

Transitive Spaces of Operators

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Abstract. We investigate algebraic and topological transitivity and, more generally, k -transitivity for linear spaces of operators. In finite dimensions, we determine minimal dimensions of k -transitive spaces for every k , and find relations between the degree of transitivity of a product or tensor product on the one hand and those of the factors on the other. We present counterexamples to some natural conjectures. Some infinite dimensional analogues are discussed. A simple proof is given of Arveson's result on the weak-operator density of transitive spaces that are masa bimodules.

A collection \mathcal{E} of operators from a vector space \mathcal{X} to another vector space \mathcal{Y} is said to be *transitive* if, given nonzero $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, there is an element $A \in \mathcal{E}$ such that $Ax = y$. If \mathcal{E} has some additional structure imposed on it, for example, if it is a ring or an algebra with $\mathcal{X} = \mathcal{Y}$, then a great deal is implied by the transitivity assumption. This paper represents an effort to extract as much information as possible from transitivity (and its strengthening given below) from the linear structure on \mathcal{E} alone.

This paper could well have been written twenty years ago. Indeed, it is in the spirit of Azoff's paper [2] which explores transitive, reflexive, and elementary spaces and the relations between these properties. When we examine infinite dimensional analogues, the Toeplitz operators are a natural source of examples. The paper of Azoff and Ptak [3] contains some related results about the space of Toeplitz operators.

Let \mathcal{H} and \mathcal{K} be Hilbert spaces. A subspace \mathcal{L} of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is *k -transitive* provided that for every choice of k linearly independent vectors x_1, \dots, x_k in \mathcal{H} and arbitrary vectors y_1, \dots, y_k in \mathcal{K} , there is an $A \in \mathcal{L}$ so that $Ax_i = y_i$ for $1 \leq i \leq k$. Thus transitivity coincides with 1- transitivity. \mathcal{L} is *topologically k -transitive* if this can be done approximately; that is, for each $\varepsilon > 0$ there exists $A \in \mathcal{L}$ such that $\|Ax_i - y_i\| < \varepsilon$, $1 \leq i \leq k$. In the finite dimensional setting,

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the two notions coincide. As with transitivity, if $k = 1$, we shall abbreviate the notation and refer to a space as being *topologically transitive*.

Starting with finite dimensions, we present a number of positive results and a lot of counterexamples to natural conjectures. Among other things, we determine the minimal dimension of k -transitive subspaces of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ in terms of the dimensions of \mathcal{H} and \mathcal{K} . We consider spans of products as well as tensor products of spaces, and study the relations between their degree of transitivity and those of their constituent spaces. We investigate the relations between the minimal and maximal ranks present in a transitive subspace of \mathbb{M}_n . In particular, we show that such a subspace must contain invertible elements. In the infinite dimensional setting, there are fewer positive results and more counterexamples. We are able to extend some of the finite dimensional results. We provide a simple proof of a result of Arveson [1] that a topologically transitive subspace of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ which is a masa bimodule is WOT-dense in $\mathcal{B}(\mathcal{H}, \mathcal{K})$.

Any transitive subalgebra of $\mathbb{M}_n = \mathcal{B}(\mathcal{H}_n)$, the space of $n \times n$ complex matrices, is equal to all of \mathbb{M}_n by Burnside's Theorem. The situation is completely different for subspaces. For every $0 \leq k < \min\{m, n\}$, there are k -transitive subspaces of \mathbb{M}_{mn} which are not $(k+1)$ -transitive (see Example 1.4). In infinite dimensions, a topologically transitive operator algebra has no proper invariant subspaces. Whether it is WOT-dense in $\mathcal{B}(\mathcal{H})$ is the famous Transitive Algebra Problem, a generalization of the Invariant Subspace Problem. Again, if we consider subspaces, there are many proper k -transitive WOT-closed subspaces.

It is a well-known result of Azoff [2] that a subspace \mathcal{L} of $\mathbb{M}_{mn} = \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)$ is k -transitive if and only if the pre-annihilator \mathcal{L}_\perp contains no non-zero elements of rank at most k . Here $(\mathbb{M}_{mn})_*$ is identified with \mathbb{M}_{nm} equipped with the trace norm via the bilinear pairing $\langle A, T \rangle = \text{Tr}(AT)$.

Indeed suppose that $0 \neq T = \sum_{i=1}^k x_i y_i^*$ belongs to \mathcal{L}_\perp , where the vectors x_i are linearly independent. Then the k -tuple (Ax_1, \dots, Ax_k) is orthogonal to (y_1, \dots, y_k) for all $A \in \mathcal{L}$. Thus \mathcal{L} is not k -transitive. Conversely, if \mathcal{L} is not k -transitive, then for some linearly independent set x_1, \dots, x_n , the k -tuples (Ax_1, \dots, Ax_k) span a proper subspace of \mathcal{H}_m^k , and thus is orthogonal to a non-zero vector (y_1, \dots, y_k) . Reversing the argument shows that $T = \sum_{i=1}^k x_i y_i^*$ belongs to \mathcal{L}_\perp .

In the infinite-dimensional setting, we identify $(\mathcal{B}(\mathcal{H}, \mathcal{K}))_*$ with the space $\mathcal{C}_1(\mathcal{K}, \mathcal{H})$ of trace class operators from \mathcal{K} to \mathcal{H} via the same bilinear pairing. The above Theorem of Azoff then applies to topological k -transitivity, and shows that a subspace \mathcal{L} of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is topologically k -transitive if and only if the pre-annihilator \mathcal{L}_\perp of \mathcal{L} contains no non-zero elements of rank less than or equal to k .

We end this introduction with a simple but handy observation which will be used implicitly throughout this paper, namely: if $\mathcal{L} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ is topologically k -transitive, and if $P \in \mathcal{B}(\mathcal{H})$ and $Q \in \mathcal{B}(\mathcal{K})$ are projections which satisfy $\min(\text{rank } P, \text{rank } Q) \geq k$, then $Q\mathcal{L}P \subseteq \mathcal{B}(P\mathcal{H}, Q\mathcal{K})$ is also topologically k -transitive. (This follows immediately from Azoff's Theorem.) If, furthermore, P and Q are finite rank, then $Q\mathcal{L}P$ is in fact k -transitive.

We are indebted to the referee for providing us with a number of very useful references and helpful comments - in particular, the argument of Theorem 2.4.

1. Dimension

In this section, we find the minimal dimension of a k -transitive subspace of \mathbb{M}_{mn} . By Azoff's Theorem as outlined above, if $k \geq \min(m, n)$, then $\mathcal{L}_\perp = 0$ and hence $\mathcal{L} = \mathbb{M}_{mn}$. As such, we always assume that $k < \min(m, n)$.

Lemma 1.1. *For every $p \geq 1$ and every $0 \leq k \leq p$, the diagonal subalgebra \mathfrak{D}_p of \mathbb{M}_p contains a subspace of dimension $p - k$ containing no non-zero element of rank at most k .*

Proof. It suffices to choose $p - k$ diagonal operators with the property that when restricted to any $p - k$ diagonal entries, they are linearly independent. For then a linear combination which has $p - k$ zeros must be zero. An example of such a sequence is $D_j = \text{diag}(1^j, 2^j, \dots, p^j)$ for $0 \leq j < p - k$. ■

The special case of the following result for $k = 1$ was established by Azoff [2].

Theorem 1.2. *A k -transitive subspace of \mathbb{M}_{mn} has dimension at least $k(m + n - k)$; and this is sharp.*

Proof. Using Azoff's Theorem, we search instead for the maximal dimension of the pre-annihilator of a k -transitive subspace \mathcal{L} . This subspace \mathcal{L}_\perp cannot intersect the closed variety \mathcal{R}_k of matrices of rank at most k except in $\{0\}$. This variety has dimension $k(m + n - k)$ (see [7, Prop. 12.2]). It follows that $\dim \mathcal{L}_\perp + \dim \mathcal{R}_k \leq mn$. Hence $\dim \mathcal{L} \geq \dim \mathcal{R}_k = k(m + n - k)$.

On the other hand, consider the subspace \mathcal{N} of \mathbb{M}_{nm} which has zeros on diagonals of length $p \leq k$ and has dimension $p - k$ on the diagonals with $p > k$ entries such that the rank of any non-zero element on one such diagonal is always at least $k + 1$. This is possible by Lemma 1.1. There are $m + n - 1$ diagonals, and $2k$ of them have length at most k . So the dimension of \mathcal{N} is

$$\begin{aligned} mn - (m + n - 1 - 2k)k - 2 \sum_{p=1}^k p &= mn - (m + n - 1 - 2k)k - (k + 1)k \\ &= mn - k(m + n - k) \end{aligned}$$

Thus $\mathcal{L} = \mathcal{N}^\perp$, the annihilator of \mathcal{N} , has dimension $k(m + n - k)$.

Consider a non-zero element N of \mathcal{N} . It must be non-zero on some diagonal. Let p_0 be the shortest non-zero diagonal. Consider the square submatrix containing the p_0 th diagonal as its main diagonal. This submatrix is triangular, and hence its rank is at least as great as the rank of the diagonal, which is at least $k + 1$. Hence \mathcal{N} contains no non-zero elements of rank at most k . Therefore \mathcal{L} is k -transitive. ■

Example 1.3. By the above theorem, any proper $(n - 1)$ -transitive subspace \mathcal{L} of \mathbb{M}_n necessarily has dimension $n^2 - 1$. It is then not hard to see that we can find invertible elements $S, T \in \mathbb{M}_n$ so that $S\mathcal{L}T = \mathfrak{sl}_n$, the space of trace zero matrices in \mathbb{M}_n .

Since multiplying a k -transitive subspace \mathcal{M} of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ by invertibles S in $\mathcal{B}(\mathcal{K}, \mathcal{K}')$ and T in $\mathcal{B}(\mathcal{H}', \mathcal{H})$ yields a k -transitive subspace of $\mathcal{B}(\mathcal{H}', \mathcal{K}')$, we shall think of \mathcal{M} and SMT as being *equivalent* insofar as transitivity is concerned.

A similar statement holds for topologically k -transitive spaces.

Example 1.4. As mentioned in the introduction, there are k -transitive subspaces of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ which are not $(k + 1)$ -transitive provided that $k < \min\{\dim \mathcal{H}, \dim \mathcal{K}\}$. For example, if $\mathcal{H} = \mathcal{H}_n$ and $\mathcal{K} = \mathcal{H}_m$, then it suffices to consider $R \in \mathbb{M}_{nm}$ of rank $k + 1$. With $\mathcal{M} = \text{span}\{R\}$, $\mathcal{L} = \mathcal{M}^\perp \subseteq \mathbb{M}_{mn}$ is such an example.

This example can easily be adapted to the infinite-dimensional setting to produce subspaces $\mathcal{L} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ which are topologically k -transitive but not topologically $(k + 1)$ -transitive.

Example 1.5. As shown in Example 3.4 of [2], the space \mathfrak{T}_n of all Toeplitz matrices $T = [t_{i-j}]$ in \mathbb{M}_n is a transitive subspace of dimension $2n - 1$. It is routine to verify that the pre-annihilator of \mathfrak{T}_n consists of those matrices in \mathbb{M}_n whose entries along any diagonal sum to zero. The rank of such a matrix is at least as big as the length of the smallest non-zero diagonal, which is at least two if the matrix is non-zero.

