1-HYPERREFLEXIVITY AND COMPLETE HYPERREFLEXIVITY

KENNETH R. DAVIDSON AND RUPERT H. LEVENE

Abstract. The subspaces and subalgebras of $\mathcal{B}(\mathcal{H})$ which are hyperreflexive with constant 1 are completely classified. It is shown that there are 1-hyperreflexive subspaces for which the complete hyperreflexivity constant is strictly greater than 1. The constants for $\mathbb{C}T \otimes \mathcal{B}(\mathcal{H})$ are analyzed in detail.

Keywords. 1-hyperreflexivity, complete hyperreflexivity, distance formula

The study of invariant subspaces and the notion of reflexivity plays a central role in operator theory. The quantitative notion of hyperreflexivity is a significant strengthening of reflexivity. And when this property holds, there are important ramifications. This is best seen in the theory of nest algebras, where one obtains a precise distance formula [1]; and for a von Neumann algebra, where hyperreflexivity is equivalent to the vanishing of a certain cohomology group [5].

Until recently, the collection of known hyperreflexive algebras has been quite limited. In addition to nest algebras and most von Neumann algebras (excluding those with certain intractible type II_1 commutants, where the problem remains open), there were not many others. The Toeplitz algebra [7] and certain free semigroup algebras including the so called noncommutative analytic Toeplitz algebras [10, 8] are hyperreflexive. However the first author and S. Power constructed CSL algebras which are not hyperreflexive [11]. The class of hyperreflexive algebras was significantly expanded by Bercovici [2] who found general properties which imply hyperreflexivity. Jaeck and Power [16] have combined these results to show that the free semigroupoid algebra associated to any finite directed graph is hyperreflexive.

These notions make perfect sense for subspaces as well as algebras. Loginov and Shulman [20] reformulated reflexivity in this context. This was done for hyperreflexivity by Larson [19]. See Hadwin [13, 14] for a quite general view of these issues. A very recent theorem of Müller

2000 Mathematics Subject Classification: 47L05.
September 7, 2005 version.

and Ptak [22] shows that every finite dimensional reflexive subspace is hyperreflexive, a surprisingly difficult result.

One focus of this paper is the case where one obtains an exact distance formula. We call a subspace 1-hyperreflexive if this holds, namely

$$\operatorname{dist}(T,\mathcal{S}) = \sup_{\|x\|=1} \operatorname{dist}(Tx,\mathcal{S}x) \quad \text{for all} \quad T\in\mathcal{B}(\mathcal{H}).$$

This can be reformulated as an interchange of sups and infs:

$$\inf_{S\in\mathcal{S}} \sup_{\|x\|=1} \|Tx - Sx\| = \sup_{\|x\|=1} \inf_{S\in\mathcal{S}} \|Tx-Sx\|.$$

We will classify these spaces.

The second focus of this paper is the notion of complete hyperreflexivity, namely the hyperreflexivity of $\mathcal{S}\,\bar{\otimes}\,\mathcal{B}(\mathcal{H})$, the WOT-closed spatial tensor product. It is an open question whether hyperreflexivity of $\mathcal{S}$ implies complete hyperreflexivity. In the case of known examples such as nest algebras and von Neumann algebras, the proofs yields the same constant for the complete case as for the algebra itself. The same is true for the Toeplitz algebra, free semigroup algebras and algebras handled by Bercovici's Theorem. We will produce examples of hyperreflexive subspaces which are completely hyperreflexive but for which the constant increases. Indeed, any one-dimensional subspace $\mathbb{C}T$ where $\operatorname{rank} T\geq 2$ will be 1-hyperreflexive but not completely 1-hyperreflexive; but it will have complete hyperreflexive constant no greater than 4.

1. Setting the Stage

Recall that subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ is reflexive if

$$\mathfrak{A} = \operatorname{Alg}(\operatorname{Lat}\mathfrak{A}) = \{T\in\mathcal{B}(\mathcal{H}) : TP = PTP \text{ for all } P\in\operatorname{Lat}\mathfrak{A}\},$$

and $\mathfrak{A}$ is hyperreflexive if there is a constant C so that for all $T\in\mathcal{B}(\mathcal{H})$,

$$\operatorname{dist}(T,\mathfrak{A}) \leq C\sup\{\|P^\perp TP\| : P\in\operatorname{Lat}\mathfrak{A}\}.$$

The inequality

$$\sup\{\|P^\perp TP\| : P\in\operatorname{Lat}\mathfrak{A}\} \leq \operatorname{dist}(T,\mathfrak{A})$$

is elementary. In the same vein, a subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is reflexive if

$$\mathcal{S} = \operatorname{Ref}(\mathcal{S}) = \{T\in\mathcal{B}(\mathcal{H}) : Tx\in\overline{\mathcal{S}x} \text{ for all } x\in\mathcal{H}\},$$

and $\mathcal{S}$ is hyperreflexive if there is a constant C so that for all $T\in\mathcal{B}(\mathcal{H})$,

$$\operatorname{dist}(T,\mathcal{S}) \leq C\sup\{\|P_{\mathcal{S}x}^\perp Tx\| : \|x\|=1\}.$$

The optimal constant $\kappa_{\mathcal{S}}$ is called the *distance constant.* We say that $\mathcal{S}$ is 1-*hyperreflexive* if $\kappa_{\mathcal{S}}=1$. We will write

$$\beta_{\mathcal{S}}(T) = \sup\{\|P_{\mathcal{S}x}^\perp Tx\| : \|x\|=1\}.$$

One trivial observation is worth recording: *if $\mathcal{S}$ is hyperreflexive with constant C, then so is $U\mathcal{S}V$ where U and V are any unitary operators.*

One purpose of this paper is to describe all subspaces which are 1-hyperreflexive. There are three known classes of algebras with distance constant 1:

A1. Nest algebras. Arveson [1] (see [6, Theorem 9.5]).

A2. $\mathbb{C}I$. Stampfli [26] (see [6, Theorem 9.15]).

A3. $\mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2)$ for any Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. [17]

Recall that a nest is a chain $\mathcal{N}$ of subspaces of a Hilbert space $\mathcal{H}$ containing both 0 and $\mathcal{H}$ which is complete with respect to intersections and closed spans. The corresponding nest algebra $\mathcal{T}(\mathcal{N})$ consists of all operators leaving the nest invariant. Thus it is reflexive by definition. The 1-hyperreflexivity of nest algebras is known as the Arveson distance formula. It plays a central role in the theory. We refer the reader to [6] for more information about nest algebras.

Stampfli shows that $\operatorname{dist}(T, \mathbb{C}I) = \frac{1}{2}\|\delta_T\|$ where δ_T is the inner derivation $\delta_T(A) = AT - TA$. He accomplishes this by proving that if $\operatorname{dist}(T, \mathbb{C}I) = \|T\| = 1$, then there is a sequence x_n of unit vectors so that

$$\lim_{n\to\infty} \|Tx_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \langle Tx_n, x_n\rangle = 0.$$

Then setting $P_n = x_n x_n^*$ yields

$$\beta_{\mathbb{C}I}(T) \geq \sup_{n\geq 1} \|P_n^\perp T P_n\| = 1 = \|T\|.$$

Example A3 cannot be extended to the direct sum of three copies of $\mathcal{B}(\mathcal{H})$ because even the 3×3 diagonal algebra is not 1-hyperreflexive. Indeed it has constant $\sqrt{3/2}$ [9]. As we will see below, the proper generalization of this example is that the space of block off-diagonal operators is 1-hyperreflexive.

When we expand our view to subspaces, these examples become:

S1. Nest bimodules. If $\mathcal{M}$ and $\mathcal{N}$ are nests and θ is an order preserving map of $\mathcal{N}$ into $\mathcal{M}$, then the WOT-closed $\mathcal{T}(\mathcal{M})$–$\mathcal{T}(\mathcal{N})$ bimodule

$$\mathfrak{X}(\theta) := \{T \in \mathcal{B}(\mathcal{H}) : TN \subset \theta(N) \quad \text{for all} \quad N \in \mathcal{N}\}$$

is 1-hyperreflexive.

S2. $\mathbb{C}T$ for T an arbitrary operator. Magajna [21].

S3. Let $\mathcal{P} = \{P_i : i \in \mathcal{I}\}$ and $\mathcal{Q} = \{Q_i : i \in \mathcal{I}\}$ be partitions of the identity of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then the subspace

$$\mathfrak{X} := \{T \in \mathcal{B}(\mathcal{H}) : Q_i T P_i = 0 \quad \text{for all} \quad i \in \mathcal{I}\}$$

is 1-hyperreflexive.

Magajna [21] generalizes Stampfli's result in a straighforward way. See the remark following Proposition 2.4.

If $\mathfrak{X}$ is a $\mathcal{T}(\mathcal{M})$–$\mathcal{T}(\mathcal{N})$ nest bimodule which is WOT-closed, then there is a unique left continuous order preserving map θ of $\mathcal{N}$ into $\mathcal{M}$ so that $\mathfrak{X} = \mathfrak{X}(\theta)$ [12] (see [6, Theorem 15.14]). In this paper, all nest bimodules will be WOT-closed; so we will just call them nest bimodules. The distance formula for nest bimodules is a routine adaptation of Power's proof [24] of Arveson's distance formula.

