ON THE TOPOLOGICAL STABLE RANK OF NON-SELFADJOINT OPERATOR ALGEBRAS

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Abstract. We provide a negative solution to a question of M. Rieffel who asked if the right and left topological stable ranks of a Banach algebra must always agree. Our example is found amongst a class of nest algebras. We show that for many other nest algebras, both the left and right topological stable ranks are infinite. We extend this latter result to Popescu's non-commutative disc algebras and to free semigroup algebras as well.

1. Introduction

The study of topological stable rank for Banach algebras originated with Rieffel [14]. Motivated by a search for stability results in C^*-algebras, he introduced topological stable rank as a non-commutative analogue of the covering dimension for compact spaces.

Given a unital Banach algebra $\mathcal{A}$, we denote by $Lg_n(\mathcal{A})$ (resp. $Rg_n(\mathcal{A})$) the set of n-tuples of elements of $\mathcal{A}$ which generate $\mathcal{A}$ as a left ideal (resp. as a right ideal). That is, $Lg_n(\mathcal{A}) = \{(a_1, a_2, ..., a_n) : a_i \in \mathcal{A}, 1 \le i \le n$ and there exists $b_1, b_2, ..., b_n \in \mathcal{A}$ such that $\sum_{i=1}^n b_i a_i = 1\}$. The *left* (resp. *right*) *topological stable rank* of $\mathcal{A}$, denoted by $\mathrm{ltsr}(\mathcal{A})$ (resp. $\mathrm{rtsr}(\mathcal{A})$), is the least positive integer n for which $Lg_n(\mathcal{A})$ (resp. $Rg_n(\mathcal{A})$) is dense in $\mathcal{A}^n$. When no such integer exists, we set $\mathrm{ltsr}(\mathcal{A}) = \infty$ (resp. $\mathrm{rtsr}(\mathcal{A}) = \infty$). If $\mathrm{ltsr}(\mathcal{A}) = \mathrm{rtsr}(\mathcal{A})$, we refer to their common value simply as the *topological stable rank* of $\mathcal{A}$, written $\mathrm{tsr}(\mathcal{A})$. If $\mathcal{A}$ is not unital, we define the left (resp. the right) topological stable rank of $\mathcal{A}$ to be that of its unitization.

For C^*-algebras, it was shown by Herman and Vaserstein [9] that topological stable rank coincides with the ring-theoretic notion of stable rank, first introduced by Bass [2]. Consider a ring $\mathcal{R}$ with identity. The *left Bass stable rank* of $\mathcal{R}$, $\mathrm{lBsr}(\mathcal{R})$, is the least positive integer m so that for each $(a_1, a_2, ..., a_{m+1}) \in Lg_{m+1}(\mathcal{R})$, there exists $(b_1, b_2, ..., b_m) \in \mathcal{R}^m$ for which $\sum_{i=1}^m (a_i + b_i a_{m+1}) \in Lg_m(\mathcal{R})$. The right Bass stable rank of $\mathcal{R}$ is analogously defined. Vaserstein [15] (see also Warfield [16]) has shown that $\mathrm{lBsr}(\mathcal{R}) = \mathrm{rBsr}(\mathcal{R})$ for all rings, and hence one normally speaks only of *Bass stable rank* $\mathrm{Bsr}(\mathcal{R})$.

For general Banach algebras $\mathcal{A}$ we have the inequality

$$\mathrm{Bsr}(\mathcal{A}) \le \min(\mathrm{ltsr}(\mathcal{A}), \mathrm{rtsr}(\mathcal{A}))$$

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(see Corollary 2.4 of [14]). Jones, Marshall and Wolff [11] have shown that the disc algebra $\mathcal{A}(\mathbb{D})$ satisfies $\mathrm{Bsr}(\mathcal{A}(\mathbb{D})) = 1$. (Recall that the disc algebra $\mathcal{A}(\mathbb{D})$ consists of those functions which are continuous on the closed unit disc of $\mathbb{C}$ and which are analytic on the open unit disc.) Rieffel [14] had shown that $\mathrm{tsr}(\mathcal{A}(\mathbb{D})) = 2$. This shows that the inequality above may be strict.

Question 1.5 of Rieffel's paper asks whether or not there exists a Banach algebra $\mathcal{A}$ for which $\mathrm{ltsr}(\mathcal{A}) \neq \mathrm{rtsr}(\mathcal{A})$. It is clear that if such an algebra is to exist, there must be something inherently different between the structure of the left and of the right ideals of $\mathcal{A}$. If $\mathcal{A}$ is a C^*-algebra, then the involution provides an anti-isomorphism between left and right ideals, and so one would expect that $\mathrm{rtsr}(\mathcal{A})$ should equal $\mathrm{ltsr}(\mathcal{A})$ for these algebras. That this is the case is the conclusion of Proposition 1.6 of [14].

Thus, the search for an algebra $\mathcal{A}$ of Hilbert space operators for which the left and right topological stable ranks differ takes us into the class of non-selfadjoint algebras. Two of the best studied such classes are nest algebras and free semigroup algebras.

We begin the second section of this paper by presenting an example of a nest algebra $\mathcal{T}(\mathcal{N})$ for which $\mathrm{ltsr}(\mathcal{T}(\mathcal{N})) = \infty$ while $\mathrm{rtsr}(\mathcal{T}(\mathcal{N})) = 2$. The example is found among atomic nest algebras, order isomorphic to the natural numbers $\mathbb{N}$, all of whose atoms are finite dimensional, with the dimensions of the atoms growing sufficiently rapidly. We then examine the left and right topological stable ranks of nest algebras in general, and show that in many other cases, the stable ranks agree and are infinite. It is not yet clear which nest algebras satisfy $\mathrm{ltsr}(\mathcal{T}(\mathcal{N})) = \mathrm{rtsr}(\mathcal{T}(\mathcal{N}))$, or indeed, which values of the left (or right) topological stable ranks are attainable.

In the third section, we deal with the case of non-commutative disc algebras and of WOT-closed free semigroup algebras. We show that the left and right topological stable ranks of such algebras are always infinite.

First let us prepare the groundwork for what will follow. We shall need the following two results due to Rieffel [14].

1.1. Theorem.

(a) *Let $\mathcal{A}$ be a Banach algebra and let J be an ideal of $\mathcal{A}$. Then* $\mathrm{ltsr}(\mathcal{A}/J) \leq \mathrm{ltsr}(\mathcal{A})$.

(b) *Let $\mathfrak{H}$ be an infinite dimensional, complex Hilbert space. Then* $\mathrm{tsr}(\mathcal{B}(\mathfrak{H})) = \infty$.

A trivial modification of Theorem 1.1(a) shows that if $\varphi : \mathcal{A} \to \mathcal{B}$ is a continuous unital homomorphism with dense range, then $\mathrm{ltsr}(\mathcal{B}) \leq \mathrm{ltsr}(\mathcal{A})$. (See Proposition 4.12 of [1] for a version of this for topological algebras.) Of course there is a corresponding result for right topological stable rank. We shall also require a slightly more general version of Proposition 1.6 of [14]. Its proof is essentially the same as the proof of that Proposition.

1.2. Lemma. *Suppose that $\mathcal{A}$ is a unital Banach algebra with a continuous involution. Let $\mathcal{B}$ be a unital (not necessarily selfadjoint) subalgebra of $\mathcal{A}$. Then* $\mathrm{ltsr}(\mathcal{B}) = \mathrm{rtsr}(\mathcal{B}^*)$. *Hence* $\mathrm{rtsr}(\mathcal{B}) = \mathrm{ltsr}(\mathcal{B}^*)$.