By Theorem 1.2, this is the minimal possible dimension of a transitive subspace of \mathbb{M}_n .

As pointed out by the referee, there are a number of other papers which investigate the relationship between the dimension of a linear subspace \mathcal{S} of matrices and the ranks of its members. For example, H. Flanders [6] has shown that if $\mathcal{S} \subseteq \mathbb{M}_n$ is a linear space and the rank of every element of \mathcal{S} is at most k , then $\dim \mathcal{S} \leq kn$, and he classified those spaces \mathcal{S} of maximal dimension. More recently, R. Meshulam and P. Šemrl [8] have shown that if $\min\{\text{rank } S : 0 \neq S \in \mathcal{S}\} > \dim \mathcal{S}$, then \mathcal{S} is reflexive as a space of operators. In other words, \mathcal{S}_\perp is spanned by its rank one elements.

2. Dually Transitive

In finite dimensions, we can consider both \mathcal{L} and \mathcal{L}_\perp as spaces of matrices. So we can ask whether both can be k -transitive. Dimension arguments show that this requires the space to be sufficiently large.

Proposition 2.1. *There is a subspace \mathcal{L} of \mathbb{M}_4 such that both \mathcal{L} and \mathcal{L}_\perp are transitive.*

Proof. It suffices to exhibit a transitive subspace \mathcal{L} so that neither \mathcal{L} nor \mathcal{L}_\perp contains a rank one element. Let Φ be a linear bijection of \mathbb{M}_2 onto itself with four distinct eigenvalues whose corresponding eigenvectors all have rank 2. Define

$$\mathcal{L} = \left\{ \begin{bmatrix} A & \Phi(B) \\ B & A \end{bmatrix} : A, B \in \mathbb{M}_2 \right\}.$$

Suppose that \mathcal{L} contains a rank 1 element $\begin{bmatrix} xu^* & xv^* \\ yu^* & yv^* \end{bmatrix}$, where x, y, u, v are vectors in \mathbb{C}^2 . Then comparing the diagonal entries shows that $xu^* = yv^*$, so that xv^* is a multiple of yu^* , say $xv^* = \lambda yu^*$. So now a comparison of the off-diagonal entries shows that $\Phi(yu^*) = \lambda yu^*$, contrary to fact. Therefore \mathcal{L} contains no rank 1 elements.

The analysis of \mathcal{L}_\perp is similar since Φ^t has the same eigenvector property. ■

Example 2.2. For example, let $\Phi \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} d & 2c \\ b & a \end{bmatrix}$. Then

$$\mathcal{L} = \begin{bmatrix} a & b & h & 2g \\ c & d & f & e \\ e & f & a & b \\ g & h & c & d \end{bmatrix} \quad \text{and} \quad \mathcal{L}_\perp = \begin{bmatrix} a & b & -h & -g \\ c & d & -f & -e \\ e & f & -a & -b \\ g/2 & h & -c & -d \end{bmatrix}$$

Here is a general technique modelled on the previous example.

Theorem 2.3. *If $1 \leq k < n/(2 + \sqrt{2})$, then \mathbb{M}_{2n} contains a subspace \mathcal{L} such that both \mathcal{L} and \mathcal{L}_\perp are k -transitive.*

Proof. The proof of Theorem 1.2 shows that there is a subspace \mathcal{N} of \mathbb{M}_n with dimension $(n - k)^2$ which contains no non-zero elements of rank k or less. Since $n > (2 + \sqrt{2})k$, it is easy to check that $(n - k)^2 \geq n^2/2$. Thus there is an injective linear map $T : \mathbb{M}_n/\mathcal{N} \rightarrow \mathcal{N}$. Let $\Phi = JTQ$, where Q is the quotient map of \mathbb{M}_n onto \mathbb{M}_n/\mathcal{N} and J is the injection of \mathcal{N} into \mathbb{M}_n .

Observe that for $A \in \mathbb{M}_n$,

$$\text{rank } A \geq k + 1 \quad \text{or} \quad \text{rank } \Phi(A) \geq k + 1.$$

Indeed, if $\Phi(A) \neq 0$, then it has rank at least $k + 1$. If $\Phi(A) = 0$, then $A \in \mathcal{N}$; so it has rank at least $k + 1$.

Define a subspace \mathcal{L} of \mathbb{M}_{2n} consisting of all elements of the form

$$\begin{bmatrix} A & B \\ \Phi(B) & \Phi(A) \end{bmatrix}$$

where $A, B \in \mathbb{M}_n$ are arbitrary. A non-zero element of \mathcal{L} has either A or B non-zero. So at least one of the four matrix entries has rank at least $k+1$. In particular, \mathcal{L}_\perp is k transitive.

Note that \mathcal{L}_\perp consists of all matrices of the form $\begin{bmatrix} \Phi^t(X) & -Y \\ \Phi^t(Y) & -X \end{bmatrix}$ where X, Y are arbitrary elements of \mathbb{M}_n . So the same argument shows that \mathcal{L}_\perp contains no non-zero elements of rank at most k . Therefore \mathcal{L} is k -transitive. ■

We thank the referee for the following argument which significantly sharpens the above result.

Theorem 2.4. *There is a subspace \mathcal{L} of \mathbb{M}_{mn} such that if \mathcal{L} is k -transitive and \mathcal{L}_\perp is l -transitive if and only if $k, l \leq \min\{m, n\}$ and $k(m+n-k) + l(m+n-l) \leq mn$.*

Proof. Recall that for $1 \leq t \leq \min\{m, n\}$, \mathcal{R}_t denotes the variety of $m \times n$ matrices whose rank does not exceed t , and that $\dim \mathcal{R}_t = t(m+n-t)$. The subspace \mathcal{L} has the property that $\mathcal{L} \cap \mathcal{R}_l = \{0\}$ and $\mathcal{L} \cap \mathcal{R}_k = \{0\}$. Thus $\dim \mathcal{L} + \dim \mathcal{R}_l \leq mn$, and $\dim \mathcal{L}_\perp + \dim \mathcal{R}_k \leq mn$. Therefore

$$mn = \dim \mathcal{L} + \dim \mathcal{L}_\perp \leq 2mn - \dim \mathcal{R}_l - \dim \mathcal{R}_k.$$

Hence we require

$$\dim \mathcal{R}_k + \dim \mathcal{R}_l \leq mn.$$

Conversely, suppose that $\dim \mathcal{R}_k + \dim \mathcal{R}_l \leq mn$. Let $d = k(m+n-k) = \dim \mathcal{R}_k$; and look for a subspace \mathcal{L} of dimension d as above. Among the Grassmanian space \mathcal{G}_d of d -dimensional subspaces of \mathbb{M}_{mn} , the subset of subspaces \mathcal{L} with $\mathcal{L} \cap \mathcal{R}_l = \{0\}$ is Zariski open and non-empty. In particular, it is a dense open subset in the usual topology. (We postpone this argument to the end of the proof.) Similarly, \mathcal{L}_\perp belongs to \mathcal{G}_{mn-d} , the space of $(mn-d)$ -dimensional subspaces of \mathbb{M}_{nm} . Since $(mn-d) + \dim \mathcal{R}_k = mn$, the property that $\mathcal{L}_\perp \cap \mathcal{R}_k = \{0\}$ is also generic. So this also corresponds to a dense open set of subspaces \mathcal{L} of \mathcal{G}_d . Thus generically, both properties hold and many such subspaces exist.

Finally, we give the geometric argument to explain the genericity. Consider the projective variety $\overline{\mathcal{R}}_l$ in the projective space $\mathbb{P}^{mn-1} = \overline{\mathbb{M}_{mn}}$. Form the space $\mathcal{G}_d \times \mathbb{P}^{mn-1}$ and consider the subset

$$\Sigma = \{(G, P) : P \in G\}.$$

Then Σ is Zariski closed because it is the set of elements of rank (at most) k . Now $\mathcal{G}_d \times \overline{\mathcal{R}}_l$ is a subset of $\mathcal{G}_d \times \mathbb{P}^{mn-1}$, and thus $(\mathcal{G}_d \times \overline{\mathcal{R}}_l) \cap \Sigma$ is Zariski closed in $\mathcal{G}_d \times \overline{\mathcal{R}}_l$. Project onto the first coordinate, and note that this is precisely the set $\{G \in \mathcal{G}_d : G \cap \overline{\mathcal{R}}_l \neq \emptyset\}$; i.e., the set of subspaces \mathcal{L} containing a non-zero

element of rank at most l . This is therefore Zariski closed and clearly it is a proper subspace. So the complement is Zariski open; in particular, it is open and dense. ■

Corollary 2.5. *There exists a subspace $\mathcal{L} \subseteq \mathbb{M}_n$ such that \mathcal{L} and \mathcal{L}_\perp are k -transitive if and only if $k < n/(2 + \sqrt{2})$.*

Proof. When $m = n$ and $k = l$, the inequality of Theorem 2.4 reduces to

$$n^2 - 4kn + 2k^2 \geq 0,$$

or $2(k - n)^2 \geq n^2$. Since $k \leq n/2$, we obtain $k \leq n(1 - 1/\sqrt{2}) = n/(2 + \sqrt{2})$. This is never an equality. ■

Example 2.6. Theorem 2.4 above implies in particular that there should exist a subspace $\mathcal{L} \subseteq \mathbb{M}_{3,4}$ of dimension 6 so that \mathcal{L} and \mathcal{L}_\perp are transitive. Let us exhibit such an example.

Consider

$$\mathcal{L} = \left\{ \begin{bmatrix} a & b & c & 2d \\ e & f & a & b \\ d & c & f & e \end{bmatrix} : a, b, c, d, e, f \in \mathbb{C} \right\}.$$

Observe that if \mathcal{L} contains a rank one matrix L , then the compression PLQ of that matrix to any 2×2 block must have determinant equal to zero. We consider two cases:

Case 0: $a = 0$. From $\begin{vmatrix} f & a \\ c & f \end{vmatrix} = 0$, we get $f = 0$. Then $\begin{vmatrix} b & 2d \\ f & b \end{vmatrix} = 0$ yields $b = 0$. So $\begin{vmatrix} b & c \\ c & f \end{vmatrix} = 0$ gives $c = 0$; $\begin{vmatrix} e & b \\ d & e \end{vmatrix} = 0$, gives $e = 0$; and $\begin{vmatrix} a & 2d \\ d & e \end{vmatrix} = 0$ shows that $d = 0$. Thus $L = 0$.

Case 1: $a = 1$. From $\begin{vmatrix} a & c \\ e & a \end{vmatrix} = 0$, we get $ec = 1$; and $\begin{vmatrix} e & f \\ d & c \end{vmatrix} = 0$ yields $df = 1$. Thus $\begin{vmatrix} c & 2d \\ f & e \end{vmatrix} = 0$ leads to the erroneous conclusion $1 = ec = 2df = 2$, a contradiction. Therefore \mathcal{L} does not contain a rank one operator.