We wish to isolate part of the "diagonal" of a nest bimodule $\mathfrak{X}$. It is convenient to describe this by identifying a certain smaller nest bimodule $\mathfrak{X}_0$. Consider a finite or countable collection of elements $\{N_i : i \in \mathcal{I}\}$ in $\mathcal{N}$ such that for every i, $N_i^+ \neq N_i$ and $\theta(N_i^+) = M_i^+$ is a successor in $\mathcal{M}$, and the restriction of θ to this collection is injective. Let $B_i = N_i^+ - N_i$ and $A_i = M_i^+ - M_i$ be the corresponding atoms of $\mathcal{N}$ and $\mathcal{M}$ respectively. Define

$$\theta_0(N) = \begin{cases} M_i & \text{if} \quad N = N_i^+,\ i \in \mathcal{I} \\ \theta(N) & \text{otherwise} \end{cases}.$$

This determines a bimodule $\mathfrak{X}_0$. Moreover,

$$\mathfrak{X} = \mathfrak{X}_0 + \sum_{i \in \mathcal{I}}^{\oplus} A_i \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) B_i$$

and we refer to $\sum_{i\in\mathcal{I}}^{\oplus} A_i \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) B_i$ as the diagonal determined by the $\mathcal{T}(\mathcal{M})$–$\mathcal{T}(\mathcal{N})$ *bimodule pair* $(\mathfrak{X}, \mathfrak{X}_0)$. We also write $\Delta(\mathfrak{X}, \mathfrak{X}_0)$ for the set $\{(A_i, B_i) : i \in \mathcal{I}\}$.

The reason that this definition is convenient is that the notion of an atom of $\mathfrak{X}$ is in part determined by the choice of the nests $\mathcal{M}$ and $\mathcal{N}$, and is not intrinsic to $\mathfrak{X}$. For example, suppose that $\mathfrak{X} = \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then we should choose any proper projections A and B. Consider the nests $\mathcal{M} = \{0, A^\perp\mathcal{H}_2, \mathcal{H}_2\}$ and $\mathcal{N} = \{0, B\mathcal{H}_1, \mathcal{H}_1\}$. Then $\mathfrak{X}$ is a $\mathcal{T}(\mathcal{M})$–$\mathcal{T}(\mathcal{N})$ bimodule with $\theta(0) = 0$ and $\theta(B\mathcal{H}_1) = \theta(\mathcal{H}_1) = \mathcal{H}_2$. But $A\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)B$ becomes the diagonal if we choose $\mathfrak{X}_0$ to be the $\mathcal{T}(\mathcal{M})$–$\mathcal{T}(\mathcal{N})$ bimodule with $\theta(0) = 0$, $\theta(B\mathcal{H}_1) = A^\perp\mathcal{H}_2$ and $\theta(\mathcal{H}_1) = \mathcal{H}_2$.

In the next section, we present two constructions of new 1-hyperreflexive subspaces. Lemma 2.1 shows that one can replace atoms of a nest bimodule with a one-dimensional subspace. Lemma 2.2 shows that in examples of type S3, one can replace the zero diagonal entries with subspaces formed by Lemma 2.1. Our goal is to show that every 1-hyperreflexive subspace is obtained in this manner.

Theorem 1.1. *Let $\mathcal{S}$ be a* WOT*-closed subspace of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then $\mathcal{S}$ is 1-hyperreflexive if and only if there are partitions of the identity $\mathcal{C} = \{C_j : j \in \mathcal{J}\}$ and $\mathcal{D} = \{D_j : j \in \mathcal{J}\}$ of $\mathcal{H}_2$ and $\mathcal{H}_1$ respectively and for each $j \in \mathcal{J}$, there are subspaces $\mathfrak{X}_j$ of $C_j\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)D_j$ obtained from the construction of Lemma 2.1 so that*

$$\mathcal{S} = \{T \in \mathcal{B}(\mathcal{H}) : C_jTD_j \in \mathfrak{X}_j \quad \textit{for all} \quad j \in \mathcal{J}\}.$$

This will be proven in section 4.

2. 1-Hyperreflexivity

In order to complete the list of 1-hyperreflexive subspaces, we need two basic constructions. The first is more surprising. It says that atoms of a nest bimodule may be replaced by 1-dimensional subspaces.

Lemma 2.1. *Let $(\mathfrak{X}, \mathfrak{X}_0)$ be a $\mathcal{T}(\mathcal{M})$–$\mathcal{T}(\mathcal{N})$ bimodule pair with diagonal $\Delta(\mathfrak{X}, \mathfrak{X}_0) = \{(A_i, B_i) : i \in \mathcal{I}\}$. Select operators $X_i \in \mathcal{B}(B_i\mathcal{H}_1, A_i\mathcal{H}_2)$ and define a subspace*

$$\mathcal{S} = \{X \in \mathfrak{X} : A_iXB_i \in \mathbb{C}X_i \textit{ for } i \in \mathcal{I}\}.$$

Then $\mathcal{S}$ is 1-hyperreflexive.

Proof. Write $A_i = M_i^+ - M_i$ and $B_i = N_i^+ - N_i$, and let θ, θ_0 be the functions such that $\mathfrak{X} = \mathfrak{X}(\theta)$ and $\mathfrak{X}_0 = \mathfrak{X}(\theta_0)$.

If $x \in \mathcal{H}_1$, let N be the smallest subspace of $\mathcal{N}$ containing x. Then $\overline{\mathfrak{X}x} = \theta(N)$. Thus if $N \neq N_i^+$ for some $i \in \mathcal{I}$, we also have $\overline{\mathcal{S}x} = \theta(N)$. When $N = N_i^+$ for some $i \in \mathcal{I}$, $\overline{\mathcal{S}x} = \theta(N_i) + [X_iB_ix]$ where $[y]$ denotes the projection onto $\mathbb{C}y$.

Suppose that $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is given. Then

$$\begin{aligned}
\beta_{\mathcal{S}}(T) &= \sup_{\|x\|=1} \|P_{\mathcal{S}x}^{\perp}Tx\| \\
&= \max\Big\{ \sup_{N\in\mathcal{N}} \|P_{\theta(N)}^{\perp}TP_N\|, \ \sup_{i\in\mathcal{I}} \sup_{\substack{x\in N_i^+ \\ \|x\|=1}} \|(P_{\theta(N_i)}^{\perp} - [X_iB_ix])Tx\| \Big\} \\
&= \max\Big\{\beta_{\mathfrak{X}}(T), \ \sup_{i\in\mathcal{I}} \beta_{\mathbb{C}X_i}(P_{\theta(N_i)}^{\perp}T|_{N_i^+})\Big\}
\end{aligned}$$

The first important observation is that $P_{\theta(N_i)}^{\perp}\mathfrak{X}P_{N_i^+} = \mathbb{C}X_i$ is a 1-hyperreflexive subspace of $\mathcal{B}(N_i^+, \theta(N_i)^{\perp})$. Thus if $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, there is a multiple t_iX_i so that

$$\|P_{\theta(N_i)}^{\perp}T|_{N_i^+} - t_iX_i\| = \beta_{\mathbb{C}X_i}(P_{\theta(N_i)}^{\perp}T|_{N_i^+}).$$

Clearly, $\|t_i X_i\|$ is uniformly bounded. Let $S \in \mathcal{S}$ be the diagonal element given by $SB_i = t_i X_i$. Replace T by $T' = T - S$. Observe that

$$\begin{aligned}\beta_{\mathcal{S}}(T') = \beta_{\mathcal{S}}(T) &= \max\big\{\beta_{\mathfrak{X}}(T),\ \sup_{i\in\mathcal{I}} \beta_{\mathbb{C}X_i}(P^{\perp}_{\theta(N_i)} T|_{N_i^+})\big\}\\ &= \max\big\{\beta_{\mathfrak{X}}(T'),\ \sup_{i\in\mathcal{I}} \|P^{\perp}_{\theta(N_i)} T' P_{N_i^+}\|\big\}\\ &= \beta_{\mathfrak{X}_0}(T').\end{aligned}$$

Since $\mathfrak{X}_0$ is 1-hyperreflexive, there is an element $X_0 \in \mathfrak{X}_0$ so that $\|T-(S+X_0)\| = \|T'-X_0\| = \beta_{\mathfrak{X}_0}(T')$. This is the desired approximant, showing that $\mathcal{S}$ is 1-hyperreflexive. ∎

The second construction is more elementary.

Lemma 2.2. *Let $\mathcal{P} = \{P_i : i \in \mathcal{I}\}$ and $\mathcal{Q} = \{Q_i : i \in \mathcal{I}\}$ be partitions of the identity of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. For each $i \in \mathcal{I}$, let $\mathfrak{X}_i$ be a 1-hyperreflexive subspace of $Q_i\mathcal{B}(\mathcal{H})P_i$. Then the subspace*

$$\mathfrak{X} := \{T \in \mathcal{B}(\mathcal{H}) : Q_i T P_i \in \mathfrak{X}_i \quad \textit{for all} \quad i \in \mathcal{I}\}$$

is 1-hyperreflexive.

Proof. This is straightforward. Observe that if $x = P_i x$ for some $i \in \mathcal{I}$ then $\overline{\mathfrak{X}x} = Q_i^{\perp}\mathcal{H}_2 + \overline{\mathfrak{X}_i x}$; while otherwise $\mathfrak{X}x = \mathcal{H}_2$. Suppose that $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and set $T_i = Q_i T|_{P_i\mathcal{H}_1}$. It is easy to see that $\beta_{\mathfrak{X}}(T) = \sup_{i\in\mathcal{I}} \beta_{\mathfrak{X}_i}(T_i)$. For each $i \in \mathcal{I}$, there is an $X_i \in \mathfrak{X}_i$ so that

$$\|T_i - X_i\| = \beta_{\mathfrak{X}_i}(T_i).$$

Then $X = T + \sum_{i\in\mathcal{I}} Q_i(X_i - T_i)P_i$ lies in $\mathfrak{X}$ and

$$\|T - X\| = \|\operatorname{diag}(T_i - X_i)\| = \beta_{\mathfrak{X}}(T).$$

Thus $\mathfrak{X}$ is 1-hyperreflexive. ∎

Next, we need a simple way to recognize a nest bimodule.

Proposition 2.3. *A subspace is a nest bimodule if and only if it is reflexive and the collection of subspaces $\{\overline{\mathcal{S}x} : x \in \mathcal{H}\}$ is totally ordered.*

Proof. If $\mathcal{S}$ is a nest bimodule, then $\overline{\mathcal{S}x} = \overline{\mathcal{T}(\mathcal{M})\mathcal{S}\overline{\mathcal{T}(\mathcal{N})x}}$. Now $\overline{\mathcal{T}(\mathcal{N})x}$ is a subspace $N \in \mathcal{N}$, which is a nested collection. Thus its image under $\mathcal{S}$ is also nested. It follows from the Erdos–Power Theorem [12] that $\mathcal{S}$ is reflexive. The map θ is given by $\theta(N) = \overline{\mathcal{S}N}$.