Proof. A simple calculation shows that $(b_1, b_2, ..., b_n) \in Lg_n(\mathcal{B})$ if and only if $(b_1^*, b_2^*, ..., b_n^*) \in Rg_n(\mathcal{B}^*)$. From this the result easily follows. □

When $\mathcal{A}$ is a subalgebra of operators on a Hilbert space $\mathfrak{H}$ (or on any other vector space for that matter), we may construct row spaces $\mathcal{R}_n(\mathcal{A})$ and column spaces $\mathcal{C}_n(\mathcal{A})$ of n-tuples of elements of $\mathcal{A}$. This allows us to view elements of $\mathcal{R}_n(\mathcal{A})$ as operators from $\mathfrak{H}^{(n)}$ to $\mathfrak{H}$, and to view elements of $\mathcal{C}_n(\mathcal{A})$ as operators from $\mathfrak{H}$ to $\mathfrak{H}^{(n)}$. To say that an element $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$ of $\mathcal{R}_n(\mathcal{A})$ lies in $Rg_n(\mathcal{A})$ is equivalent to saying that A is right invertible, i.e. that there exists $B = \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}^t \in \mathcal{C}_n(\mathcal{A})$ such that AB is the identity operator on $\mathfrak{H}$. That there exists a corresponding statement for $Lg_n(\mathcal{A})$ is clear.

Our main tool for determining the topological stable ranks of the algebras considered below is the following observation:

1.3. Remark. If an algebra $\mathcal{A}$ of operators in $\mathcal{B}(\mathfrak{H})$ contains operators $A_1, ..., A_n$ so that $[A_1 \ A_2 \ \cdots \ A_n] \in \mathcal{B}(\mathfrak{H}^{(n)}, \mathfrak{H})$ is a semi-Fredholm operator of negative semi-Fredholm index, then $\mathrm{rtsr}(\mathcal{A}) \geq n+1$. In particular, therefore, if $\mathcal{R}_n(\mathcal{A})$ contains a proper isometry, then $\mathrm{rtsr}(\mathcal{A}) \geq n+1$. This follows from basic Fredholm theory (see, for eg. [4]), as no small perturbation $[A_1' \ A_2' \ \cdots \ A_n']$ of $[A_1 \ A_2 \ \cdots \ A_n]$ will be surjective, and thus $\sum_{i=1}^n A_i' B_i \neq I$ for any choice of $B_1, B_2, ..., B_n \in \mathcal{A}$.

The corresponding result for left topological stable rank says that if $\mathcal{C}_n(\mathcal{A})$ contains a proper co-isometry, then $\mathrm{ltsr}(\mathcal{A}) \geq n+1$.

The way this observation will be used is as follows:

1.4. Proposition. *Suppose that $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{H})$ is a Banach algebra of operators and that $\mathcal{A}$ contains two isometries U and V with mutually orthogonal ranges. Then* $\mathrm{rtsr}(\mathcal{A}) = \infty$.

Proof. Once $\mathcal{A}$ contains two such isometries U and V, it is clear that for each $n \geq 1$, $\{U, VU, V^2U, ..., V^nU\}$ are $n+1$ isometries in $\mathcal{A}$ with mutually orthogonal ranges. Let $Y = [U \ VU \ V^2U \ \cdots \ V^{n-1}U] \in \mathcal{B}(\mathfrak{H}^{(n)}, \mathfrak{H})$; then Y is an isometry and $\operatorname{ran} Y$ is orthogonal to $\operatorname{ran} V^nU$, so that Y is in fact a proper isometry.

By Remark 1.3, $\mathrm{rtsr}(\mathcal{A}) \geq n+1$. Since $n \geq 1$ was arbitrary, $\mathrm{rtsr}(\mathcal{A}) = \infty$. □

Of course, if $\mathcal{A}$ contains two co-isometries with mutually orthogonal initial spaces, then by considering $\mathcal{B} = \mathcal{A}^*$, we get $\mathrm{ltsr}(\mathcal{A}) = \mathrm{rtsr}(\mathcal{B}) = \infty$.

2. Nest algebras

The first class of algebras we shall examine are *nest algebras*, which are an infinite dimensional generalization of the algebra $\mathcal{T}_n(\mathbb{C})$ of upper triangular $n \times n$ matrices. A *nest* $\mathcal{N}$ on a Hilbert space $\mathfrak{H}$ is a chain of closed subspaces of $\mathfrak{H}$ such that $\{0\}, \mathfrak{H}$ lie in $\mathcal{N}$, and $\mathcal{N}$ is closed under the operations of taking arbitrary intersections and closed linear spans of its elements. At times it is convenient to identify the nest $\mathcal{N}$ with the collection $\mathcal{P}(\mathcal{N}) = \{P(N) : N \in \mathcal{N}\}$, where, for a subspace M of $\mathfrak{H}$, $P(M)$ denotes the orthogonal projection of $\mathfrak{H}$ onto M. For each $N \in \mathcal{N}$, we may define the *successor* of N to be $N_+ := \inf\{M \in \mathcal{N} : M > N\}$. If $N_+ \neq N$, then $N_+ \ominus N$ is called an *atom* of $\mathcal{N}$. If $\mathfrak{H}$ is spanned by the atoms of $\mathcal{N}$, we say that $\mathcal{N}$ is *atomic.* If $\mathcal{N}$ admits no atoms, we say that $\mathcal{N}$ is *continuous*. Most nests are neither atomic nor continuous.

Given a nest $\mathcal{N}$, there corresponds to $\mathcal{N}$ the (WOT-closed) *nest algebra*

$$\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(\mathfrak{H}) : TN \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

A very important example of a nest algebra is the following. Suppose that $\mathfrak{H}$ is a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let $N_0 = \{0\}$, $N_\infty = \mathfrak{H}$, and for $n \geq 1$, let $N_n = \text{span}\{e_1, e_2, ..., e_n\}$. Then $\mathcal{N} = \{N_k : 0 \leq k \leq \infty\}$ is a nest. The corresponding nest algebra coincides with the set of all operators in $\mathcal{B}(\mathfrak{H})$ whose matrix with respect to this orthonormal basis is upper triangular. Because of the obvious relation of this nest to the natural numbers, we shall denote this nest algebra by $\mathcal{T}(\mathbb{N})$. It is also worth noting that if $\mathcal{N}$ is a nest on $\mathfrak{H}$, then so is $\mathcal{N}^\perp := \{N^\perp : N \in \mathcal{N}\}$. In fact, $\mathcal{T}(\mathcal{N}^\perp) = \{T^* : T \in \mathcal{T}(\mathcal{N})\} = \mathcal{T}(\mathcal{N})^*$. We denote by $\mathcal{D}(\mathcal{N}) = \mathcal{T}(\mathcal{N}) \cap \mathcal{T}(\mathcal{N})^*$ the *diagonal* of $\mathcal{T}(\mathcal{N})$. This is a von Neumann algebra. If $\mathcal{N}$ is atomic, then it is known that there exists a unique expectation of $\mathcal{T}(\mathcal{N})$ onto $\mathcal{D}(\mathcal{N})$ (see, for eg., Chapter 8 of [5]).

The following is the main result of the paper. It provides an example of a Banach algebra for which the right and left topological stable ranks differ, thereby answering Question 1.5 of [14] in the negative. We thank J. Orr for simplifying one of the calculations at the end of the proof.