A similar analysis may be applied to

$$\mathcal{L}_\perp = \left\{ \begin{bmatrix} u & y & -x \\ v & z & -w \\ w & -u & -z \\ \frac{1}{2}x & -v & -y \end{bmatrix} : u, v, w, x, y, z \in \mathbb{C} \right\},$$

to prove that \mathcal{L}_\perp does not contain any rank one matrices. From this we conclude that \mathcal{L} and \mathcal{L}_\perp are both transitive.

3. Tensor Products

In this section, we consider tensor products of k -transitive subspaces. The first lemma is well known, but is included for the convenience of the reader.

Lemma 3.1. *Let \mathcal{L} and \mathcal{M} be subspaces of \mathbb{M}_{lp} and \mathbb{M}_{mn} respectively. Then*

$$(\mathcal{L} \otimes \mathcal{M})_{\perp} = \mathcal{L}_{\perp} \otimes \mathbb{M}_{nm} + \mathbb{M}_{pl} \otimes \mathcal{M}_{\perp}.$$

Proof. Clearly $\mathcal{L}_{\perp} \otimes \mathbb{M}_{nm} + \mathbb{M}_{pl} \otimes \mathcal{M}_{\perp}$ is contained in $(\mathcal{L} \otimes \mathcal{M})_{\perp}$. The other containment follows from a simple dimension argument. Indeed, let $\dim \mathcal{L} = d_1$ and $\dim \mathcal{M} = d_2$. Then $\dim \mathcal{L} \otimes \mathcal{M} = d_1 d_2$, from which we deduce that

$$\dim(\mathcal{L} \otimes \mathcal{M})_{\perp} = lp mn - d_1 d_2.$$

Now $\dim(\mathcal{L}_{\perp} \otimes \mathbb{M}_{nm}) = (lp - d_1)mn$ and $\dim(\mathbb{M}_{pl} \otimes \mathcal{M}_{\perp}) = lp(mn - d_2)$. Since

$$\mathcal{L}_{\perp} \otimes \mathbb{M}_{nm} \cap \mathbb{M}_{pl} \otimes \mathcal{M}_{\perp} = \mathcal{L}_{\perp} \otimes \mathcal{M}_{\perp}$$

has dimension $(lp - d_1)(mn - d_2)$, it follows that

$$\begin{aligned} \dim(\mathcal{L}_{\perp} \otimes \mathbb{M}_{nm} + \mathbb{M}_{pl} \otimes \mathcal{M}_{\perp}) &= (lp - d_1)mn + lp(mn - d_2) - (lp - d_1)(mn - d_2) \\ &= lp mn - d_1 d_2. \end{aligned}$$

From this the result follows easily. ■

The main theorem of this section shows that under additional hypotheses, tensoring preserves some level of transitivity.

Theorem 3.2. *Suppose that a subspace $\mathcal{L} \subset \mathbb{M}_{lp}$ is k -transitive and is spanned by its elements of rank at most r . If a subspace $\mathcal{M} \subset \mathbb{M}_{mn}$ is rk -transitive, then $\mathcal{L} \otimes \mathcal{M}$ is k -transitive.*

Proof. Let $T = [T_{ji}] \in \mathbb{M}_{pl} \otimes \mathbb{M}_{nm}$, considered as a $p \times l$ matrix with coefficients $T_{ji} \in \mathbb{M}_{nm}$. It follows from Lemma 3.1 that $(\mathcal{L} \otimes \mathcal{M})_{\perp} = \mathcal{L}_{\perp} \otimes \mathbb{M}_{nm} + \mathbb{M}_{pl} \otimes \mathcal{M}_{\perp}$. Now $T = [T_{ji}] \in \mathcal{L}_{\perp} \otimes \mathbb{M}_{nm} = (\mathcal{L} \otimes \mathbb{M}_{nm})_{\perp}$ if and only if

$$\sum_{i=1}^l \sum_{j=1}^p l_{ij} T_{ji} = 0 \quad \text{for all } L = [l_{ij}] \in \mathcal{L},$$

while $T \in \mathbb{M}_{pl} \otimes \mathcal{M}_{\perp} = (\mathbb{M}_{pl} \otimes \mathcal{M})_{\perp}$ if and only if each $T_{ji} \in \mathcal{M}_{\perp}$.

We may decompose T as $T = R + S$ where $R = [R_{ji}] \in \mathcal{L}_{\perp} \otimes \mathbb{M}_{nm}$ and $S = [S_{ji}] \in \mathbb{M}_{pl} \otimes \mathcal{M}_{\perp}$.

For $L = [l_{ij}] \in \mathcal{L}$, consider

$$\sum_{i=1}^l \sum_{j=1}^p l_{ij} T_{ji} = \sum_{i=1}^l \sum_{j=1}^p l_{ij} (R_{ji} + S_{ji}) = \sum_{i=1}^l \sum_{j=1}^p l_{ij} S_{ji} \in \mathcal{M}_{\perp}. \quad (1)$$

Suppose that $\text{rank } T = 1$. Then there are vectors u_j and v_i for $1 \leq j \leq p$ and $1 \leq i \leq l$ so that $T_{ji} = u_j v_i^*$. Now if L is rank one, then there are scalars x_i and y_j so that $l_{ij} = x_i y_j$. In this case,

$$\sum_{i=1}^l \sum_{j=1}^p l_{ij} T_{ji} = \sum_{i=1}^l \sum_{j=1}^p x_i y_j u_j v_i^* = \left(\sum_{j=1}^p y_j u_j \right) \left(\sum_{i=1}^l \overline{x_i} v_i \right)^*$$

is a rank one matrix. (As pointed out by the referee, this step is similar to the proof that the set of rank 1 operators is stable under Schur multiplication.) It is then easy to see that if $\text{rank } T \leq k$ and $\text{rank } L \leq r$, then this sum has rank at most rk .

Clearly it suffices to satisfy equation (1) for a spanning subset of \mathcal{L} . So we may suppose that each L has rank at most r . Consider an element $T \in (\mathcal{L} \otimes \mathcal{M})_\perp$ with rank at most k . The analysis of the previous paragraph yields an element of \mathcal{M}_\perp with rank at most rk . As \mathcal{M} is rk -transitive, this means that these sums are all zero:

$$\sum_{i=1}^l \sum_{j=1}^p l_{ij} S_{ji} = 0 \quad \text{for all } L = [l_{ij}] \in \mathcal{L}.$$

Consequently, S lies in $\mathcal{L}_\perp \otimes \mathbb{M}_{nm}$. Since R is also in this set, we conclude that T belongs to $\mathcal{L}_\perp \otimes \mathbb{M}_{nm}$ as well.

But \mathcal{L}_\perp admits no non-zero operators of rank at most k , and thus neither does $\mathcal{L}_\perp \otimes \mathbb{M}_{nm}$. To see this, think of such matrices as $n \times m$ matrices with coefficients in \mathcal{L}_\perp . Any non-zero coefficient results in rank at least $k+1$. Hence $(\mathcal{L} \otimes \mathcal{M})_\perp$ contains no non-zero elements of rank at most k , and therefore $\mathcal{L} \otimes \mathcal{M}$ is k -transitive. ■

We shall revisit this result in the infinite-dimensional setting in Section 6 (see Theorem 6.12).

Definition 3.3. A subspace $\mathcal{L} \subset \mathbb{M}_{mn}$ is *fully k -transitive* if $\mathcal{L} \otimes \mathcal{M}$ is k -transitive whenever \mathcal{M} is k -transitive.

It is clear that the tensor product of fully k -transitive spaces is again fully k -transitive.

The following corollary yields a large class of fully k -transitive subspaces. This is an immediate application of Theorem 3.2 taking $r = 1$.

Corollary 3.4. *If $\mathcal{L} \subset \mathbb{M}_{lp}$ is a k -transitive subspace which is spanned by its rank one elements, then \mathcal{L} is fully k -transitive.*

Proof. $\mathcal{L} \otimes \mathcal{M}$ is k -transitive by Corollary 3.4. The tensor product of rank one elements is rank one, and so $\mathcal{L} \otimes \mathcal{M}$ is spanned by rank ones. Hence it is fully k -transitive by the same corollary. ■

Another easy consequence uses the fact that $\mathcal{L} \subset \mathbb{M}_{lp}$ is always spanned by elements of rank at most $\min\{l, p\}$.

Corollary 3.5. *If $\mathcal{L} \subset \mathbb{M}_{lp}$ is a k -transitive subspace and $\mathcal{M} \subset \mathbb{M}_{mn}$ is $\min\{kl, kp\}$ -transitive, then $\mathcal{L} \otimes \mathcal{M}$ is k -transitive.*

Fully k -transitive spaces have a certain permanence.

Proposition 3.6. *If $\mathcal{L} \subset \mathbb{M}_{lp}$ is fully k -transitive, and P, Q are idempotents in \mathbb{M}_l and \mathbb{M}_p respectively, then $P\mathcal{L}Q \subset \mathcal{B}(Q\mathcal{H}_p, P\mathcal{H}_l)$ is fully k -transitive.*

Proof. Let $\mathcal{M} \subset \mathbb{M}_{mn}$ be a k -transitive space. If $A \in (P\mathcal{L}Q)_\perp$, then

$$0 = \text{Tr}(PLQA) = \text{Tr}(LQAP) \quad \text{for all } L \in \mathcal{L}.$$

So $QAP \in \mathcal{L}_\perp$, where QAP is just A considered as an element of \mathbb{M}_{lp} rather than $\mathcal{B}(P\mathcal{H}_l, Q\mathcal{H}_p)$. Therefore

$$\begin{aligned} (P\mathcal{L}Q \otimes \mathcal{M})_\perp &= (P\mathcal{L}Q)_\perp \otimes \mathbb{M}_{nm} + Q\mathbb{M}_{pl}P \otimes \mathcal{M}_\perp \\ &\subset \mathcal{L}_\perp \otimes \mathbb{M}_{nm} + \mathbb{M}_{pl} \otimes \mathcal{M}_\perp = (\mathcal{L} \otimes \mathcal{M})_\perp \end{aligned}$$

The right hand side contains no non-zero matrices of rank at most k , and so neither does the left side. Therefore $P\mathcal{L}Q$ is fully k -transitive. \blacksquare

Example 3.7. The space of Toeplitz matrices \mathfrak{T}_m is fully transitive because the rank one matrices $[a^{i-j}]$ for $a \in \mathbb{C}$ span \mathfrak{T}_m . To see this, just observe that the entries on the first row and column determine T , and that this may be any vector in \mathbb{C}^{2m-1} . The rank one matrices mentioned above correspond to the vectors $(a^k)_{|k| < m}$. Any choice of $2m - 1$ distinct non-zero values of a yields a basis.

Example 3.8. The space $\mathfrak{sl}_d \subset \mathbb{M}_d$ from Example 1.3 is fully $(d - 1)$ -transitive since it is spanned by the rank one matrices $\{E_{11} + E_{1j} - E_{j1} - E_{jj} : 2 \leq j \leq m\}$ and $\{E_{ij} : i \neq j\}$.