Conversely, suppose that $\mathcal{S}$ is reflexive and $\{\overline{\mathcal{S}x} : x \in \mathcal{H}\}$ is totally ordered. Let $\mathcal{M} = \{\overline{\mathcal{S}x} : x \in \mathcal{H}\} \cup \mathcal{H}$. For any $S \in \mathcal{S}$ and $T \in \mathcal{T}(\mathcal{M})$ and $x \in \mathcal{H}$, $TSx \in T\overline{\mathcal{S}x} \subset \overline{\mathcal{S}x}$. As $\mathcal{S}$ is reflexive, $\mathcal{S} = \mathcal{T}(\mathcal{M})\mathcal{S}$.

Now observe that $\mathcal{S}^* = \mathcal{S}^*\mathcal{T}(\mathcal{M})^* = \mathcal{S}^*\mathcal{T}(\mathcal{M}^\perp)$ is a right nest module. So the argument of the first paragraph shows that the ranges $\overline{\mathcal{S}^*x}$ are totally ordered. Define $\mathcal{N}^\perp = \{\overline{\mathcal{S}^*x} : x \in \mathcal{H}\} \cup \mathcal{H}$. As in the second paragraph, $\mathcal{S}^* = \mathcal{T}(\mathcal{N}^\perp)\mathcal{S}^*$. Hence $\mathcal{S} = \mathcal{S}\mathcal{T}(\mathcal{N}) = \mathcal{T}(\mathcal{M})\mathcal{S}\mathcal{T}(\mathcal{N})$ is a nest bimodule. ∎

Here is an easy general condition for 1-hyperreflexivity.

Proposition 2.4. *A* WOT*-closed subspace $\mathcal{S}$ is* 1*-hyperreflexive if and only if: for every $T \in \mathcal{B}(\mathcal{H})$ with* $\operatorname{dist}(T, \mathcal{S}) = \|T\|$*, there is a sequence of unit vectors $x_n \in \mathcal{H}$ with $\lim_{n\to\infty} \|Tx_n\| = \|T\|$ such that $\lim_{n\to\infty} \|P_{\mathcal{S}x_n} T x_n\| = 0$.*

Proof. By hypothesis, for any $\varepsilon > 0$, there is a unit vector x so that $\|P^\perp_{\mathcal{S}x} T x\| > \|T\| - \varepsilon$. Therefore $\|Tx\| > \|T\| - \varepsilon$ and

$$\begin{aligned} \|P_{\mathcal{S}x} T x\|^2 &\le \|Tx\|^2 - \|P^\perp_{\mathcal{S}x} T x\|^2 \\ &\le \|T\|^2 - (\|T\| - \varepsilon)^2 < 2\|T\|\varepsilon. \end{aligned}$$

The converse is even easier. ∎

In the case of Magajna's Theorem [21], where $\mathcal{S} = \mathbb{C}A$, the condition becomes: $\lim_{n\to\infty} \|Tx_n\| = \|T\|$ and $\lim_{n\to\infty} \langle Tx_n, Ax_n \rangle = 0$. He defines a set $W_A(T)$ to be the set of scalars $\lambda \in \mathbb{C}$ for which there are unit vectors x_n with $\lim_{n\to\infty} \|Tx_n\| = \|T\|$ and $\lim_{n\to\infty} \langle Tx_n, Ax_n \rangle = \lambda$. The proof proceeds by showing that this set is convex; and if it does not contain 0, then a multiple of A may be subtracted from T to reduce its norm. There is no obvious way to define such a set for a higher dimensional algebra that will accomplish the same thing.

Theorem 2.5. *Let $\mathcal{S}$ be a subspace of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Suppose that there are projections P and Q, at least one of which is finite rank, so that the compression $Q\mathcal{S}P$ is not* 1*-hyperreflexive. Then neither is $\mathcal{S}$.*

Proof. We may assume that P is finite rank (for if it were Q, we could consider $\mathcal{S}^*$ instead). Select an element $T \in \mathcal{B}(P\mathcal{H}_1, Q\mathcal{H}_2)$ so that

$$1 = \|T\| = \operatorname{dist}(T, Q\mathcal{S}P) > \sup_{\|x\|=1,\, x\in P\mathcal{H}_1} \|P^\perp_{Q\mathcal{S}Px} T x\| =: \beta.$$

Consider QTP as an element of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then

$$\|QTP\| \ge \operatorname{dist}(QTP, \mathcal{S}) \ge \operatorname{dist}(QTP, Q\mathcal{S}P) = \|T\| = \|QTP\|.$$

So $\operatorname{dist}(QTP, \mathcal{S}) = 1$.

If $\mathcal{S}$ were 1-hyperreflexive, Proposition 2.4 provides a sequence of unit vectors $x_n \in \mathcal{H}$ so that

$$\lim_{n\to\infty} \|QTPx_n\| = \|QTP\| \quad \text{and} \quad \lim_{n\to\infty} \|P_{\mathcal{S}x_n} QTPx_n\| = 0.$$

In particular, $\lim_{n\to\infty} \|P^\perp x_n\| = 0$.

Since rank(P) is finite, there is a subsequence (which we relabel as x_n) so that $x = \lim_{n\to\infty} x_n$ exists (and lies in $P\mathcal{H}_1$). Clearly

$$\|QTPx\| = \lim_{n\to\infty} \|QTPx_n\| = \|QTP\|.$$

The projections $P_{\mathcal{S}x_n}$ need not converge to $P_{\mathcal{S}x}$; but there is a lower semicontinuity: if $y \in \overline{\mathcal{S}x}$, then $\lim_{n\to\infty} P_{\mathcal{S}x_n} y = y$. Thus

$$\lim_{n\to\infty} P_{\mathcal{S}x} P_{\mathcal{S}x_n} = \lim_{n\to\infty} P_{\mathcal{S}x_n} P_{\mathcal{S}x} = P_{\mathcal{S}x}.$$

Consequently,

$$\|P_{\mathcal{S}x} QTPx\| \leq \lim_{n\to\infty} \|P_{\mathcal{S}x_n} QTPx_n\| = 0.$$

So

$$\|P^\perp_{\mathcal{S}x} QTPx\| = \|QTPx\| = \|QTP\|.$$

Therefore $QTPx$ is orthogonal to $\mathcal{S}x$. As it is obviously orthogonal to $Q^\perp \mathcal{S}x$, it is also orthogonal to $Q\mathcal{S}x$. Hence $\|P^\perp_{Q\mathcal{S}x} Tx\| = \|T\|$ contrary to our hypothesis. This contradiction establishes the result. ∎

Remark 2.6. There is no straightforward way to quantify this. For example, if one takes $\mathcal{A}_n$ to be the algebra of 2×2 matrices of the form $\begin{bmatrix} a & n(a-b) \\ 0 & b \end{bmatrix}$, then it is easy to check using the matrix $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ that $\kappa_{\mathcal{A}_n} \geq \dfrac{n^2+1}{2n}$. However the infinite inflation $\mathcal{A}_n^{(\infty)} = \mathcal{A}_n \otimes \mathbb{C}I$ always has distance constant at most 3 by Bercovici's Theorem [2]. Indeed this even holds for the algebra of matrices of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, which is not even reflexive. So the compression even to a direct summand of $\mathcal{A}_n^{(\infty)}$ can yield an arbitrarily large distance constant, or none at all, while the distance constant for the algebra remains bounded.

Another example of the difficulty in quantifying this can be obtained as follows. Let $\mathcal{S}_n$ be the subspace of 2×4 matrices of the form $\begin{bmatrix} A & -nA \end{bmatrix}$ for $A \in \mathcal{A}$, a hyperreflexive subspace. Choose an operator $T \in \mathfrak{M}_2$ so that $\|T\| = 1 = \operatorname{dist}(T, \mathcal{A})$ while $\beta_{\mathcal{A}}(T) = 1/\kappa_{\mathcal{A}}$. It would be natural to try $T' = \begin{bmatrix} T & 0 \end{bmatrix}$ as a test case for the distance constant for $\mathcal{S}_n$. Clearly $\|T'\| = 1 = \operatorname{dist}(T', \mathcal{S}_n)$. However pick unit vectors x and y so that $Tx = y$. Observe that $x' = \begin{bmatrix} nx/\sqrt{n^2+1} \\ x/\sqrt{n^2+1} \end{bmatrix}$ belongs to $\ker \mathcal{S}_n$. Thus $\beta_{\mathcal{S}_n}(T') \geq y^*T'x' = \dfrac{n}{\sqrt{n^2+1}}$. So the *proof* of Theorem 2.5 does not reveal much about the distance constant of $\mathcal{S}_n$. Nevertheless, in this example, one can show that $\kappa_{\mathcal{S}_n} \geq \kappa_{\mathcal{A}}$.

The following result shows that, for bimodules over masas, the 1-hyperreflexive ones are obtained using Lemma 2.2 where the diagonal entries are nest bimodules. (The only 1-dimensional $\mathfrak{C}$–$\mathfrak{D}$ masa bimodule has the form $Q\mathcal{B}(\mathcal{H})P$ where Q and P are one dimensional projections. This is a nest bimodule. So type S2 reduces to type S1 in this case.)