2.1. Theorem. *Let $\mathcal{N}$ be an atomic nest which is order isomorphic to $\mathbb{N}$, with finite dimensional atoms $E_k = P(N_k) - P(N_{k-1})$ of rank n_k satisfying $n_k \geq 4\sum_{i<k} n_i$. Then*

$$\text{ltsr}(\mathcal{T}(\mathcal{N})) = \infty \quad \textit{and} \quad \text{rtsr}(\mathcal{T}(\mathcal{N})) = 2.$$

Proof. Let $\{e_{kj} : 1 \leq j \leq n_k\}$ be an orthonormal basis for the atom E_k, $k \geq 1$. We can construct two co-isometries U and V with mutually orthogonal initial spaces in $\mathcal{T}(\mathcal{N})$ by defining $U^* e_{kj} = e_{2^k 3^j\ 1}$ and $V^* e_{kj} = e_{5^k 3^j\ 1}$ for all $1 \leq j \leq n_k$ and $k \geq 1$. By the remark following Proposition 1.4, $\text{ltsr}(\mathcal{T}(\mathcal{N})) = \infty$.

It is a consequence of Proposition 3.1 of Rieffel [14], that $\text{rtsr}(\mathcal{T}(\mathcal{N})) \geq 2$.

Let Δ be the expectation $\Delta(A) = \sum_{k\geq 1} E_k A E_k$ of $\mathcal{T}(\mathcal{N})$ onto the diagonal $\mathcal{D}(\mathcal{N})$, which is a finite von Neumann algebra. Every element $D \in \mathcal{D}(\mathcal{N})$ factors as $D = UP$ where P is positive and U is unitary. Thus for any $\varepsilon > 0$, $D' = U(P + \varepsilon I)$ is an ε-perturbation which is invertible with inverse bounded by ε^{-1}.

Let A and B belong to $\mathcal{T}(\mathcal{N})$, and let $\varepsilon > 0$ be given. By the previous paragraph, there are $\varepsilon/2$-perturbations A', B' of A and B so that $A' = D_a + A_0'$ and $B' = D_b + B_0'$ where A_0', B_0' lie in the ideal $\mathcal{T}_0(\mathcal{N})$ of strictly upper triangular operators and D_a, D_b are invertible elements of $\mathcal{D}(\mathcal{N})$ with inverses bounded by $2\varepsilon^{-1}$. Let

$$A_1 = A' D_a^{-1} = I + A_0 \quad \text{and} \quad B_1 = B' D_b^{-1} = I + B_0,$$

where $A_0 = A_0' D_a^{-1}$ and $B_0 = B_0' D_b^{-1}$.

Now $A_0 = \sum_{k\geq 2} A_0 E_k$ and

$$\operatorname{rank}(A_0E_k) = \operatorname{rank}(P(N_{k-1})A_0E_k) \leq \operatorname{rank}(P(N_{k-1})) = \sum_{i<k} n_i \leq \frac{n_k}{4}.$$

The same estimate holds for B_0. Therefore we may select projections $P_k \leq E_k$ with $\operatorname{rank} P_k \leq n_k/2$ so that $A_0E_k = A_0P_k$ and $B_0E_k = B_0P_k$.

Let $U_k = P_kU_k(E_k - P_k)$ be a partial isometry with range $P_k\mathfrak{H}$. Define operators $U = \sum_{k\geq 1} U_k$ and $P = \sum_{k\geq 1} P_k$; so $P^\perp = \sum_{k\geq 1}(E_k - P_k)$ and $UU^* = P$. Take any positive number

$$0 < \delta < \frac{\varepsilon}{2}\|D_a\|^{-1}.$$

Consider $A'' = A' + \delta UD_a$. Then

$$\|A - A''\| < \|A - A'\| + \delta\|D_a\| < \varepsilon,$$

and from above, $\|B - B'\| < \varepsilon/2$. We will show that $\begin{bmatrix} A'' & B' \end{bmatrix}$ is right invertible.

Observe that $A'D_a^{-1}P^\perp = B'D_b^{-1}P^\perp = P^\perp$. Thus

$$\begin{aligned} A''D_a^{-1}P^\perp U^* + B'D_b^{-1}P^\perp(I - U^*) &= P^\perp U^* + \delta UD_aD_a^{-1}P^\perp U^* + P^\perp(I - U^*) \\ &= \delta P + P^\perp. \end{aligned}$$

It is clear that this is (right) invertible (by $\delta^{-1}P + P^\perp$), whence $\begin{bmatrix} A'' & B' \end{bmatrix}$ is also right invertible.

It follows that $Rg_2(\mathcal{T}(\mathcal{N}))$ is dense in $\mathcal{R}_2(\mathcal{T}(\mathcal{N}))$; that is, $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = 2$. □

Let us next turn our attention to general nest algebras. We can show in a large number of cases, the left and right topological stable ranks of a nest algebra agree, and that they are infinite. For the remainder of this article, we shall restrict our attention to *complex, infinite dimensional, separable Hilbert spaces.*

2.2. Proposition. *Let $\mathcal{N}$ be a nest on a Hilbert space $\mathfrak{H}$, and suppose that $\mathcal{N}$ contains a strictly decreasing sequence $\{N_k\}_{k=0}^\infty$. Then* $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$.

Proof. Let $N_\infty = \cap_{k\geq 0} N_k \in \mathcal{N}$. If $\mathfrak{K} := N_0 \ominus N_\infty$, then $\mathcal{M} = \{N \cap \mathfrak{K} : N \in \mathcal{N}\}$ is a nest, and the compression map

$$\begin{aligned} \Gamma: \ \mathcal{T}(\mathcal{N}) &\rightarrow \mathcal{T}(\mathcal{M}) \\ T &\mapsto P_{\mathfrak{K}}T_{|\mathfrak{K}} \end{aligned}$$

is a contractive, surjective homomorphism of $\mathcal{T}(\mathcal{N})$ onto $\mathcal{T}(\mathcal{M})$. By Theorem 1.1, it suffices to prove that $\operatorname{rtsr}(\mathcal{T}(\mathcal{M})) = \infty$.

If $M_k := N_k \ominus N_\infty$, then $M_k \in \mathcal{M}$ for all $k \geq 1$, and $M_0 > M_1 > M_2 > \cdots$. Let $A_k = M_{k-1} \ominus M_k$, $k \geq 1$, and choose an orthonormal basis $\{e_{kj} : 1 \leq j < n_k\}$ for A_k, where $2 \leq n_k \leq \infty$. Observe that $\cup_{k\geq 1}\{e_{kj} : 1 \leq j < n_k\}$ is then an orthonormal basis for $\mathfrak{K}$.

We then define two isometries $U, V \in \mathcal{T}(\mathcal{M})$ via:

$$Ue_{kj} = e_{2^j3^k\ 1}, \qquad Ve_{kj} = e_{5^j7^k\ 1},$$

for all $1 \le j < n_k$, $1 \le k < \infty$. Clearly U and V have mutually orthogonal ranges. By Proposition 1.4, $\operatorname{rtsr}(\mathcal{T}(\mathcal{M})) = \infty$, which – as we have seen – ensures that $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$. □

2.3. Corollary. *Let $\mathcal{N}$ be a nest on a Hilbert space $\mathfrak{H}$, and suppose that $\mathcal{N}$ contains a strictly increasing sequence $\{N_k\}_{k=0}^{\infty}$. Then* $\operatorname{ltsr}(\mathcal{T}(\mathcal{N})) = \infty$.

In the following Theorem, we refer to the *dual* of an ordinal. If $(\beta, \le)$ is an ordinal, the dual of β is the totally ordered set $(\beta^*, \le_*)$ where $\beta^* = \beta$ and $x \le_* y$ if and only if $y \le x$.

2.4. Theorem. *Let $\mathcal{N}$ be a nest on a Hilbert space $\mathfrak{H}$. If $\mathcal{N}$ satisfies any one of the following three properties, then* $\operatorname{ltsr}(\mathcal{T}(\mathcal{N})) = \operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$.