Example 3.9. Consider $\mathcal{L} = \mathfrak{sl}_d \otimes \mathbb{M}_m \subset \mathbb{M}_{dm}$. We may think of this space as those $d \times d$ matrices $A = [A_{ij}]$ with coefficients in \mathbb{M}_m such that $\sum_{i=1}^d A_{ii} = 0$. Then $\mathcal{L}_\perp = (\mathfrak{sl}_d)_\perp \otimes \mathbb{M}_m = \mathbb{C}I_d \otimes \mathbb{M}_m$. Thus the minimum rank of a non-zero element of \mathcal{L}_\perp is d . Therefore \mathcal{L} is $(d - 1)$ -transitive. It is spanned by its rank one elements since both \mathfrak{sl}_d and \mathbb{M}_m are. Hence \mathcal{L} is fully $(d - 1)$ -transitive.

Example 3.10. There are transitive spaces which are not fully transitive. Consider the space \mathcal{L} from Example 2.2. Evidently, the smallest rank of a non-zero element of \mathcal{L} is 2. We will show that $(\mathcal{L}_\perp \otimes \mathcal{L}_\perp)_\perp$ contains a rank one. By symmetry, it follows that neither $\mathcal{L} \otimes \mathcal{L}$ nor $\mathcal{L}_\perp \otimes \mathcal{L}_\perp$ is transitive. By Lemma 3.1,

$$(\mathcal{L}_\perp \otimes \mathcal{L}_\perp)_\perp = \mathcal{L} \otimes \mathbb{M}_4 + \mathbb{M}_4 \otimes \mathcal{L}.$$

To find a rank 1 in this space, we look for matrices A, \dots, H in \mathbb{M}_4 , $X_{ij} \in \mathcal{L}$ and vectors $u_i, v_j \in \mathbb{C}^4$ for $1 \leq i, j \leq 4$ so that

$$\begin{bmatrix} A & B & H & 2G \\ C & D & F & E \\ E & F & A & B \\ G & H & C & D \end{bmatrix} = \begin{bmatrix} X_{ij} + u_i v_j^* \end{bmatrix}$$

This is equivalent to solving the system

$$\begin{aligned} u_1 v_1^* - u_3 v_3^* &= X_{33} - X_{11} \in \mathcal{L} & u_2 v_4^* - u_3 v_1^* &= X_{31} - X_{24} \in \mathcal{L} \\ u_1 v_2^* - u_3 v_4^* &= X_{34} - X_{12} \in \mathcal{L} & u_2 v_3^* - u_3 v_2^* &= X_{32} - X_{23} \in \mathcal{L} \\ u_2 v_1^* - u_4 v_3^* &= X_{43} - X_{21} \in \mathcal{L} & u_1 v_3^* - u_4 v_2^* &= X_{42} - X_{13} \in \mathcal{L} \\ u_2 v_2^* - u_4 v_4^* &= X_{44} - X_{22} \in \mathcal{L} & u_1 v_4^* - 2u_4 v_1^* &= 2X_{41} - X_{14} \in \mathcal{L} \end{aligned}$$

Let e_1, \dots, e_4 be the standard basis for \mathbb{C}^4 . One can check that

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} e_4 \\ e_3 \\ e_2 \\ e_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} e_4 \\ e_3 \\ -e_2 \\ -e_1 \end{bmatrix}$$

is a non-trivial solution.

4. Spans of products

While transitive subspaces are not algebras, an algebra can be obtained by taking spans of products. Thus in the matrix case, one eventually obtains \mathbb{M}_n . In many cases, this happens very quickly. In particular, the order of transitivity increases quickly.

Proposition 4.1. *If $\mathcal{L} \subset \mathbb{M}_l$ is a transitive subspace which is spanned by its rank one elements, then $\text{span } \mathcal{L}^2 = \mathbb{M}_l$.*

Proof. \mathcal{L} has no kernel; so we may select l non-zero rank one elements $R_i = x_i y_i^*$ in \mathcal{L} such that $\{y_1, \dots, y_l\}$ forms a basis for \mathcal{H}_l . Every matrix in \mathbb{M}_l may be written as $T = \sum_{i=1}^l u_i y_i^*$. Choose $A_i \in \mathcal{L}$ so that $A_i x_i = u_i$. Then $T = \sum_{i=1}^l A_i R_i$ belongs to $\text{span } \mathcal{L}^2$. ■

Example 4.2. Consider the space \mathcal{L} from Example 2.2. Let

$$X = \begin{bmatrix} a_1 & b_1 & h_1 & 2g_1 \\ c_1 & d_1 & f_1 & e_1 \\ e_1 & f_1 & a_1 & b_1 \\ g_1 & h_1 & c_1 & d_1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} a_2 & b_2 & h_2 & 2g_2 \\ c_2 & d_2 & f_2 & e_2 \\ e_2 & f_2 & a_2 & b_2 \\ g_2 & h_2 & c_2 & d_2 \end{bmatrix}$$

be arbitrary elements of \mathcal{L} . Let $Z = XY$, and consider the diagonal elements z_{ii} of Z , $1 \leq i \leq 4$. Then

$$\begin{aligned} z_{11} &= a_1a_2 + b_1c_2 + h_1e_2 + 2g_1g_2 \\ z_{22} &= c_1b_2 + d_1d_2 + f_1f_2 + e_1h_2 \\ z_{33} &= e_1h_2 + f_1f_2 + a_1a_2 + b_1c_2 \\ z_{44} &= 2g_1g_2 + h_1e_2 + c_1b_2 + d_1d_2. \end{aligned}$$

Thus $z_{11} + z_{22} = z_{33} + z_{44}$ for all $Z = [z_{ij}] \in \mathcal{L}^2$. Since $I \in \mathcal{L}$,

$$\text{span}\{\mathcal{L}, \mathcal{L}^2\} = \text{span } \mathcal{L}^2 \neq \mathbb{M}_4.$$

Note that $\text{span}\{\mathcal{L}, \mathcal{L}^2, \mathcal{L}^3\} = \text{span } \mathcal{L}^3 = \mathbb{M}_4$.

The following concept is a substantial weakening of the notion of k -transitivity.

Definition 4.3. A subspace $\mathcal{L} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$ is *k-separating* if for every set x_1, \dots, x_k of independent vectors in \mathcal{H} , there is an $L \in \mathcal{L}$ so that $Lx_i = 0$ for $1 \leq i \leq k-1$ and $Lx_k \neq 0$.

The simple result below shows why the property of being k -separating is nice to have when considering products of spaces.

Lemma 4.4. Suppose that $\mathcal{L}_1, \mathcal{L}_2$ are subspaces of \mathbb{M}_n with \mathcal{L}_1 transitive and \mathcal{L}_2 k -separating. Then $\text{span } \mathcal{L}_1\mathcal{L}_2$ is k -transitive.

Proof. Let x_1, \dots, x_k be linearly independent vectors, and let vectors y_1, \dots, y_k be given. Use the k -separating property to select elements B_1, \dots, B_k in \mathcal{L}_2 such that $B_ix_j = \delta_{ij}z_i$, where z_i are non-zero vectors. By the transitivity of \mathcal{L}_1 , select elements A_1, \dots, A_k in \mathcal{L}_1 so that $A_iz_i = y_i$. Then $\sum_{i=1}^k A_iB_i$ belongs to $\text{span } \mathcal{L}_1\mathcal{L}_2$ and takes x_i to y_i for $1 \leq i \leq k$. Therefore this space is k -transitive. ■

Since \mathbb{M}_n is the unique n -transitive subspace of itself, we obtain:

Corollary 4.5. Suppose that $\mathcal{L}_1, \mathcal{L}_2$ are subspaces of \mathbb{M}_n with \mathcal{L}_1 transitive and \mathcal{L}_2 n -separating. Then $\text{span } \mathcal{L}_1\mathcal{L}_2 = \mathbb{M}_n$.

Lemma 4.6. Let $1 \leq k < \min\{m, n\}$. Then a k -transitive subspace \mathcal{L} of \mathbb{M}_{mn} is $(k+1)$ -separating.

Proof. Given $x_1, \dots, x_{k+1} \in \mathbb{C}^n$ linearly independent, set $\mathcal{X} = \text{span}\{x_1, \dots, x_{k+1}\}$. Then the restriction $\mathcal{M} = \mathcal{L}|_{\mathcal{X}} \subset \mathbb{M}_{m, k+1}$ is k -transitive. By Theorem 1.2, we have $\dim \mathcal{M} \geq k(m+1)$. The subspace of \mathcal{M} which vanishes on $\text{span}\{x_1, \dots, x_k\}$ has km linear conditions imposed, and hence it has dimension at least k . Thus there are elements of \mathcal{M} which annihilate x_1, \dots, x_k and are non-zero on x_{k+1} . ■

Example 4.7. The set \mathfrak{T}_n of $n \times n$ Toeplitz matrices ($n \geq 3$) is an example of a space which is 2-separating but not 3-separating. Indeed, the fact that \mathfrak{T}_n is 1-transitive implies that it is 2-separating by the above result. On the other hand, if the first and last columns of a matrix T in \mathfrak{T}_n are both zero, then all entries of T are zero, so that \mathfrak{T}_n is not 3-separating.

Example 4.8. The converse of Lemma 4.6 is false.

If \mathcal{M} is a k -transitive but not $(k+1)$ -transitive subspace of \mathbb{M}_{mn} , consider the subspace \mathcal{L} of $\mathbb{M}_{m+1,n}$ of the form $L = \begin{bmatrix} x \\ M \end{bmatrix}$ where x is an arbitrary row vector in \mathbb{C}^n and $M \in \mathcal{M}$. Then this is an n -separating space which is k -transitive but not $(k+1)$ -transitive.

In particular, if we take $\mathcal{M} = \{0\}$, then \mathcal{L} is n -separating but is not even 1-transitive.

Theorem 4.9. Suppose that subspaces $\mathcal{K} \subset \mathbb{M}_{mn}$ and $\mathcal{L} \subset \mathbb{M}_{np}$ are k -transitive and l -transitive respectively. Then the product $\text{span } \mathcal{KL}$ is $\min\{k+l, m, p\}$ -transitive.

Proof. We know that \mathcal{K}_\perp contains no non-zero element of rank at most k ; and \mathcal{L}_\perp contains no non-zero element of rank at most l .

Assume first that $l < \min\{n, p\}$. Suppose that $A \in (\mathcal{KL})_\perp$ and satisfies $1 \leq \text{rank } A \leq k+l$. Then $0 = \text{Tr}(KLA)$ for all $K \in \mathcal{K}$ and $L \in \mathcal{L}$. Hence $LA \in \mathcal{K}_\perp$ for all $L \in \mathcal{L}$. As \mathcal{L} is $(l+1)$ -separating by Lemma 4.6, select $L \in \mathcal{L}$ which is non-zero on some vector in the range of A and annihilates $\min\{l, \text{rank } A - 1\}$ independent vectors in the range of A . Then

$$1 \leq \text{rank } LA \leq (k+l) - l = k.$$

This contradicts the k -transitivity of \mathcal{K} . Thus $\text{span } \mathcal{KL}$ is $(k+l)$ -transitive whenever $k+l < \min\{m, p\}$. But if $k+l \geq \min\{m, p\}$, then this shows that $(\mathcal{KL})_\perp = \{0\}$. Hence $\text{span } \mathcal{KL} = \mathbb{M}_{mp}$ is $\min\{m, p\}$ -transitive.