Theorem 2.7. *Suppose that $\mathfrak{X}$ is a 1-hyperreflexive subspace such that the families of projections $\{P_{\mathfrak{X}x} : x \in \mathcal{H}_1\}$ and $\{P_{\mathfrak{X}^*y} : y \in \mathcal{H}_2\}$ are both commutative. Then there are abelian von Neumann algebras $\mathfrak{C}$ and $\mathfrak{D}$ in $\mathcal{B}(\mathcal{H}_2)$ and $\mathcal{B}(\mathcal{H}_1)$ respectively so that $\mathfrak{X}$ is a $\mathfrak{C}'$–$\mathfrak{D}'$ bimodule. Moreover, there are two collections of pairwise orthogonal projections*

$$\mathcal{C} = \{C_j : j \in \mathcal{J}\} \subset \mathfrak{C} \quad \textit{and} \quad \mathcal{D} = \{D_j : j \in \mathcal{J}\} \subset \mathfrak{D}$$

and nest bimodules $\mathfrak{X}_j \subset C_j\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)D_j$ so that

$$\mathfrak{X} = \{T \in \mathcal{B}(\mathcal{H}) : C_jTD_j \in \mathfrak{X}_j \quad \textit{for all} \quad j \in \mathcal{J}\}.$$

Proof. Since the projections $P_{\mathfrak{X}x}$ commute, there is an abelian von Neumann algebra $\mathfrak{C}$ in $\mathcal{B}(\mathcal{H}_2)$ containing all of them. Observe that if $X \in \mathfrak{X}$, $C \in \mathfrak{C}'$ and $x \in \mathcal{H}_1$, then

$$CXx \in CP_{\mathfrak{X}x}\mathcal{H}_2 = P_{\mathfrak{X}x}C\mathcal{H}_2 \subset \overline{\mathfrak{X}x}.$$

As $\mathfrak{X}$ is reflexive, $\mathfrak{C}'\mathfrak{X} = \mathfrak{X}$. Similarly there is an abelian von Neumann algebra $\mathfrak{D}$ in $\mathcal{B}(\mathcal{H}_1)$ containing $\{P_{\mathfrak{X}^*y} : y \in \mathcal{H}_2\}$, and $\mathfrak{X} = \mathfrak{X}\mathfrak{D}'$.

For each non-zero vector $x \in \mathcal{H}_1$ such that $\overline{\mathfrak{X}x} \neq \mathcal{H}_2$, let C_x be the smallest projection in $\mathfrak{C}$ such that

(1) $C_x(\mathfrak{X}x)^\perp = (\mathfrak{X}x)^\perp$, and
(2) for all $y \in \mathcal{H}_1$, either $C_x(\mathfrak{X}y)^\perp = 0$ or $C_x(\mathfrak{X}y)^\perp = (\mathfrak{X}y)^\perp$.

There is such a smallest projection because the product of any two projections with this property also has the property; and so does the (decreasing) limit of any sequence of such projections.

Let D_x be the projection onto

$$\operatorname{span}\{y \in \mathcal{H}_1 : C_x(\mathfrak{X}y)^\perp \neq 0\}.$$

Then $D_x \in \mathfrak{D}$. Indeed, if $C_x(\mathfrak{X}y)^\perp \neq 0$ and $z = Dy$ for $D \in \mathfrak{D}'$, then $\mathfrak{X}z \subset \mathfrak{X}y$ and hence $(\mathfrak{X}z)^\perp \supset (\mathfrak{X}y)^\perp \neq 0$. Hence $C_x(\mathfrak{X}z)^\perp \neq 0$ and $z = D_xz$. Thus the range of D_x is $\mathfrak{D}'$–invariant; so D_x lies in $\mathfrak{D}'' = \mathfrak{D}$.

Observe that for vectors x and y, either $C_x = C_y$ or $C_xC_y = 0$. Indeed, this follows from the minimality of C_x and C_y. For if $C_xC_y \neq 0$, then either $C_y(\mathfrak{X}x)^\perp = 0$ and thus $C_xC_y^\perp$ will be a smaller projection satisfying the two conditions, a contradiction; or $C_y(\mathfrak{X}x)^\perp = (\mathfrak{X}x)^\perp$ so that C_xC_y is such a projection. This latter condition is not contradictory only if $C_x \leq C_y$. But by symmetry, we also obtain $C_y \leq C_x$, whence equality.

Since D_x is a function only of C_x, not x itself, we obtain that $D_x = D_y$ when $C_x = C_y$. If $C_xC_y = 0$, then $D_xD_y = 0$ also. To see this, it suffices to show that if

$$C_x(\mathfrak{X}u)^\perp = (\mathfrak{X}u)^\perp \quad \text{and} \quad C_y(\mathfrak{X}v)^\perp = (\mathfrak{X}v)^\perp,$$

then $\langle u, v\rangle = 0$. But if this inner product is non-zero, the two $\mathfrak{D}'$-modules $\overline{\mathfrak{D}'u}$ and $\overline{\mathfrak{D}'v}$ have non-trivial intersection, say containing a non-zero vector w. The set of vectors z with $C_x(\mathfrak{X}z)^\perp = (\mathfrak{X}z)^\perp$ was shown to be invariant under multiplication by $\mathfrak{D}'$ and is clearly norm closed. Thus w is in this set, so that $C_x(\mathfrak{X}w)^\perp = (\mathfrak{X}w)^\perp$. Similarly $C_y(\mathfrak{X}w)^\perp = (\mathfrak{X}w)^\perp$; and thus $C_xC_y \neq 0$.

Therefore the collection of all projections D_x may be enumerated as a family $\{D_j : j \in \mathcal{J}\}$ of pairwise orthogonal projections with corresponding projections $\{C_j : j \in \mathcal{J}\}$. If $\overline{\mathfrak{X}x} \neq \mathcal{H}_2$, then there is an $j \in \mathcal{J}$ so that $D_j = D_x$ and $C_j = C_x$. Hence $\overline{\mathfrak{X}x}$ contains $C_j^\perp\mathcal{H}_2$. Define spaces $\mathfrak{X}_j = C_j\mathfrak{X}D_j$ for $j \in \mathcal{J}$. For every vector $0 \neq x \in D_j\mathcal{H}_1$, $\overline{\mathfrak{X}x}$ contains $C_j^\perp\mathcal{H}_2$. Since $\mathfrak{X}$ is reflexive and is a $\mathfrak{C}'$–$\mathfrak{D}'$ module, it must contain $C_j^\perp\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)D_j$ and $\mathfrak{X}_j$. It follows that

$$\mathfrak{X} = \{X \in \mathcal{B}(\mathcal{H}) : C_jXD_j \in \mathfrak{X}_j,\ j \in \mathcal{J}\}.$$

The last step is to show that each $\mathfrak{X}_j$ is a nest bimodule. Fix $j \in \mathcal{J}$ and work in $\mathcal{B}(D_j\mathcal{H}_1, C_j\mathcal{H}_2)$. Let $\mathfrak{C}_j$ and $\mathfrak{D}_j$ be the restrictions of the abelian von Neumann algebras. By Proposition 2.3, it suffices to show that the projections $Q_x = P_{\overline{\mathfrak{X}_jx}}$ in $\mathfrak{C}_j$ are nested for $x \in D_j\mathcal{H}_1$.

Suppose to the contrary that there are vectors x and y for which this fails. So $Q_xQ_y^\perp \neq 0$ and $Q_x^\perp Q_y \neq 0$. Now $Q_x\mathcal{H}_2 = \overline{\mathfrak{X}_j\mathfrak{D}_jx}$ depends only on the projection P_x onto $\overline{\mathfrak{D}_j'x}$ in $\mathfrak{D}_j$. We first show that we may suppose $P_xP_y = 0$. Indeed, if this is not the case, we may choose vectors x_1, y_1, z so that x_1 has support $P_xP_y^\perp$, y_1 has support $P_yP_x^\perp$ and z has support P_xP_y. Observe that

$$Q_x = Q_{x_1} \vee Q_z \quad \text{and} \quad Q_y = Q_{y_1} \vee Q_z,$$

whence

$$Q_x^\perp = Q_{x_1}^\perp Q_z^\perp \quad \text{and} \quad Q_y^\perp = Q_{y_1}^\perp Q_z^\perp. \tag{3}$$

So

$$0 \neq Q_x Q_y^\perp = (Q_{x_1} \vee Q_z) Q_{y_1}^\perp Q_z^\perp = Q_{x_1} Q_{y_1}^\perp Q_z^\perp .$$

So $Q_{x_1} Q_{y_1}^\perp \neq 0$ and likewise $Q_{x_1}^\perp Q_{y_1} \neq 0$. This reduces us to vectors with disjoint support projections.

Consider the restriction of $\mathfrak{X}_j$ to $(P_{x_1} \oplus P_{y_1})\mathcal{H}_1$ and compress the range to

$$Q_{x_1}^\perp \vee Q_{y_1}^\perp = Q_{x_1} Q_{y_1}^\perp + Q_{x_1}^\perp Q_{y_1} + Q_{x_1}^\perp Q_{y_1}^\perp .$$

Then $(Q_{x_1}^\perp \vee Q_{y_1}^\perp)\mathfrak{X}_i(P_{x_1} + P_{y_1})$ has the form

$$\begin{bmatrix} * & 0 \\ 0 & * \\ 0 & 0 \end{bmatrix}$$

where the $*$ entries are non-zero and independent. If $Q_{x_1}^\perp Q_{y_1}^\perp \neq 0$, we may choose vectors in each subspace and compress. This will be a 3×2 masa bimodule of the same form, which has distance constant $\sqrt{9/8}$ (see [9]). So by Theorem 2.5, $\mathfrak{X}_i$ would not be 1-hyperreflexive. Thus $Q_{x_1}^\perp Q_{y_1}^\perp = 0$.

To recap, this shows that if x_1 and y_1 have disjoint supports, then either $Q_{x_1}^\perp$ and $Q_{y_1}^\perp$ are comparable or they are orthogonal. Equation (3) shows that this remains true if we drop the condition on disjoint supports.

Fix x_1 as above with $Q_{x_1}^\perp \neq 0$, and consider the span Q of all projections $Q_z^\perp$ which are comparable to $Q_{x_1}^\perp$. For each vector y such that $Q_y^\perp Q \neq 0$, one has $Q_y^\perp Q_z^\perp \neq 0$ for some $Q_z^\perp$ which is comparable to $Q_{x_1}^\perp$. So either $Q_y^\perp \leq Q_z^\perp \leq Q$; or $Q_y^\perp \geq Q_z^\perp$ and thus $Q_y^\perp Q_{x_1}^\perp \neq 0$. Hence $Q_y^\perp$ is comparable to $Q_{x_1}^\perp$ and thus is less than Q. So Q is a projection satisfying conditions (1) and (2), for which C_j is the minimal choice. But we are working in $\mathcal{B}(C_j\mathcal{H}_1, D_j\mathcal{H}_2)$ so $Q \leq C_j$ is automatic. Thus $Q = C_j$.