(a) *$\mathcal{N}$ has an infinite dimensional atom.*
(b) *$\mathcal{N}$ is uncountable.*
(c) *$\mathcal{N}$ is countable, but is not order isomorphic to an ordinal or its dual.*

Proof. (a) Choose $N \in \mathcal{N}$ so that $\dim(N_+ \ominus N) = \infty$, and set $E = N_+ \ominus N$. Then E is a semi-invariant subspace for $\mathcal{T}(\mathcal{N})$, and the map $\gamma : \mathcal{T}(\mathcal{N}) \to \mathcal{B}(E)$ defined by $X \mapsto P(E)X|_E$ is a surjective homomorphism. Now $\operatorname{tsr}(\mathcal{B}(E)) = \infty$, by Theorem 1.1(b). Furthermore, by Theorem 1.1(a), since $\mathcal{B}(E)$ is a homomorphic image of $\mathcal{T}(\mathcal{N})$, $\operatorname{ltsr}(\mathcal{T}(\mathcal{N})) \ge \operatorname{ltsr}(\mathcal{B}(E)) = \infty$, and similarly $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) \ge \operatorname{rtsr}(\mathcal{B}(E)) = \infty$, completing the proof.

(b,c) In each of these cases, the conditions on $\mathcal{N}$ guarantee the existence of both a strictly increasing sequence $\{N_k\}_{k=1}^{\infty}$ and a strictly decreasing sequence $\{M_k\}_{k=1}^{\infty}$ of subspaces in $\mathcal{N}$. The result now follows immediately from Proposition 2.2 and Corollary 2.3. □

2.5. Corollary. *Let $\mathcal{N}$ be a nest. Then* $\max(\operatorname{ltsr}(\mathcal{T}(\mathcal{N}), \operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$.

Proof. Taking into account the above results, the only case left to consider is that where $\mathcal{N}$ is a countably infinite nest, order isomorphic to an ordinal or the dual of an ordinal. As such, $\mathcal{N}$ either contains an interval $[N_1, N_\infty)$ which is order isomorphic to $\mathbb{N}$, or an interval $(N_{-\infty}, N_{-1}]$ which is order isomorphic to $-\mathbb{N}$. From Proposition 2.2 and Corollary 2.3 we deduce that $\max(\operatorname{ltsr}(\mathcal{T}(\mathcal{N}), \operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$. □

We have thus reduced the problem of determining the topological stable ranks of nest algebras to the problem of determining the right topological stable rank of a countable, atomic nest $\mathcal{N}$, order isomorphic to an ordinal, all of whose atoms are finite dimensional. Theorem 2.1 shows that in this case it is possible to have $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = 2$. We shall see below that this example may be extended to a more general class of nest algebras whose nests are totally ordered like ω (the first infinite ordinal), and for which there is an arithmetically increasing sequence of atoms whose ranks grow geometrically fast (see Theorem 2.11). Having said this, the exact nature of the nests for which the right topological stable rank is finite is not completely understood. We begin by establishing a couple of conditions on a nest $\mathcal{N}$ which will guarantee that the right topological stable rank of $\mathcal{T}(\mathcal{N})$ is infinite.

The proofs of the results depend upon the existence of certain surjective homomorphisms of nest algebras established in [6]. Since they play such a key role, we briefly recall the construction of these homomorphisms as outlined in that paper.

2.6. **The Davidson-Harrison-Orr Construction.** Let Ω be an interval of $\mathbb{Z}$, and suppose that $\Omega = \cup_{n=1}^{\infty} \Omega_n$, where $\Omega_n \subseteq \Omega_{n+1}$, $n \geq 1$ are subintervals of Ω. Suppose also that $\mathcal{M}$ is a nest, order isomorphic to Ω via an order isomorphism λ. Let E_n denote the subinterval of $\mathcal{M}$ corresponding via λ to the interval Ω_n, $n \geq 1$.

Consider next a nest $\mathcal{N}$ containing countably many subintervals F_n acting on pairwise orthogonal subspaces such that $P_{F_n \mathfrak{H}} \mathcal{T}(\mathcal{N})|_{F_n \mathfrak{H}}$ is unitarily equivalent to $P_{E_n \mathfrak{H}} \mathcal{T}(\mathcal{M})|_{E_n \mathfrak{H}}$ via a unitary conjugation $Ad_{U_n} : P_{F_n \mathfrak{H}} \mathcal{T}(\mathcal{N})|_{F_n \mathfrak{H}} \to P_{E_n \mathfrak{H}} \mathcal{T}(\mathcal{M})|_{E_n \mathfrak{H}}$. Let $\alpha_n : \mathcal{T}(\mathcal{N}) \to P_{F_n \mathfrak{H}} \mathcal{T}(\mathcal{N})|_{F_n \mathfrak{H}}$ be the natural compression maps; and let $\beta_n : P_{E_n \mathfrak{H}} \mathcal{T}(\mathcal{M})|_{E_n \mathfrak{H}} \to \mathcal{T}(\mathcal{M})$ be the inclusion maps for $n \geq 1$. Let $\varphi_n : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{M})$ be the maps $\varphi_n = \beta_n \circ Ad_{U_n} \circ \alpha_n$ for $n \geq 1$, so that φ_n is a homomorphism for all n.

Letting $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, we have that

$$\begin{array}{rccc} \varphi: & \mathcal{T}(\mathcal{N}) & \to & \mathcal{T}(\mathcal{M}) \\ & T & \mapsto & \textsc{wot}\text{-}\lim_{n \in \mathcal{U}} \varphi_n(T) \end{array}$$

defines a continuous epimorphism of $\mathcal{T}(\mathcal{N})$ onto $\mathcal{T}(\mathcal{M})$ ([6], Corollary 5.3 and Theorem 6.8).

For example, suppose that $\mathcal{M}$ is the maximal atomic nest, ordered like ω^*, so that $\mathcal{T}(\mathcal{M}) \simeq \mathcal{T}(\mathbb{N})^*$. Set $\Omega = -\mathbb{N}$, $\Omega_n = \{-2^n, -2^n+1, ..., -3, -2, -1\}$, and let E_n denote the corresponding subinterval of $\mathcal{M}$. Thus $\mathcal{T}(\mathcal{M})|_{E_n \mathfrak{H}} \simeq \mathcal{T}_{2^n}(\mathbb{C})$, the upper triangular $2^n \times 2^n$ matrices over $\mathbb{C}$. Choose integers $r_1 < r_2 < r_3 < \cdots$ such that $r_n - r_{n-1} > 2^n$. If $F_n = \text{span}\{e_{r_n+1}, e_{r_n+2}, ..., e_{r_n+2^n}\}$, then the F_n's are pairwise orthogonal and $\mathcal{T}(\mathbb{N})|F_n\mathfrak{H} \simeq \mathcal{T}_{2^n}(\mathbb{C})$ as well, and so we can find a unitary matrix $U_n : F_n \mathfrak{H} \to E_n \mathfrak{H}$ such that $P_{F_n \mathfrak{H}} \mathcal{T}(\mathbb{N})|_{F_n \mathfrak{H}} = U_n^* (P_{E_n \mathfrak{H}} \mathcal{T}(\mathcal{M})|_{E_n \mathfrak{H}}) U_n$. With $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$,

$$\varphi(T) = \textsc{wot}\text{-}\lim_{n \in \mathcal{U}} \varphi_n(T)$$

implements a continuous epimorphism of $\mathcal{T}(\mathbb{N})$ onto $\mathcal{T}(\mathcal{M}) \simeq \mathcal{T}(\mathbb{N})^*$.