We obtain a similar conclusion if $k < \min\{m, n\}$. If $k = \min\{m, n\}$ and $l = \min\{n, p\}$, then $\mathcal{K} = \mathbb{M}_{mn}$ and $\mathcal{L} = \mathbb{M}_{np}$. Thus $\text{span } \mathcal{KL} = \mathbb{M}_{mp}$ is $\min\{m, p\}$ transitive. ■

For \mathcal{M} a subspace of \mathbb{M}_n , let $\mathcal{M}^* = \{M^* : M \in \mathcal{M}\}$, again considered as a subspace of \mathbb{M}_n .

Proposition 4.10. If $\mathcal{L} \subset \mathbb{M}_n$ is transitive and spanned by its rank r elements and $\mathcal{M}^* \subset \mathbb{M}_n$ is r -separating, then $\text{span } \mathcal{LM} = \mathbb{M}_n$.

Proof. If $L \in \mathcal{L}$ has rank at most r , then for any vector $u \in L\mathcal{H}$ and $0 \neq x \in \mathcal{H}$, we will show that $ux^* \in \mathcal{LM}$. Indeed, we may write $L = \sum_{i=1}^s u_i v_i^*$ where $u_1 = u$, $s \leq r$ and $\{v_1, \dots, v_s\}$ are linearly independent. Select $M \in \mathcal{M}$ so that $M^* v_1 = x$ and $M^* v_i = 0$ for $2 \leq i \leq s$. Then

$$LM = \sum_{i=1}^s u_i v_i^* M = \sum_{i=1}^s u_i (M^* v_i)^* = ux^*.$$

As the ranges of elements (of rank at most r) of \mathcal{L} span \mathcal{H} , the result follows. ■

If $\mathcal{M} \subseteq \mathbb{M}_n$ is r -transitive for some $r \geq 1$, then so is \mathcal{M}^* . This is easily seen by considering Azoff's characterization of r -transitivity in terms of the pre-annihilator of \mathcal{M} . It then follows from Lemma 4.6 that \mathcal{M}^* is $(r+1)$ -separating. When $r = 0$, no such statement holds. So we obtain:

Corollary 4.11. *If $\mathcal{L} \subset \mathbb{M}_n$ is transitive and spanned by its rank r elements and $\mathcal{M} \subset \mathbb{M}_n$ is $\max\{r-1, 1\}$ -transitive, then $\text{span } \mathcal{L}\mathcal{M} = \mathbb{M}_n$.*

This allows an extension of Proposition 4.1 and Example 4.2.

Corollary 4.12. *If $\mathcal{L} \subset \mathbb{M}_n$ is transitive and is spanned by its rank r elements, then $\text{span } \mathcal{L}^{r+1} = \mathbb{M}_n$.*

Proof. By Theorem 4.9, $\text{span } \mathcal{L}^r$ is r -transitive. So by the previous lemma, we conclude that $\text{span } \mathcal{L}^{r+1} = \mathbb{M}_n$. ■

5. Invertibles

Proposition 5.1. *If $\mathcal{L} \subset \mathcal{M}_n$ is a subspace consisting of singular matrices, then \mathcal{L} is not transitive.*

Proof. Let k be the largest rank of an element of \mathcal{L} , and fix $A \in \mathcal{L}$ with $\text{rank } A = k$. Transitivity is unchanged if \mathcal{L} is multiplied on either side by invertible operators. So after such a change, we may suppose that $A = A^2 = A^*$ is a projection.

Decompose $\mathcal{H} = A\mathcal{H} \oplus (I - A)\mathcal{H}$. With this decomposition, each element $L \in \mathcal{L}$ has the form $L \simeq \begin{bmatrix} L_0 & E \\ F & L_1 \end{bmatrix}$. Since $L + tA = \begin{bmatrix} L_0 + tI_k & E \\ F & L_1 \end{bmatrix}$, the $1, 1$ entry is invertible for large t , and so it factors as

$$\begin{bmatrix} L_0 + tI_k & 0 \\ F & I_{n-k} \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & L_1 - F(L_0 + tI_k)^{-1}E \end{bmatrix} \begin{bmatrix} I_k & (L_0 + tI_k)^{-1}E \\ 0 & I_{n-k} \end{bmatrix}.$$

From this, it follows that

$$k \geq \text{rank}(L + tA) = k + \text{rank}(L_1 - F(L_0 + tI_k)^{-1}E).$$

Therefore $L_1 = F(L_0 + tI_k)^{-1}E$. As the right side tends to 0 as $t \rightarrow \infty$, $L_1 = 0$ for all $L \in \mathcal{L}$.

This shows that $(I - A)\mathcal{L}(I - A) = 0$ and so \mathcal{L} is not transitive. ■

Proposition 5.2. *If $\mathcal{L} \subset \mathbb{M}_n$ is transitive, let r be the minimal rank of non-zero elements of \mathcal{L} and let s be the largest rank of singular elements of \mathcal{L} . Then $r + s \geq n$.*

Proof. Let $F \in \mathcal{L}$ with $\text{rank } F = r$. Suppose first that there is an invertible element $A \in \mathcal{L}$ such that $0 \neq \lambda \in \sigma(A^{-1}F)$. Then

$$0 = \det(\lambda I - A^{-1}F) = \det(A^{-1}) \det(\lambda A - F).$$

As $\det(A^{-1}) \neq 0$, $\lambda A - F$ is singular and thus has rank at most s . But clearly it has rank at least $n - r$. So $r + s \geq n$.

Otherwise, for every invertible A in \mathcal{L} , $A^{-1}F$ is nilpotent. By our Proposition 5.1, \mathcal{L} contains invertible elements. Select an invertible $B \in \mathcal{L}$ so that among the elements of the form $A^{-1}F$ for invertible A in \mathcal{L} , the operator $F_0 = B^{-1}F$ is nilpotent of the greatest index, say $m+1$. That is, $F_0^m \neq 0 = F_0^{m+1} = (A^{-1}F)^{m+1}$ for all $A \in \mathcal{L}$ which are invertible.

For any $L \in \mathcal{L}$, and sufficiently small μ , the operator $B - \mu L$ is invertible. So

$$0 = ((B - \mu L)^{-1}F_0)^{m+1} = (I - \mu B^{-1}L)^{-1}F_0((I - \mu B^{-1}L)^{-1}F_0)^m$$

Therefore expanding $(I - \mu B^{-1}L)^{-1} = \sum_{k \geq 0} \mu^k (B^{-1}L)^k$,

$$0 = F_0((I - \mu B^{-1}L)^{-1}F_0)^m = \sum_{k \geq 0} \mu^k X_k.$$

All coefficients of this power series must vanish, and in particular

$$0 = X_1 = F_0 B^{-1} L F_0^m + F_0^2 B^{-1} L F_0^{m-1} + \cdots + F_0^m B^{-1} L F_0.$$

Multiply on the left by F_0^{m-1} to obtain $F_0^m B^{-1} L F_0^m = 0$ for all $L \in \mathcal{L}$. This means that $\mathcal{L} \text{Ran } F_0^m \subset \ker F_0^m B^{-1}$. Both $\text{Ran } F_0^m$ and $\ker F_0^m B^{-1} = B \ker F_0^m$ are proper subspaces, so this contradicts the transitivity of \mathcal{L} . This contradiction establishes the result. \blacksquare

6. Infinite dimensional results

In this section we examine to what degree the results of the previous sections extend to the infinite dimensional setting. As we shall see, there are more negative results than positive results. We begin with an infinite dimensional version of Proposition 5.1.

Recall that if $T \in \mathcal{B}(\mathcal{H})$, then an element λ of the spectrum $\sigma(T)$ of T is called a *Riesz point* if λ is an isolated point of $\sigma(T)$, and if $\bigvee_{k \geq 1} \ker(\lambda I - T)^k$ is finite dimensional. In particular, λ is not an element of the essential spectrum $\sigma_e(T)$ of T .

Theorem 6.1. *Let \mathcal{L} be a subspace of $\mathcal{B}(\mathcal{H})$ consisting of singular operators. If \mathcal{L} contains an operator A with 0 as a Riesz point of its spectrum, then \mathcal{L} is not topologically transitive.*

Proof. After a similarity, $A \simeq \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix}$ where $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ and \mathcal{M} is the spectral subspace $\ker A^p$ for all sufficiently large p . So A_1 is invertible in $\mathcal{B}(\mathcal{M}^\perp)$, and $A^p \simeq \begin{bmatrix} 0 & 0 \\ 0 & A_1^p \end{bmatrix}$.

Any $L \in \mathcal{L}$ has the form $L \simeq \begin{bmatrix} L_0 & E \\ F & L_1 \end{bmatrix}$. As in the proof of Proposition 5.1, the $2, 2$ entry of $L_t := L + tA^p$ is invertible for all t sufficiently large. Thus L_t is Fredholm of index 0. Since $0 \in \sigma(L_t)$, it must have non-trivial kernel. However, L_t factors as above as

$$\begin{bmatrix} I_{\mathcal{M}} & E(L_1 + tA_1^p)^{-1} \\ 0 & I_{\mathcal{M}^\perp} \end{bmatrix} \begin{bmatrix} L_0 - E(L_1 + tA_1^p)^{-1}F & 0 \\ 0 & I_{\mathcal{M}^\perp} \end{bmatrix} \begin{bmatrix} I_{\mathcal{M}} & 0 \\ F & L_1 + tA_1^p \end{bmatrix}.$$

The middle factor must have kernel. Letting $t \rightarrow \infty$ shows that L_0 is singular.

This shows that $P_{\mathcal{M}}\mathcal{L}P_{\mathcal{M}}$ consists of singular matrices. Hence it is not transitive by Proposition 5.1. Since the rank of $P_{\mathcal{M}}$ is finite, \mathcal{L} is not topologically transitive, as observed in the last paragraph of the introduction. ■

Example 6.2. The set \mathfrak{K} of compact operators is transitive and singular.

To get even closer to the hypotheses of Theorem 6.1, consider the set $\mathcal{L} = \mathbb{C}S^* + \mathfrak{K}$, where S^* is the backward shift. Every element is singular, and there are elements $A \in \mathcal{L}$ such that $0 \notin \sigma_e(A)$ and $\ker A \neq 0$. Nevertheless, this is a transitive space.

The spaces \mathfrak{T} and \mathfrak{H} of Toeplitz and of Hankel operators appear frequently as counterexamples to possible extensions of our finite dimensional results. Let dm denote normalized Lebesgue measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The set $\{e_n = e^{in\theta}\}_{n \in \mathbb{Z}}$ forms an orthonormal basis for the Hilbert space $L^2(\mathbb{T}) = L^2(\mathbb{T}, dm)$. Let us denote by $\mathcal{M}^\infty(\mathbb{T})$ the space $\{M_f : f \in L^\infty(\mathbb{T}, dm)\}$ of multiplication operators on $L^2(\mathbb{T})$. It is well-known and routine to verify that $T = [t_{ij}]$ lies in $\mathcal{M}^\infty(\mathbb{T})$ if and only if $T \in \mathcal{B}(L^2(\mathbb{T}))$ and $t_{ij} = t_{i+k, j+k}$ for all $i, j, k \in \mathbb{Z}$.