Return to $Q_{y_1}^\perp$, which is orthogonal to $Q_{x_1}^\perp$. By the previous paragraph, there is a vector z_1 so that $Q_{z_1}^\perp$ is comparable to $Q_{x_1}^\perp$ and $Q_{y_1}^\perp Q_{z_1}^\perp \neq 0$. We deduce that $Q_{z_1}^\perp > Q_{y_1}^\perp$. Observe that, since $Q_{x_1} Q_{y_1}^\perp \neq 0$ and $Q_{z_1} Q_{y_1}^\perp = 0$, if we replace x_1 by $x_2 = P_{z_1}^\perp x_1$ then $Q_{x_2} Q_{y_1}^\perp \neq 0$. Since $Q_{x_2} \leq Q_{x_1}$, $Q_{x_2}^\perp Q_{y_1} \neq 0$ too. Similarly we can replace y_1 by $y_2 = P_{z_1}^\perp y_1$ and still maintain the fact that

$$Q_{x_2} Q_{y_2}^\perp \neq 0 \quad \text{and} \quad Q_{x_2}^\perp Q_{y_2} \neq 0.$$

Moreover since $Q_{z_1}^\perp Q_{x_2}^\perp \neq 0$ and $Q_{z_1}^\perp Q_{y_2}^\perp \neq 0$, it follows again that $Q_{z_1}^\perp \geq Q_{x_2}^\perp + Q_{y_2}^\perp$. But now x_2, y_2 and z_1 have disjoint supports.

Pick vectors $u \in Q_{x_2}^{\perp} Q_{y_2} \mathcal{H}_2$ and $v \in Q_{x_2} Q_{y_2}^{\perp} \mathcal{H}_2$; and consider the compression $\mathcal{S} = P_{\text{span}\{u,v\}} \mathfrak{X}_j P_{\text{span}\{x_2,y_2,z_1\}}$. This has the form $\begin{bmatrix} 0 & * & 0 \\ * & 0 & 0 \end{bmatrix}$ with respect to this decomposition and the two non-zero entries are not dependent. Thus this has distance constant $\sqrt{9/8}$. By Theorem 2.5, $\mathfrak{X}_j$ is not 1-hyperreflexive. This contradiction establishes the fact that $\mathfrak{X}_j$ is a nest bimodule. ∎

For future use, we record one fact that is a consequence of the proof. The $\mathfrak{C}_i$–$\mathfrak{D}_i$ module $\mathfrak{X}_i$ not only has the property that the range projections $\{P_{\mathfrak{X}_i x}\}$ and $\{P_{\mathfrak{X}^* u}\}$ are commutative. It also has the minimality hypothesis that the projections C_x satisfying (1) and (2) above are I or 0. When $\mathfrak{X}_i$ was not a nest bimodule, this allowed us to find compressions which were 2×3 or 3×2 submodules which are evidently not 1-hyperreflexive.

Corollary 2.8. *Assume that $\mathfrak{X}$ is a $\mathfrak{C}'$–$\mathfrak{D}'$ bimodule for two abelian von Neumann algebras $\mathfrak{C}$ and $\mathfrak{D}$, Assume also that no proper projection $C \in \mathfrak{C}$ has the property that $C(\mathfrak{X}x)^{\perp}$ is either 0 or $(\mathfrak{X}x)^{\perp}$ for each $x \in \mathcal{H}_2$. If $\mathfrak{X}$ is not a nest bimodule, then either*

(a) *there are orthogonal projections $D_1, D_2 \in \mathfrak{D}$ and $C_1, C_2, C_3 \in \mathfrak{C}$ so that $(C_1 + C_2 + C_3)\mathfrak{X}(D_1 + D_2)$ has the form $\begin{bmatrix} * & 0 \\ 0 & * \\ 0 & 0 \end{bmatrix}$ where the $*$ entries are non-zero and independent, or*

(b) *there are orthogonal projections $D_1, D_2, D_3 \in \mathfrak{D}$ and $C_1, C_2 \in \mathfrak{C}$ so that $(C_1 + C_2)\mathfrak{X}(D_1 + D_2 + D_3)$ has the form $\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \end{bmatrix}$ where the $*$ entries are non-zero and independent.*

3. 1-Hyperreflexive subspaces in low dimensions

In order to conveniently eliminate subspaces as failing to be 1-hyperreflexive, we need some low dimensional examples. In this section, we characterize the subspaces of 2×2 and 2×3 matrices which are 1-hyperreflexive.

Theorem 3.1. *Let $\mathcal{S}$ be a 1-hyperreflexive subspace of $\mathfrak{M}_2$. Then it is one of the following:*

(1) $\dim \mathcal{S}$ *is* 0,1 *or* 4.
(2) $\dim \mathcal{S} = 3$ *and there are unit vectors x and y so that* $\mathcal{S} = \{T \in \mathfrak{M}_2 : \langle Tx, y \rangle = 0\}$.
(3) $\dim \mathcal{S} = 2$ *and*

(a) $\dim \operatorname{Ran} \mathcal{S} = 1$, *so that* $\mathcal{S} = Q\mathfrak{M}_2$ *for some projection* Q.
(b) $\dim \ker \mathcal{S} = 1$, *so that* $\mathcal{S} = \mathfrak{M}_2 P$ *for some projection* P.
(c) *there are unitaries* U *and* V *so that* $\mathcal{S} = U\mathfrak{D}_2 V$, *where* $\mathfrak{D}_2$ *is a masa.*

Cases $(1)_0$, $(1)_4$, (2), $(3a)$ *and* $(3b)$ *are nest bimodules; and case* $(1)_1$ *is* 1*-dimensional. Case* (3c) *is type* S3.

Proof. The cases of $\dim \mathcal{S} = 0$ or 4 are trivial, and $\dim \mathcal{S} = 1$ is Magajna's Theorem.

If $\dim \mathcal{S} = 3$ and there is a vector x so that $\dim \mathcal{S}x = 1$, then by a dimension count, one concludes that $\mathcal{S} = \{T \in \mathfrak{M}_2 : \langle Tx, y\rangle = 0\}$ where y is chosen orthogonal to $\mathcal{S}x$. Evidently $\mathcal{S}u$ belongs to $\{0, \mathcal{S}x, \mathcal{H}\}$ depending on whether $u = 0$, $u \in \mathbb{C}^* x$, or not, respectively. By Proposition 2.3, $\mathcal{S}$ is 1-hyperreflexive.

On the other hand, if $\dim \mathcal{S}x = 2$ for all $x \neq 0$, then $\operatorname{Ref}(\mathcal{S}) = \mathfrak{M}_2$. So $\mathcal{S}$ is not even reflexive.

Now consider $\dim \mathcal{S} = 2$. Cases (3a) and (3b) are evidently 1-hyperreflexive. They are both nest bimodules. So we assume that $\mathcal{S}$ has no proper kernel or cokernel.

As in the 3-dimensional case, there must be a vector x_1 so that $\mathcal{S}x_1 = \mathbb{C}y_1$ is 1-dimensional. If all vectors $x \notin \mathbb{C}x_1$ had 2-dimensional range under $\mathcal{S}$, the 3-dimensional case again shows that $\operatorname{Ref}(\mathcal{S})$ would be a 3-dimensional nest bimodule. So there is a second vector x_2 independent of x_1 so that $\mathcal{S}x_2 = \mathbb{C}y_2$. If y_2 is a multiple of y_1, it follows that $\mathcal{S}\mathcal{H} = \mathbb{C}y_1$, which is case (a). Thus we have y_1 and y_2 independent.

Next observe that the functionals $\varphi_i(S) = \langle Sx_i, y_i\rangle$ must be independent. For otherwise, $\mathcal{S}$ would be 1-dimensional. Consider $\mathcal{S}$ with respect to an orthogonal basis x_1, x_1' for the domain and y_1, y_1' for the range. Then $\varphi_1'(S) = \langle Sx_1', y_1'\rangle$ is easily seen to be independent of φ_1. In this basis, we now have independent entries in the $1,1$ and $2,2$ positions and 0 in the $2,1$ position. The $1,2$ entry must be a linear combination of the other two. Hence there are scalars r and s so that

$$\mathcal{S} = \begin{bmatrix} a & ar+bs \\ 0 & b \end{bmatrix}.$$

We may further simplify this to the case of $r \geq 0$ and $s \geq 0$ as follows. Write $r = |r|e^{i\rho}$ and $s = |s|e^{i\sigma}$. Then

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\sigma} \end{bmatrix} \begin{bmatrix} a & ar+bs \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\rho} \end{bmatrix} = \begin{bmatrix} a & a|r| + be^{i(\sigma-\rho)}|s| \\ 0 & be^{i(\sigma-\rho)} \end{bmatrix}.$$

In this basis, we see that $x_2 = \left(\begin{smallmatrix} -r \\ 1 \end{smallmatrix}\right)$ and $y_2 = \left(\begin{smallmatrix} s \\ 1 \end{smallmatrix}\right)$.

Observe that $\mathcal{S}x = \mathcal{H}$ except when x is a multiple of either x_1 or x_2. Hence if we select a unit vector y_2' orthogonal to y_2,

$$\beta_{\mathcal{S}}(T) = \max_{i=1,2} \|P^{\perp}_{\mathcal{S}x_i} T x_i\| = \max_{i=1,2} |\langle T x_i, y_i' \rangle|$$

The proof will be complete once we show that 1-hyperreflexivity implies that $r = s = 0$. Suppose that we have a unitary operator U such that $\operatorname{dist}(U, \mathcal{S}) = 1$. We will have $\beta_{\mathcal{S}}(U) < 1$ if and only if $\langle U x_i, y_i \rangle \neq 0$ for $i = 1, 2$.