2.7. Corollary. $\text{ltsr}(\mathcal{T}(\mathbb{N})) = \text{rtsr}(\mathcal{T}(\mathbb{N})) = \infty$.

Proof. By Corollary 2.3, $\text{ltsr}(\mathcal{T}(\mathbb{N})) = \infty$. Let $\varphi : \mathcal{T}(\mathbb{N}) \to \mathcal{T}(\mathbb{N})^*$ be the epimorphism described in the Section 2.6. By Theorem 1.1 and Lemma 1.2, $\text{rtsr}(\mathcal{T}(\mathbb{N})) \geq \text{rtsr}(\mathcal{T}(\mathbb{N})^*) = \text{ltsr}(\mathcal{T}(\mathbb{N})) = \infty$. □

2.8. Remark. More generally, suppose that $\mathcal{N}$ is a countable nest, order isomorphic to an ordinal, and that $\mathcal{N}$ contains intervals of length $n_1 < n_2 < n_3 < \cdots$, such that the interval with length n_k has consecutive atoms of size $(d_{n_k}, d_{n_{k-1}}, ..., d_1)$. Without loss of generality, we may assume that the subspaces upon which these intervals act are mutually orthogonal. The above construction can be used to produce an epimorphism of $\mathcal{T}(\mathcal{N})$ onto $\mathcal{T}(\mathcal{M})$, where $\mathcal{M}$ is a nest of order type ω^* (and whose atoms have dimensions $(..., d_4, d_3, d_2, d_1))$. By Proposition 2.2, $\operatorname{rtsr}(\mathcal{T}(\mathcal{M})) = \infty$, and thus by Theorem 1.1, $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$ as well.

2.9. Example. Let $\mathcal{N}$ be the nest order isomorphic to ω, whose atoms $(A_n)_{n=1}^\infty$ have dimensions $1, 2, 1, 3, 2, 1, 4, 3, 2, 1, ..., n, n-1, n-2, ..., 3, 2, 1, n+1, n, n-1, ...$ Then $\operatorname{ltsr}(\mathcal{T}(\mathcal{N})) = \operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$.

The next result is an immediate consequence of the Remark 2.8.

2.10. Corollary. *Let $\mathcal{N}$ be a countable nest, order isomorphic to an ordinal. Suppose that $\mathcal{N}$ contains intervals E_j of length n_j, where $n_j < n_{j+1}$ for all j, and such that* $\max\{\dim A : A \in E_j$ *an atom*$\} < K$ *for some constant $K \geq 1$ independent of j. Then* $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$.

We remark that in Remark 2.8 and in Corollary 2.10, the assumption that $\mathcal{N}$ be countable and order isomorphic to an ordinal is stronger than what is needed to obtain an epimorphism of $\mathcal{T}(\mathcal{N})$ onto a nest algebra $\mathcal{T}(\mathcal{M})$ with right topological stable rank equal to ∞. On the other hand, it simplifies the exposition, and the right topological stable rank of $\mathcal{T}(\mathcal{N})$ in all other cases has been dealt with already.

Theorem 2.1 shows that if $\mathcal{N}$ is a nest, ordered like the natural numbers, whose atoms grow geometrically fast in dimension, then $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = 2$. The conditions on the rate of growth of the dimensions of the atoms can be somewhat relaxed. The following observation will prove useful.

Let $B = \begin{bmatrix} B_1 & B_2 \\ 0 & B_4 \end{bmatrix} \in \mathcal{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ be an operator where B_1, B_4 are invertible. Then B is invertible with $B^{-1} = \begin{bmatrix} B_1^{-1} & -B_1^{-1}B_2B_4^{-1} \\ 0 & B_4^{-1} \end{bmatrix}$. Thus if there exists a constant $H > 0$ so that $\|B_1^{-1}\| \leq H$, $\|B_4^{-1}\| \leq H$, then $\|B^{-1}\| \leq 2H + H^2\|B\|$.

Using induction, it is not hard to see that if

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ & A_{22} & \dots & A_{2n} \\ & & \ddots & \\ & & & A_{nn} \end{bmatrix}$$

is an operator in $\mathcal{B}(\oplus_{k=1}^n \mathfrak{H}_k)$ and if each A_{kk} is invertible with $\|A_{kk}^{-1}\| \leq H$ for some $H > 0$, then $\|A^{-1}\| \leq L$ for some constant L that depends only upon H, n and $\|A\|$.

2.11. Theorem. *Suppose that $\mathcal{N}$ is a nest ordered like ω, all of whose atoms are finite dimensional. Let A_n, $n \geq 1$, denote the atoms of $\mathcal{N}$, and let $r_n = \dim A_n$ for $n \geq 1$. Set $R(k) = \max_{1 \leq i \leq k} r_i$ for $k \geq 1$. Suppose that there exists a $\gamma > 0$ and an integer $J > 0$ such that*

$$R((k+1)J) \geq (1+\gamma)R(kJ) \quad \textit{for all} \quad k \geq 1.$$

Then

$$\operatorname{ltsr}(\mathcal{T}(\mathcal{N})) = \infty \quad \textit{and} \quad \operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = 2.$$

Proof. By Corollary 2.3, $\operatorname{ltsr}(\mathcal{T}(\mathcal{N})) = \infty$.

Suppose that $\gamma > 0$ and $J \geq 1$ are chosen as in the statement of the Theorem, and that

$$\begin{aligned} R((k+1)J) &= \max\{r_i : 1 \leq i \leq (k+1)J\} \\ &= \max\{r_i : kJ \leq i \leq (k+1)J\} \geq (1+\gamma)R(kJ). \end{aligned}$$

Choose an integer $p \geq 1$ so that $\dfrac{(1+\gamma)^p}{p} \geq 5J$. For $k \geq 1$, set

$$F_k = \sum_{(k-1)pJ < i \leq kpJ} P(A_i) \quad \text{and} \quad Q_k = \sum_{i=1}^{k} F_k.$$

In essence, we are grouping together blocks of length pJ of $\mathcal{N}$ into a single "megablock". The compression of $\mathcal{T}(\mathcal{N})$ to any such "megablock" is an upper triangular $pJ \times pJ$ operator matrix whose entries are finite dimensional matrices.

Observe that if $M_k := \operatorname{rank} F_k$, then $M_k \leq pJR(kpJ)$ (since the maximum rank among the atoms of F_k is $R(kpJ)$) and that

$$\begin{aligned} M_{k+1} &= \operatorname{rank} F_{k+1} \geq R((k+1)pJ) \\ &\geq (1+\gamma)^p R(kpJ) \geq 5pJR(kpJ) \geq 5M_k. \end{aligned}$$

Hence $M_k \geq 4\sum_{i<k} M_i$ for each $k \geq 2$.

The remainder of the proof will be a modification of the proof of Theorem 2.1. One significant difference is that we will work with blocks of length pJ of $\mathcal{N}$. Unfortunately, this creates a few extra problems that requires a slightly trickier approach.

Let $A, B \in \mathcal{T}(\mathcal{N})$ be given, and let $\varepsilon > 0$. Let $\Delta(A) = \sum_{k\geq 1} E_k A E_k$ be the expectation of A onto the diagonal $\mathcal{D}(\mathcal{N})$ of $\mathcal{T}(\mathcal{N})$. As noted in the proof of Theorem 2.1, $\mathcal{D}(\mathcal{N})$ is a finite von Neumann algebra and so $\Delta(A) = UP$ for some unitary U and positive operator P lying in $\mathcal{D}(\mathcal{N})$. But then $\Delta(A)' = U(P + \varepsilon I)$ is an ε-perturbation of $\Delta(A)$ which is invertible with inverse bounded above by ε^{-1}.

Let $A' = \Delta(A)' + (A - \Delta(A))$. Note that the compression of A' to $F_k\mathfrak{H}$ is a $pJ \times pJ$ block-upper triangular matrix whose diagonal entries are all invertible with inverses bounded above by ε^{-1}. By the comments preceding this Theorem, $F_k A' F_k$ is invertible with $\|(F_k A' F_k)^{-1}\| \leq L_A$, where L_A is a constant depending only upon ε, pJ, and $\|A\|$. A similar construction applied to B yields an operator B' such that $F_k B' F_k$ is invertible with $\|(F_k B' F_k)^{-1}\| \leq L_B$ for all $k \geq 1$, where L_B is a constant depending only upon ε, pJ and $\|B\|$.