We denote by $H^2(\mathbb{T})$ the Hardy space $\overline{\text{span}}\{e_n\}_{n=0}^\infty$ of analytic functions in $L^2(\mathbb{T})$. If P denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$, then the Toeplitz operators are the elements of $\mathfrak{T} = P\mathcal{M}^\infty|_{H^2(\mathbb{T})}$; and with $Q = (I - P)$, the set of Hankel operators is $\mathfrak{H} = Q\mathcal{M}^\infty|_{H^2(\mathbb{T})}$.

That the Toeplitz operators are topologically transitive may be found in [3]. We include a slightly different proof to that found there.

Example 6.3. The spaces \mathfrak{H} and \mathfrak{T} of Hankel and Toeplitz operators are topologically transitive.

We show that the space $\mathcal{M}^\infty P$ is topologically transitive. Since \mathfrak{T} and \mathfrak{H} are compressions of $\mathcal{M}^\infty P$, it immediately follows that they too are topologically transitive.

Consider $A \in (\mathcal{M}^\infty P)_\perp$ with $\text{rank } A = 1$. Then $A = fg^*$ for some $f \in H^2(\mathbb{T})$ and $g \in L^2(\mathbb{T})$. Since $A \neq 0$, neither f nor g is zero. The condition $A \in (\mathcal{M}^\infty P)_\perp$ implies that $\text{tr}(M_h PA) = \text{tr}(M_h fg^*) = \langle hf, g \rangle = 0$ for all $h \in L^\infty(\mathbb{T})$. That is, $\int_{\mathbb{T}} hf\bar{g} = 0$ for all $h \in L^\infty(\mathbb{T})$. Since $f\bar{g} \in L^1(\mathbb{T}) = (L^\infty(\mathbb{T}))_\perp$, it follows that $f\bar{g} = 0$ a.e.. The classical F. and M. Riesz Theorem (see [5, Theorem 6.13]) asserts that as $0 \neq f \in H^2(\mathbb{T})$, the set $\{z \in \mathbb{T} : f(z) = 0\}$ has measure 0. From this it follows that $g = 0$ a.e., a contradiction. Thus $\mathcal{M}^\infty P$ is topologically transitive.

We observe that an analogous argument may be used to establish the fact that $Q\mathcal{M}^\infty$ is also topologically transitive.

Note also that $\mathfrak{T}e_0 = \{T_h e_0 = Ph : h \in L^\infty(\mathbb{T})\}$ is dense in $H^2(\mathbb{T})$, but is not everything. Thus \mathfrak{T} is an example of a topologically transitive space which is not transitive.

Definition 6.4. A subspace $\mathcal{L} \subseteq \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be *totally separating* if \mathcal{L} is k -separating for all $k \geq 2$.

There is no point in defining *totally topologically transitive* in the analogous way, because this would just say that \mathcal{L} is SOT-dense in $\mathcal{B}(\mathcal{H})$. A natural modification of Example 4.8 shows that for each $k \geq 1$ there are totally separating spaces which are topologically k -transitive but not topologically $(k+1)$ -transitive.

Proposition 6.5. Let \mathcal{L}_1 and \mathcal{L}_2 be subspaces of $\mathcal{B}(\mathcal{H})$, and suppose that \mathcal{L}_1 is topologically transitive.

- (a) If \mathcal{L}_2 is k -separating for some $k \geq 1$, then $\text{span } \mathcal{L}_1 \mathcal{L}_2$ is topologically k -transitive.
- (b) If \mathcal{L}_2 is totally separating, then $\text{span } \mathcal{L}_1 \mathcal{L}_2$ is dense in the strong operator topology on $\mathcal{B}(\mathcal{H})$.

Proof. (a) Choose $x_1, \dots, x_k \in \mathcal{H}$ linearly independent. Let $\varepsilon > 0$, and choose any $y_1, \dots, y_k \in \mathcal{H}$. Since \mathcal{L}_2 is k -separating, we can find operators L_1, \dots, L_k in \mathcal{L}_2 so that $L_i x_i \neq 0$, but $L_i x_j = 0$ for all $1 \leq i \neq j \leq k$. Since \mathcal{L}_1 is topologically transitive, we can find $K_1, \dots, K_k \in \mathcal{L}_1$ with $\|K_j(L_j x_j) - y_j\| < \varepsilon/k$ for $1 \leq j \leq k$.

Let $T = \sum_{j=1}^k K_j L_j \in \mathcal{L}_1 \mathcal{L}_2$. Then

$$\|Tx_j - y_j\| = \left\| \left(\sum_{i=1}^n K_i L_i \right) x_j - y_j \right\| = \|K_j L_j x_j - y_j\| < \varepsilon/k,$$

for each $1 \leq j \leq k$. Since $\{x_j\}_{j=1}^k$ linearly independent and $\{y_j\}_{j=1}^k$ are arbitrary, $\text{span } \mathcal{L}_1 \mathcal{L}_2$ is topologically k -transitive.

(b) By part (a), $\text{span } \mathcal{L}_1 \mathcal{L}_2$ is topologically k -transitive for all $k \geq 1$. By the comments preceding the proposition, this says that $\text{span } \mathcal{L}_1 \mathcal{L}_2$ is dense in the strong operator topology in $\mathcal{B}(\mathcal{H})$. ■

Example 6.6. The Hankel operators \mathfrak{H} and the Toeplitz operators \mathfrak{T} are products of two topologically 1-transitive spaces, but they are not topologically 2-transitive nor even 2-separating.

Indeed, $\mathfrak{H} = (Q\mathcal{M}^\infty)(\mathcal{M}^\infty P)$ and $\mathfrak{T} = (P\mathcal{M}^\infty)(\mathcal{M}^\infty P)$. In finite dimensions, we have seen that the product of two transitive spaces is 2-transitive. We have also seen that in the finite dimensional setting, 1-transitive spaces are automatically 2-separating.

A typical operator in \mathfrak{H} admits an infinite matrix representation of the form

$$H = \begin{bmatrix} a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & \dots & \\ a_3 & \dots & \dots & \dots \\ \vdots & & & \end{bmatrix}$$

relative to the bases $\{e_n\}_{n=0}^\infty$ for $H^2(\mathbb{T})$ and $\{e_n\}_{n=-1}^\infty$ for $(H^2(\mathbb{T}))^\perp$. Therefore $He_0 = 0$ implies $H = 0$, and hence \mathfrak{H} is neither 2-separating nor topologically 2-transitive.

Similarly, if $T \in \mathfrak{T}$ and $Te_n = 0$, then $Te_i = 0$ for $0 \leq i < n$. So \mathfrak{T} is also neither 2-separating nor topologically 2-transitive. This can be contrasted with Example 4.7 where it is shown that \mathfrak{T}_n is 2-separating but not 3-separating.

The following technical result will be used in the proof of Proposition 6.8.

Lemma 6.7. *Suppose $\mathcal{L}_2 \subseteq \mathcal{B}(\mathcal{H})$ contains a sequence $(F_m)_{m=1}^\infty$ of operators with*

- 1) $\text{rank } F_m = m$, $m \geq 1$;
- 2) $\ker F_{m+1} \subseteq \ker F_m$ for all $m \geq 1$; and
- 3) $\bigcap_{m \geq 1} \ker F_m = \{0\}$.

If \mathcal{L}_1 is topologically transitive, then $\text{span } \mathcal{L}_1 \mathcal{L}_2$ is dense in the weak operator topology on $\mathcal{B}(\mathcal{H})$.

Proof. Choose a sequence $(F_m)_{m=1}^\infty \subseteq \mathcal{L}_2$ as in the statement of the Lemma. For each $m \geq 1$, let $\mathcal{H}_m = (\ker F_m)^\perp$, so that $\mathcal{H}_m \subseteq \mathcal{H}_{m+1}$, and $\dim \mathcal{H}_m = m$ for all m . Fix $e_1 \in \mathcal{H}_1$ with $\|e_1\| = 1$, and for $m \geq 2$, choose $e_m \in \mathcal{H}_m \ominus \mathcal{H}_{m-1}$ with $\|e_m\| = 1$. The third hypothesis above guarantees that $\{e_m\}_{m=1}^\infty$ spans \mathcal{H} , and thus forms an orthonormal basis for \mathcal{H} . Our goal is to show that if P_m is the orthogonal projection of \mathcal{H} onto \mathcal{H}_m , then $\overline{\text{span } \mathcal{L}_1 \mathcal{L}_2}^{\text{wot}}$ contains $\mathcal{B}(\mathcal{H})P_m$ for all $m \geq 1$. Since the latter set is clearly dense in the weak operator topology, so is the former.

Let $T \in \mathcal{B}(\mathcal{H})$ be arbitrary, and set $\varepsilon > 0$. As $e_m \in \mathcal{H}_m$, $z_m := F_m e_m \neq 0$. Now fix $R_m \in \mathcal{L}_1$ so that $\|R_m F_m e_m - T e_m\| < \varepsilon/m$. Next, because $e_{m-1} \in \mathcal{H}_{m-1}$, we have that $F_{m-1} e_{m-1} \neq 0$. Select $R_{m-1} \in \mathcal{L}$ so that

$$\|R_{m-1} z_{m-1} - (T e_{m-1} - R_m F_m e_{m-1})\| < \varepsilon/m.$$

Observe that $R_{m-1} F_{m-1} e_m = 0$, so that

$$\|(R_{m-1} F_{m-1} + R_m F_m) e_m - T e_m\| < \varepsilon/m.$$

More generally, having chosen $R_m, R_{m-1}, \dots, R_{m-k}$, we can choose $R_{m-(k+1)}$ in \mathcal{L}_1 so that

$$\|R_{m-(k+1)}F_{m-(k+1)}e_{m-(k+1)} - (T - \sum_{j=m-k}^m R_j F_j)e_{m-(k+1)}\| < \varepsilon/m.$$

It follows that $Q_m = \sum_{j=1}^m R_j F_j$ satisfies $\|Q_m e_r - T e_r\| < \varepsilon/m$ for $1 \leq r \leq m$. Since $Q_m = Q_m P_m$, $\|Q_m - T P_m\| < \varepsilon$. Finally, since Q_m belongs to $\text{span } \mathcal{L}_1 \mathcal{L}_2$ and $\varepsilon > 0$ is arbitrary, $\mathcal{B}(\mathcal{H})P_m \subset \overline{\text{span } \mathcal{L}_1 \mathcal{L}_2}$. Therefore $\overline{\text{span } \mathcal{L}_1 \mathcal{L}_2}^{\text{wot}} = \mathcal{B}(\mathcal{H})$. ■

Proposition 6.8. *Suppose that $\mathcal{L} \subseteq \mathcal{B}(\mathcal{H})$ is topologically transitive, and let \mathfrak{H} denote the space of Hankel operators considered as a subspace of $\mathcal{B}(\mathcal{H})$. Then*

$$\overline{\text{span } \mathcal{L} \mathfrak{H}}^{\text{wot}} = \overline{\text{span } \mathfrak{H} \mathcal{L}}^{\text{wot}} = \mathcal{B}(\mathcal{H}).$$

In particular, $\text{span } \mathfrak{H}^2$ is weak operator dense in $\mathcal{B}(\mathcal{H})$.