Define a unitary $U = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ where $\alpha = \sin\theta$ and $\beta = \cos\theta$ satisfy

$$0 < \alpha < (r+s)\beta \quad \text{and} \quad \langle U x_2, y_2 \rangle \neq 0.$$

Then $\beta_{\mathcal{S}}(U) < 1$. We claim that $\operatorname{dist}(U, \mathcal{S}) = 1$.

Suppose to the contrary that there are scalars a and b so that

$$\left\| \begin{bmatrix} \alpha - a & -\beta - ra - sb \\ \beta & \alpha - b \end{bmatrix} \right\| < 1.$$

We may suppose that a and b are real since the complex conjugate will have the same norm, and one can average to replace a and b by their real parts while decreasing the norm. Clearly a and b are strictly positive, for otherwise either the first column or second row will have norm at least one. The first row will have norm less than 1, and so

$$(\alpha - a)^2 + (-\beta - ra - sb)^2 < 1$$

whence

$$2a\alpha > 2\beta(ra + sb) + (ra + sb)^2 + a^2.$$

Similarly the second column leads to the inequality

$$2b\alpha > 2\beta(ra + sb) + (ra + sb)^2 + b^2.$$

Multiply the first by r and the second by s, add them and divide by $2(ra + sb)$ to obtain

$$\alpha > (r+s)\beta + \frac{1}{2}(ra + sb)(r + s) + \frac{ra^2 + sb^2}{2(ra + sb)} > (r+s)\beta$$

This is a contradiction, which establishes our claim. Thus 1-hyperreflexivity shows that $r = s = 0$, which is case (c). ∎

Theorem 3.2. *Let $\mathcal{S}$ be a 1-hyperreflexive subspace of $\mathfrak{M}_{2,3}$. Then it is one of the following:*

(1) *$\mathcal{S}$ is a nest bimodule.*
(2) $\dim \mathcal{S} = 1$.

(3) $\dim \mathcal{S} = 4$ *and there are orthonormal bases so that it consists of all matrices of the form* $\begin{bmatrix} a & 0 & c \\ 0 & b & d \end{bmatrix}$. *This is type* S3.

(4) $\dim \mathcal{S} = 3$ *and there are orthonormal bases so that it consists of all matrices of the form* $\begin{bmatrix} a & 0 & 0 \\ 0 & b & c \end{bmatrix}$. *This is type* S3.

(5) $\dim \mathcal{S} = 3$ *and there are orthonormal bases so that it consists of all matrices of the form* $\begin{bmatrix} aT & \begin{matrix} b \\ c \end{matrix} \end{bmatrix}$ *for some* $T \in \mathfrak{M}_2$ *of rank* 2. *This is an instance of the construction of Lemma* 2.2.

Proof. Suppose that $\mathcal{S}$ is reflexive and $\dim \mathcal{S} \geq 2$ but $\mathcal{S}$ is not a nest bimodule. By Proposition 2.3, the subspaces $\{\mathcal{S}x : x \in \mathcal{H}\}$ are not totally ordered. Then there are two vectors x_1 and x_2 so that $\mathcal{S}x_1 = \mathbb{C}y_1$ and $\mathcal{S}x_2 = \mathbb{C}y_2$, where y_1 and y_2 are independent. So $\dim \mathcal{S} \leq 4$.

First assume that the two functionals $\varphi_i(S) = \langle Sx_i, y_i \rangle$ are independent on $\mathcal{S}$. By Theorem 2.5, the compression of the domain to $\operatorname{span}\{x_1, x_2\}$ must be 1-hyperreflexive. Thus by Theorem 3.1, this forces x_1 and x_2 to be orthogonal and likewise y_1 and y_2 are orthogonal. Thus $\mathcal{S}$ has the form $\begin{bmatrix} a & 0 & ? \\ 0 & b & ? \end{bmatrix}$.

When φ_1 and φ_2 are dependent, let P be the projection onto the $\operatorname{span}\{x_1, x_2\}$. Then $\mathcal{S}P$ is one dimensional, so equals $\mathbb{C}T$ for some 2×2 matrix T of rank 2. In this case, $\dim \mathcal{S} \leq 3$ and has the form $\begin{bmatrix} aT & \begin{matrix} ? \\ ? \end{matrix} \end{bmatrix}$.

When $\dim \mathcal{S} = 4$, the functionals are indeed independent. Therefore this puts $\mathcal{S}$ into the predicted form $\begin{bmatrix} a & 0 & c \\ 0 & b & d \end{bmatrix}$.

When $\dim \mathcal{S} = 3$ and φ_1 and φ_2 are dependent, the unknowns are independent variables and we have the form $\begin{bmatrix} aT & \begin{matrix} b \\ c \end{matrix} \end{bmatrix}$.

When $\dim \mathcal{S} = 3$ and φ_1 and φ_2 are independent, at least one of the coefficients marked ? will be independent of a and b. By symmetry, we may suppose the form $\begin{bmatrix} a & 0 & L(a,b,c) \\ 0 & b & c \end{bmatrix}$ where c is independent of a and b and $L(a,b,c) = ra + sb + tc$ is linear. Compress to $\mathcal{S}P_1$ where P_1 is the projection onto the subspace $\operatorname{span}\{x_1, sx_2 + tx_3\}$. This yields $\mathcal{S}P_1 \simeq \begin{bmatrix} a & t(ra+d) \\ 0 & d \end{bmatrix}$ where $d = sb + tc$. By Theorem 3.1, we deduce that $t = 0$.

Similarly compress by the projection P_2 onto span $\big\{(x_1+x_2)/\sqrt{2}, x_3\big\}$ to obtain $\mathcal{S}P_2 \simeq \begin{bmatrix} a/\sqrt{2} & ra+sb \\ b/\sqrt{2} & c \end{bmatrix}$. Again by Theorem 2.5 and Theorem 3.1, this three dimensional space must have a vector

$$v = \alpha \frac{x_1+x_2}{\sqrt{2}} + \beta x_3$$

with one-dimensional range. Clearly $\beta \neq 0$. Thus the second coordinate of SP_2v is arbitary independent of the first coordinate because c is arbitrary. So the first coordinate, namely $(\alpha/\sqrt{2}+r\beta)a + (s\beta)b$, needs to be zero for all a and b. This forces $s = 0$.

So $\mathcal{S} = \begin{bmatrix} a & 0 & ra \\ 0 & b & c \end{bmatrix}$. Now a third application of Theorem 2.5 and Theorem 3.1, compressing to the subspace span$\{x_1, x_3\}$, shows that $r = 0$. This puts $\mathcal{S}$ in the desired form.

Now consider the case of $\dim \mathcal{S} = 2$ with φ_1 and φ_2 dependent. There is a norm one element $S \in \mathcal{S}$ such that $SP = 0$. Thus we can choose an orthonormal basis x_1, x_2, x_3 for the domain such that span$\{x_1, x_2\} = P\mathcal{H}_1$ and an orthonormal basis for the range, y_1, y_2 so that $Sx_3 = y_1$. Then $\mathcal{S}$ has the form $\begin{bmatrix} aT & \begin{matrix} b \\ ra \end{matrix} \end{bmatrix}$. Since T has rank 2, choose a unit vector $x = Px$ so that $z = Tx = \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)$ is neither collinear nor orthogonal to y_1; i.e. $z_1 z_2 \neq 0$. Compress the domain to span$\{x, x_3\}$ via the projection P_1. Then $\mathcal{S}P_1 = \begin{bmatrix} az_1 & b \\ az_2 & ra \end{bmatrix}$. If $r \neq 0$, this is not reflexive as the subspace ranges: 0, $\mathbb{C}z$ and $\mathcal{H}_2$ are nested. While if $r = 0$, then $\mathcal{S}P_1$ is diagonalizable but not with orthonormal bases. Thus by Theorem 3.1, it is not 1-hyperreflexive. Hence by Theorem 2.5, neither is $\mathcal{S}$.

Thus we may suppose that φ_1 and φ_2 are independent. So by the earlier analysis, $\mathcal{S}$ has the form $\begin{bmatrix} a & 0 & ra+sb \\ 0 & b & ta+ub \end{bmatrix}$. Compressing the domain to

$$P\mathcal{H} = \operatorname{span}\left\{\frac{x_1+x_2}{\sqrt{2}}, \alpha x_1 - \alpha x_2 + \beta x_3\right\}$$

where $2\alpha^2 + \beta^2 = 1$, will yield $\begin{bmatrix} a/\sqrt{2} & (\alpha+r\beta)a + s\beta b \\ b/\sqrt{2} & t\beta a + (u\beta - \alpha)b \end{bmatrix}$. This is never 1-hyperreflexive for all choices of parameters α and β. Indeed, the subspace will have kernel only if $s = t = 0$ and $\alpha + r\beta = -\alpha + u\beta$. In the other cases, one looks for vectors with one dimensional range.

Generically there are only two such vectors but they are usually neither parallel nor orthogonal. ∎

4. The Noncommuting case

We now have the tools we need to consider 1-hyperreflexive subspaces for which the projections $P_{\mathcal{S}x}$ do not commute. Once we understand exactly how this can occur, we will be able to complete the proof of Theorem 1.1.

Lemma 4.1. *Let $\mathcal{S} \subset \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a 1-hyperreflexive subspace. Suppose that the orthogonal projections $Q_x = P_{\mathcal{S}x}$ and $Q_y = P_{\mathcal{S}y}$ do not commute. Then $Q = (Q_x \vee Q_y) - (Q_x \wedge Q_y)$ has rank two, and $Q\mathcal{S}|_{\mathrm{span}\{x,y\}}$ is one dimensional.*

Proof. By Theorem 2.5, the subspace $Q\mathcal{S}|_{\mathrm{span}\{x,y\}}$ is 1-hyperreflexive. So we may work with this space, so that $\mathcal{H}_1 = \mathrm{span}\{x, y\}$, $Q_x \wedge Q_y = 0$ and $Q = Q_x \vee Q_y = I_{\mathcal{H}_2}$.