Thus we can write $A' = D_a + A_0'$, $B' = D_b + B_0'$, where $D_a = \sum_{k\geq 1} F_k A' F_k$, $D_b = \sum_{k\geq 1} F_k B' F_k$ are invertible elements of $\mathcal{D} = \sum_{k\geq 1}(F_k \mathcal{T}(\mathcal{N}) F_k)$, where $\|D_a^{-1}\| \leq L_A$, $\|D_b^{-1}\| \leq L_B$ and $A_0' := A' - D_a$, $B_0' := B' - D_b$ lie in the ideal

$$\mathcal{T}_0'(\mathcal{N}) = \{T \in \mathcal{T}(\mathcal{N}) : \sum_{k\geq 1} F_k T F_k = 0\}.$$

We wish to multiply A' and B' on the right by invertible elements of $\mathcal{T}(\mathcal{N})$ to make the block diagonal equal to the identity and the first super-diagonal with respect to the block decomposition equal to 0. To accomplish this, let $\Delta_1(T) = \sum_{k\geq 1} F_k T F_{k+1}$ for $T \in \mathcal{T}(\mathcal{N})$. Then define

$$X = D_a^{-1} \exp(-\Delta_1(A')D_a^{-1}) \quad \text{and} \quad Y = D_b^{-1} \exp(-\Delta_1(B')D_b^{-1}).$$

Then $A'X = I + X_0$ and $B = I + Y_0$ where X_0 and Y_0 lie in the ideal

$$\mathcal{T}_0'(\mathcal{N})^2 = \{T \in \mathcal{T}(\mathcal{N}) : \sum_{k\geq 1} F_k T F_k = 0 = \sum_{k\geq 1} F_k T F_{k+1}\}.$$

So we may write $X_0 = \sum_{k\geq 3} Q_{k-2} X_0 F_k$ and $Y_0 = \sum_{k\geq 3} Q_{k-2} Y_0 F_k$.

Arguing as in the proof of Theorem 2.1, we see that

$$\operatorname{rank}(X_0F_k) + \operatorname{rank}(Y_0F_k) \leq \frac{1}{2} \operatorname{rank} F_{k-1}.$$

Thus we can choose projections $P_k \leq F_k$ with $\operatorname{rank} P_k \leq \frac{1}{2} \operatorname{rank} F_{k-1}$ so that $X_0F_k = X_0P_k$ and $Y_0F_k = Y_0P_k$. The problem with a trivial modification of the argument is that we can no longer assert that P_k and the partial isometries constructed in the proof of Theorem 2.1 belong to the nest algebra. However, because we have made some extra room to maneuver, we can construct partial isometries U_k mapping $P_k\mathfrak{H}$ into $P_{k-1}^\perp F_{k-1}\mathfrak{H}$; and these operators evidently lie in the nest algebra. Set $U = \sum_{k\geq 3} U_k$. Moreover

$$X_0U^* = \sum_{k\geq 3} Q_{k-2} X_0 P_k \sum_{j\geq 3} P_j U_j^* = \sum_{k\geq 3} Q_{k-2}(X_0 P_k U_k^*) F_{k-1}.$$

So this operator also lies in $\mathcal{T}(\mathcal{N})$. Similarly Y_0U^* belongs to the nest algebra. Moreover the range of U is orthogonal to the range of $P = \sum_{k\geq 3} P_k$, and therefore $X_0U = Y_0U = 0$.

Now let $A'' = A' + \delta X_0 U^* X^{-1}$ and $B'' = B' - \delta Y_0 U^* Y^{-1}$, where $\delta > 0$ is chosen to be sufficiently small so that $\|A - A''\| < \varepsilon$ and $\|B - B''\| < \varepsilon$. Observe that

$$\begin{aligned} A''X(I - \delta^{-1}U) &+ B''Y(I + \delta^{-1}U) \\ &= (I + X_0 + \delta X_0U^*)(I - \delta^{-1}U) + (I + Y_0 - \delta Y_0 U^*)(I + \delta^{-1}U) \\ &= (I + X_0 + \delta X_0 U^* - \delta^{-1}U - X_0) + (I + Y_0 - \delta Y_0 U^* + \delta^{-1}U - Y_0) \\ &= 2I + \delta(X_0 - Y_0)U^*. \end{aligned}$$

With δ sufficiently small, this is invertible, and hence $\begin{bmatrix} A'' & B'' \end{bmatrix}$ is right invertible.

Therefore $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = 2$. □

One very interesting consequence of Theorems 2.1 and 2.11 is that they allow us to resolve (in certain cases) a question of Davidson, Harrison and Orr (see [6], Section 8) regarding epimorphisms of nest algebras onto $\mathcal{B}(\mathfrak{H})$.

2.12. Proposition. *Let $\mathcal{N}$ be a nest of the type described in Theorem 2.11. Let $\mathcal{A}$ be an operator algebra with* $\mathrm{rtsr}(\mathcal{A}) = \infty$. *Then there is no epimorphism of $\mathcal{T}(\mathcal{N})$ onto $\mathcal{A}$. In particular, this holds if $\mathcal{A}$ is any one of the following:*

(a) $\mathcal{B}(\mathfrak{H})$*;*
(b) $\mathcal{T}(\mathcal{V})$*, where $\mathcal{V}$ is an uncountable nest; or*
(c) $\mathcal{T}(\mathcal{M})$*, where $\mathcal{M}$ is a countable nest which is not isomorphic to an ordinal.*

Proof. Observe that $\mathrm{rtsr}(\mathcal{B}(\mathfrak{H})) = \mathrm{rtsr}(\mathcal{T}(\mathcal{V})) = \mathrm{rtsr}(\mathcal{T}(\mathcal{M})) = \infty$ by Theorem 2.4. If such an epimorphism were to exist, then by Theorem 1.1, it would follow that $\mathrm{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$, which is a contradiction. □

We finish this section by mentioning a few unresolved questions dealing with the stable rank of nest algebras.

There are still a number of nests for which we have been unable to determine the left and right topological stable ranks. When the nest is ordered like ω, it is clear that the value of the right topological stable rank of the corresponding nest algebra depends upon how fast the atoms grow. If the atoms of $\mathcal{N}$ are bounded in dimension, then $\mathrm{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$. If the dimensions of the atoms grow at an exponential rate, then the right topological stable rank is 2. What happens when the rate of growth lies between these two extremes? A key case which we have been unable to resolve and which would very likely shed light upon the general problem is the following:

Question 1. Suppose that $\mathcal{N}$ is a nest, ordered like ω, whose atoms $(A_n)_{n=1}^{\infty}$ satisfy $\dim\, A_n = n$, $n \geq 1$. What is $\mathrm{rtsr}(\mathcal{T}(\mathcal{N}))$?

We note that by Corollary 2.3, $\mathrm{ltsr}(\mathcal{T}(\mathcal{N})) = \infty$.

Observe that in all of our examples, $\mathrm{rtsr}(\mathcal{T}(\mathcal{N})) \in \{2, \infty\}$.

Question 2. Does there exist a countable nest $\mathcal{N}$, order isomorphic to an ordinal (in particular - order isomorphic to ω), all of whose atoms are finite dimensional, for which the value of $\mathrm{rtsr}(\mathcal{T}(\mathcal{N}))$ is other than 2 or ∞?