Proof. For each $n \geq 1$, the rank n operators

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 0 & 0 & \dots \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & \dots \\ \vdots & & & & & & & \dots \\ 1 & 1 & \dots & & & & & \\ 1 & 0 & & & & & & \\ 0 & \dots & & & & & & \end{bmatrix}$$

lie in \mathfrak{H} and satisfy the conditions of Lemma 6.7. Thus we may conclude that $\overline{\text{span } \mathcal{L} \mathfrak{H}}^{\text{wot}} = \mathcal{B}(\mathcal{H})$.

Now \mathcal{L} topologically transitive implies that \mathcal{L}^t is topologically transitive. Since $\mathfrak{H} = \mathfrak{H}^t$, it follows that $(\mathfrak{H} \mathcal{L})^t = \mathcal{L}^t \mathfrak{H}$ is weak operator dense in $\mathcal{B}(\mathcal{H})$, whence $\overline{\text{span } \mathfrak{H} \mathcal{L}}^{\text{wot}} = \mathcal{B}(\mathcal{H})$. ■

Let us next consider $\text{span } \mathfrak{T}^2$. Let $E_{ij} = e_i e_j^*$, $i, j \geq 0$ denote the matrix units of $\mathcal{B}(H^2(\mathbb{T}))$. Letting $S = PM_z|_{H^2(\mathbb{T})} \in \mathfrak{T}$, S is unitarily equivalent to the unilateral forward shift, and a routine calculation reveals that for $i, j \geq 0$,

$$E_{i,j} = S^i(I - SS^*)(S^*)^j = S^i(S^*)^j - S^{i+1}(S^*)^{j+1}.$$

Since $S^k, (S^*)^l \in \mathfrak{T}$ for all $k, l \geq 0$, $E_{ij} \in \text{span } \mathfrak{T}^2$ for all $i, j \geq 0$. Thus the norm closure of $\text{span } \mathfrak{T}^2$ contains all compact operators, and is therefore transitive.

So far we have not been able to determine whether $\mathcal{L} \subseteq \mathcal{B}(H^2(\mathbb{T}))$ topologically transitive implies that $\overline{\text{span } \mathcal{L} \mathfrak{T}}^{\text{wot}} = \mathcal{B}(\mathcal{H})$.

As we have seen in Section 4, if $\mathcal{L} \subseteq \mathbb{M}_n$ is transitive, then $\text{span } \mathcal{L}^r = \mathbb{M}_n$ for some $1 \leq r \leq n$. It would be interesting to find estimates for

$$\kappa_n := \min\{1 \leq r \leq n : \text{span } \mathcal{L}^r = \mathbb{M}_n \text{ for all } \mathcal{L} \subseteq \mathbb{M}_n \text{ transitive}\}.$$

In particular, is $(\kappa_n)_{n=1}^\infty$ bounded? In the infinite dimensional setting, does there always exist some $r \geq 1$ so that $\mathcal{L} \subseteq \mathcal{B}(\mathcal{H})$ topologically transitive implies that $\overline{\text{span } \mathcal{L}^r}^{\text{WOT}} = \mathcal{B}(\mathcal{H})$? More generally, does there exist $s \geq 1$ so that if $\mathcal{L}_1, \dots, \mathcal{L}_s$ are topologically transitive subspaces of $\mathcal{B}(\mathcal{H})$, then $\overline{\text{span } \mathcal{L}_1 \mathcal{L}_2 \cdots \mathcal{L}_s}^{\text{WOT}} = \mathcal{B}(\mathcal{H})$? Observe that if \mathcal{L} is a unital subalgebra of $\mathcal{B}(\mathcal{H})$, then $\mathcal{L}^s = \mathcal{L}$ for all $s \geq 1$, and so an affirmative answer to this question would resolve the Transitive Algebra Problem. However, since we are only asking for each \mathcal{L}_j to be a topologically transitive *subspace*, it seems plausible that one might be able to find a counterexample for each $s \geq 1$. Finally, by Example 6.6, if such an s exists, then $s \geq 3$.

Example 6.9. A subspace of $\mathcal{B}(\mathcal{H})$ can be topologically $(n-1)$ -transitive but not n -separating.

Let $\{e_n = z^n : n \in \mathbb{Z}\}$ be the standard orthonormal basis for $L^2(\mathbb{T})$, and set $\mathcal{W}_n = \text{span}\{e_1, e_2, \dots, e_n\}$. Consider the space $\mathcal{L}_n \subseteq \mathcal{B}(\mathcal{W}_n, L^2(\mathbb{T}))$ of operators of the form $A = [a_{ij}]$ where $\sum_{k=1}^n a_{i+k,k} = 0$, $i \in \mathbb{Z}$. To see that \mathcal{L}_n is not n -separating, it suffices to observe that if the first $(n-1)$ columns of \mathcal{L}_n are zero, then the last column of \mathcal{L}_n is necessarily zero as well.

The proof that \mathcal{L}_n is topologically $(n-1)$ -transitive relies upon the structure of $(\mathcal{L}_n)_\perp$. As in our analysis of the finite dimensional setting, we identify $(\mathcal{L}_n)_\perp$ with the set of trace class operators $B = [b_{ij}] \subseteq \mathcal{B}(\mathcal{W}_n, L^2(\mathbb{T}))$ which satisfy $\sum_{i=1}^n \sum_{j \in \mathbb{Z}} a_{ij} b_{ji} = 0$ for all $A = [a_{ij}] \in \mathcal{L}_n$. A routine calculation then shows that $B \in (\mathcal{L}_n)_\perp$ implies that $b_{k,i+k} = b_{1,i+1}$ for all $2 \leq k \leq n$.

If we consider the rows of B as vectors in $L^2(\mathbb{T})$, this says that $B \in (\mathcal{L}_n)_\perp$ if and only if $B^t = [f \quad zf \quad z^2 f \quad \dots \quad z^{n-1} f]$ for some $f \in L^2(\mathbb{T})$. If \mathcal{L}_n were not topologically $(n-1)$ -transitive, then we could find such a $B \neq 0$ with $\text{rank } B$ at most $n-1$. In particular, $\text{rank } B^t = \text{rank } B \leq n-1$ and so $\ker B^t \neq 0$. Choose a vector $0 \neq \sum_{i=1}^n \alpha_i e_i \in \ker B^t$. Then $(\sum_{i=1}^n \alpha_i z^{i-1})f = 0$ a.e. Since $\sum_{i=1}^n \alpha_i z^{i-1}$ is a non-trivial polynomial, it has at most finitely many zeroes. Therefore $f = 0$ a.e., contradicting $B \neq 0$. Hence \mathcal{L}_n is topologically $(n-1)$ -transitive.

Example 6.10. The intersection of a descending sequence of WOT-closed transitive spaces need not be topologically transitive.

As before, we let $\{e_n : n \in \mathbb{Z}\}$ be the standard basis for $L^2(\mathbb{T})$. Let R_n denote the orthogonal projection of $L^2(\mathbb{T})$ onto $\text{span}\{e_k : |k| \leq n\}$. Then $R_n \mathcal{M}^\infty|_{R_n L^2(\mathbb{T})}$ considered as a subspace of $\mathcal{B}(R_n L^2(\mathbb{T}))$ is clearly unitarily equivalent to the Toeplitz matrices on \mathcal{H}_{2n+1} , and so it is transitive (see Example 1.5).

Let $\mathcal{R}_n = \{X \in \mathcal{B}(L^2(\mathbb{T})) : R_n X R_n \in R_n \mathcal{M}^\infty R_n\}$. Then \mathcal{R}_n is transitive and WOT-closed. Indeed, decompose $L^2(\mathbb{T}) = R_n L^2(\mathbb{T}) \oplus (R_n L^2(\mathbb{T}))^\perp$. Then with respect to this decomposition, an element of \mathcal{R}_n has the form $\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$ where $X_1 \in R_n \mathcal{M}^\infty|_{R_n L^2(\mathbb{T})}$ and the other entries are arbitrary. As the matrix entries are independent and each corner is transitive, it follows easily that \mathcal{R}_n is transitive.

Observe, however, that $\bigcap_{n=1}^\infty \mathcal{R}_n = \mathcal{M}^\infty(\mathbb{T})$. Since $\mathcal{M}^\infty(\mathbb{T})$ has many proper closed invariant subspaces, it is not topologically transitive. So the intersection of a descending sequence of transitive spaces need not be topologically transitive.

Note that there are limits to the decreasing intersection of transitive spaces $\mathcal{L} = \bigcap_{n \geq 1} \mathcal{L}_n$. For if P and Q are rank n projections, then $P\mathcal{L}_nQ$ is a transitive subspace of $\mathcal{B}(Q\mathcal{H}, P\mathcal{H})$ for all n , and so has dimension at least $2n - 1$. Thus the same is true for the intersection. Moreover since a decreasing sequence of subspaces of a finite dimensional space is eventually constant, we see that $P\mathcal{L}Q$ is transitive whenever P and Q are finite rank. Our example shows that this estimate is sharp because the compression using $P\mathcal{H} = Q\mathcal{H} = \text{span}\{e_i : 0 \leq i < n\}$ yields \mathfrak{T}_n as the intersection; and it has dimension exactly $2n - 1$.

Theorem 6.11. *Suppose that $\mathcal{L}, \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ are topologically transitive. If \mathcal{L} is contained in the WOT-closed span of its rank one elements, then the norm closure of $\text{span } \mathcal{LM}$ is transitive.*

Proof. For each rank one element $uv^* \in \mathcal{L}$ and any element $M \in \mathcal{M}$, the product \mathcal{LM} contains $uv^*M = u(M^*v)^*$. By the topological transitivity of \mathcal{M} , the norm closure of \mathcal{LM} contains ue_j^* where $\{e_j\}$ is an orthonormal basis for \mathcal{H} . As \mathcal{L} is topologically transitive and WOT-spanned by rank ones, the collection of such vectors u densely spans \mathcal{H} , from which it follows that the norm closure of $\text{span } \mathcal{LM}$ contains the compact operators. ■

We now give the infinite dimensional analogue of Theorem 3.2. The proof is different, and provides an alternate proof in the finite dimensional case.

Theorem 6.12. *Let $\mathcal{L} \subset \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{M} \subset \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$. Suppose that \mathcal{L} is topologically k -transitive, that it is contained in the WOT-closed span of its elements of rank at most r , and that \mathcal{M} is topologically rk -transitive. Then the spatial tensor product $\mathcal{L} \otimes \mathcal{M}$ is topologically k -transitive.*

Proof. Let $\{L_p : p \in \mathcal{P}\}$ be elements of \mathcal{L} of rank at most r with WOT-closed span containing all of \mathcal{L} . Suppose that $R \in \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{K}_2, \mathcal{H}_1 \otimes \mathcal{K}_1)$ has rank at most k and is orthogonal to $\mathcal{L} \otimes \mathcal{M}$ in the trace pairing. This is equivalent to $\langle L_p \otimes M, R \rangle = 0$ for all $p \in \mathcal{P}$ and all $M \in \mathcal{M}$.