As Q_x and Q_y do not commute, there is a unit vector $u = Q_x u$ so that $u \neq v = Q_y u \neq 0$. Let Q_0 be the projection onto $\mathrm{span}\{u, v\}$. Choose a unit vector u' in $\mathrm{span}\{u, v\}$ orthogonal to u. Also choose an orthonormal basis $\{x, x'\}$ for $\mathrm{span}\{x, y\}$. Then there are constants γ and δ so that y is a non-zero multiple of $\gamma x + x'$ and v is a non-zero multiple of $\delta u + u'$.

Consider the compression $Q_0\mathcal{S}$. Since $Q_0\mathcal{S}x = \mathbb{C}u$ and $Q_0\mathcal{S}y = \mathbb{C}v$, we obtain that $Q_0\mathcal{S} \subset \mathrm{span}\{ux^*, vy^*\}$. With respect to the orthonormal bases $\{x, x'\}$ and $\{u, u'\}$, $\mathrm{span}\{ux^*, vy^*\}$ has the form

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} \overline{\gamma}\delta & \overline{\gamma} \\ \delta & 1 \end{bmatrix}.$$

This space is 1-hyperreflexive. By Theorem 3.1, it is either one-dimensional or the two bases $\{x, y\}$ and $\{u, v\}$ are orthogonal. As the latter is not the case, this compression is one-dimensional, say multiples of an operator T.

Now suppose that at least one of Q_x or Q_y has rank at least 2, say Q_x. Then we may select another unit vector u'' orthogonal to u, u' and the range of Q_y. Thus u and $Q_x u''$ will be independent, and thus the compression of $\mathcal{S}x$ to $\mathrm{span}\{u, u''\}$ is two-dimensional. Consider the projection Q_1 onto $\mathrm{span}\{u, u', u''\}$. Then $Q_1\mathcal{S}$ has the form $\begin{bmatrix} \multicolumn{2}{c}{aT} \\ b & 0 \end{bmatrix}$. By Theorem 3.2, this is not 1-hyperreflexive. This contradiction shows that Q_x and Q_y both have rank one.

Therefore $Q_x \vee Q_y$ has rank two, and we have already shown that $Q\mathcal{S}|_{\mathrm{span}\{x,y\}}$ is one dimensional. ∎

Lemma 4.2. *Let $\mathcal{S} \subset \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a 1-hyperreflexive subspace. Suppose that the projection Q_x does not commute with Q_y nor Q_z, where $\{x, y, z\}$ are independent. Then $Q_x \wedge Q_y = Q_x \wedge Q_z$. Let*

$$Q = (Q_x \vee Q_y \vee Q_z) - (Q_x \wedge Q_y \wedge Q_z).$$

Then the compression $Q\mathcal{S}|_{\mathrm{span}\{x,y,z\}}$ is one-dimensional.

Proof. We will work with the compression of the domain of $\mathcal{S}$ to $\mathrm{span}\{x, y, z\}$ and the range to $Q\mathcal{H}_2$. Let x, x' and u, u' be orthonormal bases for $\mathrm{span}\{x, y\}$ and the range of $R_1 = (Q_x \vee Q_y) - (Q_x \wedge Q_y)$, respectively. By the previous lemma, $R_1\mathcal{S}|_{\mathrm{span}\{x,y\}} = \mathbb{C}T$ where T is a 2×2 matrix of rank two. Let $R_0 = Q_x \wedge Q_y$. For each non-zero vector $\alpha x + \beta x'$, we have $Q_{\alpha x+\beta x'} = R_0 + [T(\alpha x + \beta x')]$.

Suppose that $R_0 \neq 0$. Then $0 = R_0 \wedge Q_z$. However we know from the previous lemma that $Q_x \wedge Q_z$ is codimension one in Q_x. Hence we deduce that $\mathrm{rank}\, R_0 = 1$, say spanned by a unit vector v (which is orthogonal to u, u'). Moreover Q_z is a projection of rank 2 onto a subspace that does not contain v. But $Q_x \wedge Q_z$ is rank one, so maps onto the span of some vector $w = \gamma v+\delta Tx$, with $\delta \neq 0$. If Q_z commutes with R_0, then $R_0 Q_z = 0$. This would force $\gamma = 0$ and so $Q_z \geq [Tx]$. But then $Q_x = R_0 + [Tx]$ would commute with Q_z. So Q_z does not commute with R_0.

Suppose that there were two independent vectors $\alpha_1 x + \beta_1 x'$ and $\alpha_2 x + \beta_2 x'$ so that both $Q_{\alpha_1 x+\beta_1 x'}$ and $Q_{\alpha_2 x+\beta_2 x'}$ commute with Q_z. Then Q_z would also commute with

$$Q_{\alpha_1 x+\beta_1 x'} \wedge Q_{\alpha_2 x+\beta_2 x'} = R_0,$$

contrary to fact. It follows that that Q_z does not commute with all but at most one of the projections $Q_{\alpha x+\beta x'}$. Therefore the previous lemma shows that $Q_z \wedge Q_{\alpha x+\beta x'}$ is rank one. For $\beta \neq 0$, this will not be the vector w. So the range of Q_z contains a second independent vector in the range of $R_0 + R_1$. Hence $Q_z \leq R_0 + R_1$. As this range does not contain v, $Q_1 Q_z$ maps onto the range of Q_1.

With respect to the bases x, x', z (which is not orthogonal) and u, u', v (which is orthonormal), $\mathcal{S}$ has the form

$$\begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{aT}} & b \\ & & c \\ d & e & L(b,c) \end{bmatrix}$$

where $L(b, c)$ is a linear function.

The discussion above shows that b and c are independent. It is also the case that they are independent of a. Indeed, the first two rows are 1-hyperreflexive. So it follows from Theorem 3.2 as this space must be three dimensional. Likewise the d and e are not dependent on a because Q_x and $Q_{x'}$ are rank 2. So by the same reasoning, they are also independent of each other.

Restrict the domain of $\mathcal{S}$ to the subspace $\operatorname{span}\{\alpha x+\beta x', z\}$ and write $u'' = T(\alpha x + \beta x')$ and $d' = \alpha d + \beta e$. Then we obtain

$$\begin{bmatrix} au'' & \begin{matrix} b \\ c \end{matrix} \\ d' & L(b,c) \end{bmatrix}.$$

By Theorem 3.2, since this is a 1-hyperreflexive space and is 4 dimensional, it must be the case that d' is independent of a, b, c, and the functional $L = 0$. This means that $Q_z = R_1$, a contradiction.

All of this analysis leads to the conclusion that in fact $R_0 = 0$, which is to say that $Q_x \wedge Q_y = Q_x \wedge Q_z$.

Therefore each projection Q_x, Q_y and Q_z is one dimensional. The restriction of the domain of $\mathcal{S}$ to $\operatorname{span}\{x, z\}$ is one dimensional. Thus for $S \in \mathcal{S}$, Sz is a linear function of Sx, as is Sx'. Selecting S_0 with $S_0x \neq 0$, define an operator $T' = \begin{bmatrix} S_0x & S_0x' & S_0z \end{bmatrix}$. Then $\mathcal{S} = [aT']$. ∎

Lemma 4.3. *Let $\mathcal{S} \subset \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a 1-hyperreflexive subspace. Suppose that the orthogonal projection $Q_{x_0} = P_{\mathcal{S}x_0}$ does not commute with some $Q_y = P_{\mathcal{S}y}$. Set $Q_0 = (Q_{x_0} \wedge Q_y)^\perp$; and let P_0 be the projection onto the closed span of all vectors y such that Q_y does not commute with Q_{x_0}. Then $P_0x_0 = x_0$ and $Q_0\mathcal{S}P_0$ is one dimensional, say $\mathbb{C}T$.*

Let Q be the projection onto the range of T and let P be the projection onto the range of T^*. So T is injective on $P\mathcal{H}_1$ with range dense in $Q\mathcal{H}_2$. The projection Q commutes with Q_y for every $y \in \mathcal{H}_1$; and QQ_y is 0, Q or is the rank one projection $[Ty]$. Likewise P commutes with $P_v = P_{\mathcal{S}^*v}$ for every $v \in \mathcal{H}_2$; and PP_v is 0, P or is $[T^*v]$.

Proof. By Lemma 4.2, for any two vectors y and z such that Q_y and Q_z fail to commute with Q_{x_0}, we see that

$$Q_{x_0} \wedge Q_y = Q_{x_0} \wedge Q_z = Q_0^\perp$$

and $Q_0\mathcal{S}|_{\operatorname{span}\{x_0,y,z\}}$ is one dimensional. Moreover for each vector x in $\operatorname{span}\{x_0, y, z\}$, $Q_x = Q_0^\perp + [Tx]$. As T has rank at least two, it follows that there is a set of vectors x dense in $\operatorname{span}\{x_0, y, z\}$ such that Q_x does not commute with Q_{x_0}. Thus the closure of the set of vectors $\{x \in \mathcal{H}_1 : Q_xQ_{x_0} \neq Q_{x_0}Q_x\}$ is a vector space, and thus a subspace.

Also observe that if Q_y does not commute with Q_{x_0}, then $Ty \neq 0$. Therefore the closure of vectors x so that Q_x does not commute with Q_y is the same set!

Continuing the analysis of the previous paragraph, notice that the operator T may be defined on each subspace $\operatorname{span}\{x_0, y, z\}$. The subspace $\mathbb{C}Tx_0$ does not depend on the choice of y and z. So it is possible to normalize the choices by fixing $Tx_0 = u_0$. Then we obtain a unique value for Ty for each $y = P_0 y$.

Select any element $S \in \mathcal{S}$ such that $Q_0^\perp S x_0 = T x_0$. Then $Q_0^\perp S y$ is a multiple cTy of Ty. In fact it must be exactly Ty. To see this, consider $y_t = (1-t)x_0 + ty$. Then

$$Q_0^\perp S y_t = (1-t)Tx_0 + ctTy.$$

This has to be a multiple of Ty_t for all t. Since Tx_0 and Ty are not collinear, it follows that $c = 1$.