The above analysis suggests that it is not the exact dimensions of the atoms which is significant, but rather the rate at which these dimensions grow. If $\mathrm{rtsr}(\mathcal{T}(\mathcal{N})) = m$ for some $3 \leq m < \infty$, then by a straightforward adapation of Proposition 6.1 of [14] to general Banach algebras, $\mathrm{rtsr}(\mathcal{T}(\mathcal{N}) \otimes \mathbb{M}_n) = \lceil (\mathrm{rtsr}(\mathcal{T}(\mathcal{N})) - 1)/n \rceil + 1$, and hence $\mathrm{rtsr}(\mathcal{T}(\mathcal{N}) \otimes \mathbb{M}_n) = 2$ for sufficiently large values of n. (Here $\lceil k \rceil$ denotes the least integer greater than or equal to k.) But $\mathcal{T}(\mathcal{N}) \otimes \mathbb{M}_n \simeq \mathcal{T}(\mathcal{M})$, where $\mathcal{M}$ is a nest, order isomorphic to $\mathcal{N}$, whose atoms have dimension n times the dimension of the corresponding atoms of $\mathcal{N}$. As such, the rate of growth of the atoms of $\mathcal{M}$ is identical to that of $\mathcal{N}$. We suspect that this should imply that $\mathrm{rtsr}(\mathcal{T}(\mathcal{N})) = 2$, but we have not been able to prove this.

If $\mathcal{A}$ is any unital Banach algebra with $\mathrm{rtsr}(\mathcal{A}) = \infty$, then it follows from the previous paragraph that $\mathrm{rtsr}(\mathcal{A} \otimes \mathbb{M}_n) = \infty$ for all $n \geq 1$. We obtain the following result for Banach algebras which was established for C*-algebras by Rieffel [14, Theorem 6.4]. We shall first fix a basis $\{e_n\}_{n=1}$ for $\mathfrak{H}$, and denote by E_{ij} the matrix unit $e_i e_j^* \in \mathcal{K}(\mathfrak{H})$. If $\mathcal{A}$ is a unital Banach algebra, consider any Banach algebra cross norm on $\mathcal{A} \otimes \mathcal{K}(\mathfrak{H})$ for which $\mathcal{A}$ is imbedded isometrically (but not unitally) as a corner $\mathcal{A} \otimes E_{11}$, each matrix algebra $\mathcal{A}\otimes\mathbb{M}_n$ is identified with $\big(\sum_{i=1}^n E_{ii}\big)\mathcal{A}\otimes\mathcal{K}(\mathfrak{H})\big(\sum_{j=1}^n E_{jj}\big)$, and the union of these matrix algebras is norm dense in $\mathcal{A}\otimes\mathcal{K}(\mathfrak{H})$.

2.13. Proposition. *Let $\mathcal{A}$ be a Banach algebra with identity. Then*

$$\operatorname{ltsr}(\mathcal{A}\otimes\mathcal{K}(\mathfrak{H})) = \operatorname{rtsr}(\mathcal{A}\otimes\mathcal{K}(\mathfrak{H})) \in \{1,2\},$$

and it equals 1 *if and only if* $\operatorname{tsr}(\mathcal{A}) = 1$.

Proof. The argument that the (left or right) topological stable rank is at most 2 is done by Rieffel [14]. He also shows that $\operatorname{ltsr}(\mathcal{A}) = 1$ and $\operatorname{rtsr}(\mathcal{A}) = 1$ are both equivalent to the density of the invertible elements. If the invertibles are dense in $\mathcal{A}$, Rieffel shows that they are also dense in $\mathcal{A}\otimes\mathbb{M}_n$ for all $n\geq 1$. From this, it is easy to see that the invertibles are dense in the unitization $(\mathcal{A}\otimes\mathcal{K}(\mathfrak{H}))^\sim$. To complete the proof, it suffices to show that if the invertibles are dense in $(\mathcal{A}\otimes\mathcal{K}(\mathfrak{H}))^\sim$, then they are also dense in $\mathcal{A}$.

Fix $A\in\mathcal{A}$ with $\|A\|\leq 1/2$. Let $\mathcal{C}$ denote the circle centred at 0 of radius 3/4. Define $M = \sup\{\|(zI-A)^{-1}\| : z\in\mathcal{C}\} \geq 4$. Then $A' = A\otimes E_{11} + I\otimes E_{11}^\perp$ belongs to $(\mathcal{A}\otimes\mathcal{K}(\mathfrak{H}))^\sim$. For any $0<\varepsilon<(6\pi M^2)^{-1}<1/4$, choose $B\in(\mathcal{A}\otimes\mathcal{K}(\mathfrak{H}))^\sim$ so that $\|A'-B\|<\varepsilon$.

The spectrum of A' is $\sigma(A') = \sigma(A)\dot\cup\{1\}$. By [10, Theorem 1.1], $\sigma(B)$ is disjoint from $\mathcal{C}$. By the Riesz functional calculus, there is an idempotent

$$P = \int_{\mathcal{C}} (zI-B)^{-1}\,dz$$

which commutes with B. This idempotent is close to $E := I\otimes E_{11}$ because of the following estimates. For $z\in\mathcal{C}$,

$$\begin{aligned}
\|(zI-B)^{-1}\| &= \big\|\big((zI-A')-(B-A')\big)^{-1}\big\| \\
&= \Big\|(zI-A')^{-1}\sum_{n\geq 0}\big((B-A')(zI-A')^{-1}\big)^n\Big\| \\
&\leq \frac{M}{1-M\varepsilon} < 2M.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|P-E\| &= \Big\|\int_{\mathcal{C}}(zI-B)^{-1}-(zI-A')^{-1}\,dz\Big\| \\
&\leq 2\pi\frac{3}{4}\sup_{z\in\mathcal{C}}\|(zI-B)^{-1}(A'-B)(zI-A')^{-1}\| \\
&\leq \frac{3\pi}{2}2M\varepsilon M = 3\pi M^2\varepsilon =: \varepsilon' < \frac{1}{2}.
\end{aligned}$$

Now a standard argument shows that $S = PE + (I-P)E^\perp$ is an invertible element of $(\mathcal{A}\otimes\mathcal{K}(\mathfrak{H}))^\sim$ such that $SE = PS$ and

$$\|S-I\| = \|(P-E)(E-E^\perp)\| = \|P-E\| \leq \varepsilon'.$$

Thus $B' = S^{-1}BS$ is close to B and has the form $B' = B_1 \otimes E_{11} + E_{11}^{\perp} B_2 E_{11}^{\perp}$. Indeed,

$$\begin{aligned} \|B' - B\| &\leq \|S^{-1}\| \, \|(S-I)B - B(S-I)\| \\ &\leq \frac{1}{1-\varepsilon'} 2\|B\|\varepsilon' \leq \frac{1+2\varepsilon}{1-\varepsilon'} =: \varepsilon''. \end{aligned}$$

Thus we obtain that $\|A - B_1\| < \varepsilon + \varepsilon''$ and B_1 is invertible in $\mathcal{A}$. Since ε'' tends to 0 as ε does, we conclude that the invertibles are dense in $\mathcal{A}$. □

Another interesting and open problem concerns the Bass stable rank of nest algebras. For the nests of Theorem 2.1 or more generally for those of Theorem 2.11, it follows from the inequality mentioned in the introduction that $\mathrm{Bsr}(\mathcal{T}(\mathcal{N})) \leq \min(\mathrm{ltsr}(\mathcal{T}(\mathcal{N})), \mathrm{rtsr}(\mathcal{T}(\mathcal{N})) = 2$. Nevertheless, an explicit calculation of $\mathrm{Bsr}(\mathcal{T}(\mathcal{N}))$ for this or indeed for any nest algebra seems to be an rather difficult problem.

Question 3. Find $\mathrm{Bsr}(\mathcal{T}(\mathbb{N}))$, or indeed $\mathrm{Bsr}(\mathcal{T}(\mathcal{N})))$ of any nest algebra.

