigma \oplus \lambda$ and λ which are bounded below, and thus their ranges are unitarily equivalent to $\ell^2(\mathbb{F}_n^+)$. In particular, $\sigma \oplus \lambda$ has a spanning set of wandering vectors.

Therefore there is an isometric isomorphism of $\mathfrak{L}_n$ onto $\mathfrak{S}_{\sigma\oplus\lambda}$ which is also a weak-$*$ homeomorphism [**6**]. The WOT and weak-$*$ topologies coincide on $\mathfrak{S}_{\sigma\oplus\lambda}$. Therefore it is also completely isometrically isomorphic and weak-$*$ homeomorphic to $\mathfrak{S}_{\sigma^{(\infty)}\oplus\lambda}$.

Follow the isomorphism of $\mathfrak{L}_n$ onto $\mathfrak{S}_{\sigma^{(\infty)}\oplus\lambda}$ with the projection onto the first summand. This is a weak-$*$ continuous contractive algebra homomorphism into $\mathfrak{S}_{\sigma^{(\infty)}}$, which is completely isometrically isomorphic and weak-$*$ homeomorphic to $\mathfrak{T}_\sigma$. Denote the composition mapping $\mathfrak{L}_n$ into $\mathfrak{T}_\sigma$ by τ.

The key point is that τ is injective. First observe that since $\sigma \oplus \lambda$ has a spanning set of wandering vectors, the isomorphism from $\mathfrak{L}_n$ to $\mathfrak{S}_{\sigma^{(\infty)}\oplus\lambda}$ takes isometries to isometries. The restriction to the invariant subspace $\mathcal{H}_\sigma^{(\infty)}$ clearly preserves this, as does the restriction down to $\mathcal{H}_\sigma$. Thus τ preserves isometries.

Since τ is weak-$*$ continuous, the kernel is a weak-$*$ closed ideal $\mathfrak{J}$ in $\mathfrak{L}_n$. By [**10**, Theorem 2.1], $\mathfrak{J}$ consists of all elements of $\mathfrak{L}_n$ with range contained in $\overline{\mathfrak{J}\ell^2(\mathbb{F}_n^+)}$ and this is a subspace which is invariant for both $\mathfrak{L}_n$ and its commutant $\mathfrak{R}_n$, the right regular representation algebra. By the Beurling Theorem for $\mathfrak{R}_n$ [**1, 9**], this subspace is the sum of ranges of isometries in $\mathfrak{L}_n$. In particular, if $\mathfrak{J}$ is non-zero, then it contains an isometry. By the previous paragraph, this does not occur. Hence τ is injective.

On the other hand, for any weak-$*$ continuous functional $\psi \in \mathfrak{M}_\sigma$, $\psi(\tau(L_0)) = 0$ by hypothesis. Hence $\tau(L_0) = 0$. This contradiction shows that $\mathfrak{M}_\sigma$ contains φ_0, and so is weak-$*$ type L, and $\mathfrak{M}_\sigma = \mathfrak{M}_\lambda$. ∎

Corollary 1.2. *If σ is an absolutely continuous representation of $\mathcal{E}_n$, then $\mathfrak{T}_\sigma$ is completely isometrically isomorphic and weak-$*$ homeomorphic to $\mathfrak{L}_n$.*

Example 1.3. The example of Read [**20**] and the similar examples in [**5**] are representations of von Neumann type. Moreover, the proof shows that the weak-$*$ closure of $\sigma(\mathfrak{A}_n)$ is all of $\mathcal{B}(\mathcal{H})$. So these algebras are not weak-$*$ type L, and thus are not absolutely continuous. This answers Question 5.12 in [**8**].

Corollary 1.4. *For a representation σ of $\mathcal{E}_n$, the following are equivalent:*

(1) *σ is absolutely continuous.*
(2) *σ is weak type L.*
(3) *σ is weak-$*$ type L.*

This allows us to make the following definition:

Definition 1.5. If σ is an absolutely continuous representation of $\mathcal{E}_n$, there is a least cardinal $\kappa \in \{1, 2, \dots, \aleph_0\}$ so that $\sigma^{(\kappa)}$ has wandering vectors (and hence is spanned by wandering vectors). We say that such a representation is *type L_κ*.

The big open question about wandering vectors can now be rephrased as asking whether every absolutely continuous representation is type L_1?

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