Fix $p \in \mathcal{P}$ and write $L_p = \sum_{s=1}^r u_s v_s^*$. Also let e be any fixed unit vector in \mathcal{H}_1 . We calculate

$$\begin{aligned} 0 &= \langle L_p \otimes M, R \rangle = \sum_{s=1}^r \langle (u_s v_s^* \otimes I)(I \otimes M), R \rangle \\ &= \sum_{s=1}^r \langle (u_s e^* \otimes I)(I \otimes M)(e v_s^* \otimes I), R \rangle \\ &= \sum_{s=1}^r \langle I \otimes M, (e v_s^* \otimes I) R (u_s e^* \otimes I) \rangle \\ &= \langle I \otimes M, e e^* \otimes \sum_{s=1}^r R_s \rangle \end{aligned}$$

$$= \langle ee^* \otimes M, ee^* \otimes \sum_{s=1}^r R_s \rangle = \langle M, \sum_{s=1}^r R_s \rangle$$

where $R_s \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$ is given by $ee^* \otimes R_s := (ev_s^* \otimes I)R(u_s e^* \otimes I)$; and therefore is an operator of rank at most k . Thus $\sum_{s=1}^r R_s$ has rank at most rk and is orthogonal to \mathcal{M} . Hence $\sum_{s=1}^r R_s = 0$.

Now let $P \in \mathcal{B}(\mathcal{K}_1)$ and $Q \in \mathcal{B}(\mathcal{K}_2)$ and observe that

$$\begin{aligned} 0 &= (I \otimes P)(ee^* \otimes \sum_{s=1}^r R_s)(I \otimes Q) \\ &= \sum_{s=1}^r (ev_s^* \otimes I)(I \otimes P)R_s(I \otimes Q)(u_s e^* \otimes I) \end{aligned}$$

Therefore reversing the computation above using $(I \otimes P)R(I \otimes Q)$ in place of R yields for all $p \in \mathcal{P}$ and $M \in \mathcal{M}$ that

$$\begin{aligned} 0 &= \langle L_p \otimes M, (I \otimes P)R(I \otimes Q) \rangle \\ &= \langle L_p \otimes I, (I \otimes MP)R(I \otimes Q) \rangle. \end{aligned}$$

For convenience, we assume that \mathcal{K}_1 and \mathcal{K}_2 have the same dimension, so that there is a unitary operator U in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$. Now take P and Q to be rank one projections $P = ff^*$ and $Q = gg^*$. Choose $M_n \in \mathcal{M}$ so that $M_n f \rightarrow Uf$. Then

$$\begin{aligned} 0 &= \langle L_p \otimes I, (I \otimes M_n P)R(I \otimes Q) \rangle \\ &= \langle L_p \otimes I, (I \otimes M_n f f^*)R(I \otimes g g^*) \rangle \\ &\rightarrow \langle L_p \otimes I, (I \otimes U f f^*)R(I \otimes g g^*) \rangle \\ &= \langle L_p \otimes g(g^* U f)(U f)^*, (I \otimes U f f^*)R(I \otimes g g^*) \rangle \\ &= (g^* U f) \langle L_p \otimes g f^*, (I \otimes f f^*)R(I \otimes g g^*) \rangle \\ &= (g^* U f) \langle L_p \otimes g f^*, R_{f,g} \otimes f g^* \rangle = (g^* U f) \langle L_p, R_{f,g} \rangle \end{aligned}$$

where $R_{f,g} \otimes f g^* := (I \otimes f f^*)R(I \otimes g g^*)$ defines $R_{f,g} \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$.

The pairs f, g with $g^* U f \neq 0$ are dense in $\mathcal{K}_1 \times \mathcal{K}_2$; so we conclude that $\langle L_p, R_{f,g} \rangle = 0$ for all pairs. Evidently $\text{rank } R_{f,g} \leq \text{rank } R \leq k$, and it is orthogonal to \mathcal{L} . Hence $R_{f,g} = 0$ for all pairs. That is, $(I \otimes f f^*)R(I \otimes g g^*) = 0$ for all unit vectors f and g . This clearly implies that $R = 0$. Hence $\mathcal{L} \otimes \mathcal{M}$ is topologically k -transitive. \blacksquare

As in the finite-dimensional setting, we may define a subspace $\mathcal{L} \subseteq \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ to be *fully k -transitive* if the spatial tensor product $\mathcal{L} \otimes \mathcal{M}$ is k -transitive for all k -transitive subspaces \mathcal{M} of $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$. The analogue of Corollary 3.4 follows as before.

Corollary 6.13. *If $\mathcal{L} \subseteq \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a k -transitive subspace which is contained in the WOT-closed span of its rank one elements, then \mathcal{L} is fully k -transitive.*

If the subspace has additional structure, such as being a module over a masa, then stronger results may hold. For example, a non-trivial result of Arveson [1] (see also [4, Theorem 15.9]) shows:

Theorem 6.14. *Let \mathcal{D}_i be masas in $\mathcal{B}(\mathcal{H}_i)$. A topologically transitive subspace \mathcal{L} of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ which is a \mathcal{D}_2 - \mathcal{D}_1 bimodule is WOT-dense in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.*

We provide a new, more elementary proof. Actually Arveson's proof works for the weak-* topology, whereas this proof is only valid for the WOT-topology.

Lemma 6.15. *Let (X_i, μ_i) be regular Borel measures. Let $k(x, y) = \sum_{i=1}^m \alpha_i(x)\beta_i(y)$ be given in $L^2(\mu_1 \times \mu_2)$ where $\alpha_i \in L^2(\mu_1)$ and $\beta_i \in L^2(\mu_2)$. For any a in the essential range of k and any $\varepsilon > 0$, there is a measurable rectangle $A_1 \times A_2$ with $0 < \mu_i(A_i) < \infty$ such that $|k(x, y) - a| < \varepsilon$ for all $(x, y) \in A_1 \times A_2$.*

Proof. Choose a measurable rectangle $Y_1 \times Y_2$ of finite positive measure on which k is bounded and still has a in its essential range. It is a standard argument to approximate each α_i and β_i uniformly (and in L^2) by simple functions on Y_1 and Y_2 respectively. Combining these simple functions allows us to approximate $k\chi_{Y_1 \times Y_2}$ uniformly by a finite linear combination of characteristic functions of measurable rectangles. We may then pick a rectangle on which k takes values close to a . ■

Remark 6.16. The lemma fails for arbitrary functions in $L^2(\mu_1 \times \mu_2)$. For example, take $\mu_1 = \mu_2$ to be Lebesgue measure on $[0, 1]$. Let A be a compact nowhere dense subset of $[0, 1]$ with positive measure. Then $k(x, y) = \chi_A(x - y)$ has 1 in its essential range. However if $k = 1$ on a measurable rectangle $A_1 \times A_2$, then $A_1 - A_2 \subset A$ is nowhere dense. It is a well known fact that the difference of two measurable sets of positive measure has interior. So $A_1 \times A_2$ has measure 0.

Proof of Theorem 6.14. By the spectral theorem for masas, we may suppose that there are regular Borel spaces (X_i, μ_i) so that \mathcal{D}_i are unitarily equivalent to $L^\infty(\mu_i)$ acting by multiplication on $\mathcal{H}_i = L^2(\mu_i)$. If \mathcal{L} is not WOT-dense, then there is a finite rank operator $F \in \mathcal{L}_\perp$. Moreover, it is evident that \mathcal{L}_\perp is a \mathcal{D}_1 - \mathcal{D}_2 bimodule. Our goal is to show that using F and the bimodule property, we may find a rank one element of \mathcal{L}_\perp . This will contradict topological transitivity.

Observe that F may be written as an integral operator with kernel $k(x, y) = \sum_{i=1}^m \alpha_i(x)\beta_i(y)$ where $\alpha_i \in L^2(\mu_1)$ and $\beta_i \in L^2(\mu_2)$. Since $F \neq 0$, there is a non-zero value a in the essential range of k . By Lemma 6.15, there is a measurable rectangle $A_1 \times A_2$ of finite non-zero measure so that $|k(x, y) - a| < |a|/2$ for all $(x, y) \in A_1 \times A_2$.

Let $h(x, y) = \chi_{A_1 \times A_2} k(x, y)^{-1}$. Then h lies in $L^\infty(\mu_1 \times \mu_2)$. Therefore h is a limit in $L^\infty(\mu_1 \times \mu_2)$ of a sequence of simple functions of the form $h_k = \sum_{j=1}^{m_k} f_{kj}(x)g_{kj}(y)$. It follows by routine calculations that $\sum_{j=1}^{m_k} M_{f_{kj}\chi_{A_1}} F M_{g_{kj}\chi_{A_2}}$ has kernel $\sum_{j=1}^{m_k} f_{kj}(x)k\chi_{A_1 \times A_2} g_{kj}(y)$. This converges in $L^2(\mu_1 \times \mu_2)$ to $\chi_{A_1 \times A_2}$. Thus the corresponding operators converge in norm to the rank one integral operator with kernel $\chi_{A_1 \times A_2}$. This produces a rank one element of \mathcal{L}_\perp . ■

The following result is very easy. Recall that a masa is atomic if it consist of all diagonal operators with respect to some orthonormal basis.

Proposition 6.17. *Suppose that a topologically transitive subspace $\mathcal{L} \subseteq \mathcal{B}(\mathcal{H})$ is a left or right module over an atomic masa \mathfrak{D} . Then $\overline{\mathcal{L}}^{\text{WOT}} = \mathcal{B}(\mathcal{H})$.*

Proof. First we suppose that \mathcal{L} is a right \mathfrak{D} -module. Let \mathfrak{D} be diagonal with respect to $\{e_n : n \geq 1\}$; and let $P_n = e_n e_n^*$. Since \mathcal{L} is topologically transitive, $\overline{\mathcal{L}P_n} = \mathcal{B}(\mathcal{H})P_n$. Summing over n yields a WOT-dense subspace.

By considering \mathcal{L}^t which is also topologically transitive, we obtain the left \mathfrak{D} -module case. ■

Example 6.18. A right or left module over a non-atomic masa need not be WOT-dense.

Example 6.3 showed that $\mathcal{M}^\infty P \subset \mathcal{B}(H^2, L^2(\mathbb{T}))$ is topologically transitive. It is evidently a left \mathcal{M}^∞ module, and a proper WOT-closed subspace. The adjoint $P\mathcal{M}^\infty \subset \mathcal{B}(L^2(\mathbb{T}), H^2)$ is similarly a right \mathcal{M}^∞ module, and a proper WOT-closed subspace.

Proposition 6.19. *Suppose that \mathcal{L} is a left module over a masa \mathfrak{D} , and that \mathcal{L} is topologically transitive. Then $\overline{\text{span } \mathcal{L}\mathfrak{D}}^{\text{WOT}} = \overline{\text{span } \mathcal{L}^2}^{\text{WOT}} = \mathcal{B}(\mathcal{H})$.*

Proof. Evidently $\text{span } \mathcal{L}\mathfrak{D}$ is a \mathfrak{D} -bimodule. So it is WOT-dense by Arveson's Theorem. Therefore

$$\overline{\text{span } \mathcal{L}^2}^{\text{WOT}} = \overline{\text{span } \mathcal{L}\mathfrak{D}\mathcal{L}}^{\text{WOT}} = \overline{\text{span } \mathcal{B}(\mathcal{H})\mathcal{L}}^{\text{WOT}} = \mathcal{B}(\mathcal{H}). \quad \blacksquare$$

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