3. Non-commutative operator algebras generated by isometries

Let us now consider operator algebras generated by free semigroups of isometries. The theory here divides along two lines; the norm-closed version, often referred to as *non-commutative disc algebras*, and the WOT-closed versions, known simply as *free semigroup algebras*. The latter algebras include the *non-commutative Toeplitz algebras*, to be described below.

Let $n \geq 1$. The non-commutative disc algebra $\mathfrak{A}_n$, introduced by Popescu [12, 13], is (completely isometrically isomorphic to) the norm-closed subalgebra of $\mathcal{B}(\mathfrak{H})$ generated by the identity operator I and n isometries $S_1, S_2, ..., S_n$ with pairwise orthogonal ranges. It is shown in [13] that the complete isometric isomorphism class of $\mathfrak{A}_n$ is independent of the choice of the isometries, and that $\mathfrak{A}_n$ is completely isometrically isomorphic to $\mathfrak{A}_m$ if and only if $m = n$. Note that for each $1 \leq j \leq n$, $S_j^* S_j = I \geq \sum_{i=1}^n S_i S_i^*$, and when $\sum_{i=1}^n S_i S_i^* = I$, the C^*-algebra generated by $\{S_1, S_2, ..., S_n\}$ is the Cuntz algebra $\mathcal{O}_n$. When $\sum_{i=1}^n S_i S_i^* < I$, the C^*-algebra generated by $\{S_1, S_2, ..., S_n\}$ is the Cuntz-Toeplitz algebra $\mathcal{E}_n$.

Given isometries $S_1, S_2, ..., S_n$ with pairwise orthogonal ranges as above, the WOT-closure $\mathfrak{S}_n$ of the corresponding disc algebra $\mathfrak{A}_n$ is known as a *free semigroup algebra*. These were first described in [8]. Of particular importance is the following example. Let $\mathbb{F}_n^+$ denote the free semigroup on n generators $\{1, 2, ..., n\}$. Consider the Hilbert space $\mathfrak{K}_n = \ell^2(\mathbb{F}_n^+)$ with orthonormal basis $\{\xi_w : w \in \mathbb{F}_n^+\}$. For each word $v \in \mathbb{F}_n^+$, we may define an isometry $L_v \in \mathcal{B}(\mathfrak{K}_n)$ by setting $L_v \xi_w = \xi_{vw}$ (and extending by linearity and continuity to all of $\mathfrak{K}_n$). The identity operator is $L_{\varnothing}$. Then $L_1, L_2, ..., L_n$ are n isometries with orthogonal ranges, and the WOT-closed algebra $\mathfrak{L}_n$ generated by $I, L_1, L_2, ..., L_n$ is called the non-commutative Toeplitz algebra.

A theorem of Davidson, Katsoulis, and Pitts [7] shows that if $\mathfrak{S}_n$ is a free semigroup algebra, then there exists a projection $P \in \mathfrak{S}_n$ such that $\mathfrak{S} = \mathfrak{M}P \oplus P^{\perp}\mathfrak{M}P^{\perp}$, where $\mathfrak{M}$ is the von Neumann algebra generated by $\mathfrak{S}_n$, and $\mathfrak{S}P^{\perp} = P^{\perp}\mathfrak{S}P^{\perp}$ is completely isometrically isomorphic to $\mathfrak{L}_n$.

3.1. Theorem. *Let $n \geq 2$.*

(a) *If $\mathfrak{A}_n$ is the non-commutative disc algebra on n-generators, then* $\operatorname{tsr}(\mathfrak{A}_n) = \infty$.
(b) *If $\mathfrak{S}_n$ is a free semigroup algebra on n-generators, then* $\operatorname{tsr}(\mathfrak{S}_n) = \infty$.

Proof. First observe that both $\mathfrak{A}_n$ and $\mathfrak{S}_n$ are generated by $n \geq 2$ isometries with mutually orthogonal ranges. By Proposition 1.4, $\operatorname{rtsr}(\mathfrak{A}_n) = \operatorname{rtsr}(\mathfrak{S}_n) = \infty$.

We now consider the left topological stable rank of these two algebras.

Let $V_1, V_2, ..., V_n \in \mathcal{B}(\mathfrak{H})$ be isometries with mutually orthogonal ranges. Let $A_i = \frac{1}{n} V_i^*$, $1 \leq i \leq n$. Then $\sum_{i=1}^n A_i^* A_i = \frac{1}{n} I$ is a strict contraction. By Proposition 2 of [3], there exists a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ and pure isometries $\{W_i\}_{i=1}^n \subseteq \mathcal{B}(\mathfrak{K})$ with pairwise orthogonal ranges so that $\mathfrak{H}^\perp \in \operatorname{Lat} W_i$ and $P_{\mathfrak{H}} W_i|_{\mathfrak{H}} = A_i$, $1 \leq i \leq n$.

(a) The norm-closed algebra $\mathcal{B}_n \subseteq \mathcal{B}(\mathfrak{K})$ generated by $\{I, W_1, W_2, ..., W_n\}$ satisfies $\mathcal{B}_n \simeq \mathfrak{A}_n$. The compression map

$$\begin{array}{rccc} \gamma: & \mathcal{B}_n & \rightarrow & \mathcal{B}(\mathfrak{H}) \\ & X & \mapsto & P_{\mathfrak{H}} X|_{\mathfrak{H}} \end{array}$$

is a (completely contractive) homomorphism, as $\mathfrak{H}^\perp \in \operatorname{Lat} W_i$ for all i. Thus $\operatorname{rtsr}(\mathcal{B}_n) \geq \operatorname{rtsr}(\overline{\gamma(\mathcal{B}_n)})$. But $\gamma(W_i) = \frac{1}{n} V_i^*$ for all $1 \leq i \leq n$. Thus $\overline{\gamma(\mathcal{B}_n)}$ contains $n \geq 2$ co-isometries with mutually orthogonal initial spaces, and hence $\operatorname{ltsr}(\mathfrak{A}_n) = \operatorname{ltsr}(\mathcal{B}_n) \geq \operatorname{ltsr}(\overline{\gamma(\mathcal{B}_n)}) = \infty$.

(b) This proof is almost identical. Since the $\{W_i\}_{i=1}^\infty$ are *pure* co-isometries, the WOT-closed algebra $\mathfrak{W}_n$ generated by $\{I, W_1, W_2, ..., W_n\}$ is a multiple of $\mathfrak{L}_n$, i.e. $\mathfrak{W}_n = (\mathfrak{L}_n^*)^{(k)}$ for some $1 \leq k \leq \infty$. Thus $\operatorname{rtsr}(\mathfrak{W}_n) = \operatorname{ltsr}((\mathfrak{L}_n)^{(k)}) = \operatorname{ltsr}(\mathfrak{L}_n)$. But the argument above used with the corresponding compression map

$$\begin{array}{rccc} \gamma: & \mathfrak{W}_n & \rightarrow & \mathcal{B}(\mathfrak{H}) \\ & X & \mapsto & P_{\mathfrak{H}} X|_{\mathfrak{H}} \end{array}$$

shows that $\operatorname{rtsr}(\mathfrak{W}_n) = \infty$, since $\overline{\gamma(\mathfrak{W}_n)}$ contains at least $n \geq 2$ isometries with mutually orthogonal ranges.

Hence $\operatorname{ltsr}(\mathfrak{L}_n) = \infty$. But by the Structure Theorem for free semigroup algebras mentioned above [7], either there is a homomorphism of $\mathfrak{S}_n$ onto $\mathfrak{L}_n$ or $\mathfrak{S}_n$ is a von Neumann algebra containing two isometries with orthogonal ranges. Either way, $\operatorname{rtsr}(\mathfrak{S}_n) = \operatorname{ltsr}(\mathfrak{S}_n) = \infty$. □

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