We deduce that $Ty = Q_0^\perp S y$ for all vectors y such that Q_y does not commute with Q_{x_0}. It follows now that T extends to the closed span $P_0\mathcal{H}_1$ of these vectors as a bounded operator with $\|T\| \le \|S\|$.

Now consider a vector $y = P_0 y$ such that Q_y commutes with Q_{x_0}. With y_t defined as above, we see that $Ty_t = Q_0^\perp S y_t \neq 0$ for most values of t. It follows that $Q_{y_t} = Q_0^\perp + [Ty_t]$. In particular,

$$Q_y \le \limsup Q_{y_t} = Q_0^\perp + [Ty].$$

If $Ty \neq 0$, then $Q_y = Q_0^\perp + [Ty]$. But if $Ty = 0$, we can only deduce that $Q_y \le Q_0^\perp$.

We now define Q to be the projection onto the range of T. So $Q \le Q_0$ and it commutes with all Q_y such that $y = P_0 y$. Similarly, define P to be the projection onto the range of T^*.

Next suppose that $P_0^\perp z \neq 0$. Then Q_z commutes with Q_y for all $y = P_0 y$ for which $Ty \neq 0$. Therefore it commutes with their intersection $Q_0^\perp$ and their span $Q_0^\perp + Q$. Therefore $Q_z Q$ is a projection which commutes with $[Ty]$ for all $y = P_0 y$. As the range of T is dense in $Q\mathcal{H}_2$, it follows that $Q_z Q$ is either 0 or Q.

Similarly, consideration of $\mathcal{S}^*$ shows that if there are vectors u and v such that the range projections P_u and P_v onto $\overline{\mathcal{S}^* u}$ and $\overline{\mathcal{S}^* v}$ do not commute, one likewise finds projections P_0 and Q_0 so that $P_0 \mathcal{S}^* Q_0 = \mathbb{C} P_0 T^* Q_0$. However once one finds such a form, one also sees that the projections Q_x and Q_y for $x, y \in P_0\mathcal{H}_1$ would also fail to commute, and that we have already identified this subspace in the previous analysis.

What we can conclude is that for $u = P_0 u$ such that $T^* u \neq 0$, $P_u = P_0^\perp + [T^* u]$; and if $T^* u = 0$, then $P_u \le P_0^\perp$. Also if $P_0^\perp v \neq 0$, then P_v commutes with $P_0^\perp$ and P, and $P_v P$ is either 0 or P. ∎

Lemma 4.4. *Let $\{(P_i, Q_i) : i \in \mathcal{I}\}$ be the collection of all pairs of projections $P \in \mathcal{B}(\mathcal{H}_1)$ and $Q \in \mathcal{B}(\mathcal{H}_2)$ obtained as in Lemma 4.3 from a pair of vectors x, y such that Q_x and Q_y do not commute. Then $\mathcal{P} = \{P_i : i \in \mathcal{I}\}$ and $\mathcal{Q} = \{Q_i : i \in \mathcal{I}\}$ are families of pairwise orthogonal projections which commute with every P_x and Q_x respectively. For $x \in \mathcal{H}_1$, there is at most one $i \in \mathcal{I}$ such that Q_iQ_x is neither 0 nor Q_i. Moreover, $\mathcal{S}$ is a $\mathfrak{C}$–$\mathfrak{D}$ bimodule, where $\mathfrak{C}$ and $\mathfrak{D}$ are the abelian von Neumann algebras generated by the $\{P_i\}$ and $\{Q_i\}$ respectively.*

Proof. As before, we write $Q_x = [\mathcal{S}x]$ and $P_u = [\mathcal{S}^*u]$. For each pair of vectors $x, y \in \mathcal{H}_1$ such that Q_x and Q_y do not commute, Lemma 4.3 provides projections $P \in \mathcal{B}(\mathcal{H}_1)$ and $Q \in \mathcal{B}(\mathcal{H}_2)$ so that $Q\mathcal{S}P = \mathbb{C}T$ is 1-dimensional. Moreover every Q_z commutes with Q; and Q_zQ is 0, Q or $[Tz]$.

This immediately implies that if x', y' is another such pair, then the corresponding projections P' and Q' either equal P and Q or they are orthogonal. Thus there is a set $\{(P_i, Q_i) : i \in \mathcal{I}\}$ consisting of all such pairs. We write $Q_i\mathcal{S}P_i = \mathfrak{X}_i = \mathbb{C}T_i$.

Let $x \in \mathcal{H}_1$. Suppose that for some $i \in \mathcal{I}$, $Q_iQ_x = [T_ix] \neq 0$. Then there is another vector y so that $Q_iQ_y = [T_iy]$ does not commute with $[T_ix]$. Hence by Lemma 4.1, $Q_x = (Q_x \wedge Q_y) + [T_ix]$. Moreover, if z is any other vector such that Q_z does not commute with Q_x, then $Q_z = (Q_x \wedge Q_y) + [T_iz]$. Hence for all $z = P_jz$ where $j \neq i$, Q_z commutes with Q_x and indeed with all Q_y for which $Q_iQ_y = [T_iy] \neq 0$. It follows that Q_zQ_i is 0 or Q_i, not a one dimensional projection. Likewise Q_xQ_j is 0 or Q_j.

In particular, Q_i commutes with every Q_x. It follows that if $S \in \mathcal{S}$ and $x \in \mathcal{H}_1$, then

$$Q_iSx \in Q_iQ_x\mathcal{H}_2 \subset Q_x\mathcal{H}_2 = \overline{\mathcal{S}x}.$$

By the reflexivity of $\mathcal{S}$, $Q_iS \in \mathcal{S}$. Consequently $\mathfrak{C}\mathcal{S} = \mathcal{S}$. Consideration of $\mathcal{S}^*$ yields a similar conclusion on the right. ∎

Lemma 4.5. *Let $\mathcal{S}$ be a 1-hyperreflexive subspace of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Let $\{(P_i, Q_i) : i \in \mathcal{I}\}$ be the collection of all pairs of projections obtained in Lemma 4.4. Define a space of operators by*

$$\mathfrak{X} = \mathcal{S} + \text{WOT–}\sum_{i \in \mathcal{I}} Q_i\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)P_i.$$

Then $\mathfrak{X}$ is reflexive and the projections $R_x := [\mathfrak{X}x]$ commute with R_y and Q_y for $y \in \mathcal{H}_1$.

Proof. Suppose that $X \in \operatorname{Ref}(\mathfrak{X})$ and let $D = \sum_{i\in\mathcal{I}} Q_i X P_i$. Then $D \in \mathfrak{X}$, and to show that $X \in \mathfrak{X}$ it suffices to prove that $X - D \in \mathcal{S}$. Suppose that $x \in \mathcal{H}_1$ is a vector such that $Q_i Q_x \in \{0, Q_i\}$ for all $i \in \mathcal{I}$. Then in particular, if $P_i x \neq 0$, since T_i is injective on $P_i\mathcal{H}_1$, $Q_i Q_x \mathcal{H}_2$ contains the non-zero vector $T_i P_i x$; and so $Q_x Q_i = Q_i$. Therefore

$$\mathfrak{X}x = \mathcal{S}x + \sum_{\{i:P_i x\neq 0\}} Q_i\mathcal{H}_2 = \mathcal{S}x.$$

Otherwise there is a unique $i_0 \in I$ for which $Q_{i_0}Q_x = [T_{i_0}P_{i_0}x] \neq 0$. As before, for all other i for which $P_i x \neq 0$, $Q_x Q_i = Q_i$. Also $Q_{i_0}\mathcal{S}P_{i_0}^{\perp}x = 0$ (since otherwise it is $Q_{i_0}\mathcal{H}_2$). Thus

$$\mathfrak{X}x = \mathcal{S}x + \sum_{\{i:P_i x\neq 0\}} Q_i\mathcal{H}_2 = \mathcal{S}x + Q_{i_0}\mathcal{H}_2.$$

Since $\mathcal{S}$ is a $\mathfrak{C}$–$\mathfrak{D}$ bimodule,

$$\mathcal{S}P_{i_0}^{\perp}x = Q_{i_0}^{\perp}\mathcal{S}P_{i_0}^{\perp}x \subset Q_{i_0}^{\perp}\mathcal{S}x \subset \mathcal{S}x.$$

Therefore

$$(X-D)x = (X-D)P_{i_0}^{\perp}x + (X-D)P_{i_0}x \in Q_{i_0}^{\perp}\mathcal{S}x \subset \mathcal{S}x.$$

Hence $X - D$ belongs to $\mathcal{S}$ as claimed.

Since Q_{i_0} commutes with Q_x,

$$R_x = Q_x \vee Q_{i_0} = Q_x + Q_{i_0} - Q_{i_0}Q_x.$$

R_x therefore commutes with all Q_j. If $y \in \mathcal{H}_1$, either Q_y commutes with Q_x and so with R_x or $Q_y = Q_x - [T_{i_0}x] + [T_{i_0}y]$ which evidently also commutes with R_x. Finally we can conclude that R_x also commutes with R_y. ∎

Proof of Theorem 1.1.
Let $\mathcal{S}$ be a 1-hyperreflexive subspace of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. As in Lemma 4.4, define $\mathcal{P} = \{P_i : i \in \mathcal{I}\}$ and $\mathcal{Q} = \{Q_i : i \in \mathcal{I}\}$, and select operators T_i such that $Q_i\mathcal{S}P_i = \mathbb{C}T_i$. Set

$$\mathfrak{X} = \mathcal{S} + \text{WOT–}\sum_{i\in\mathcal{I}} Q_i\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)P_i$$

as in the previous lemma.

Let $\mathfrak{C}$ be the abelian von Neumann algebra generated by the set $\mathcal{Q} \cup \{R_x : x \in \mathcal{H}_1\}$, and let $\mathfrak{D}$ be generated by $\mathcal{P} \cup \{[\mathfrak{X}^* u] : u \in \mathcal{H}_2\}$. As in the proof of Theorem 2.7, the reflexive subspace $\mathfrak{X}$ is a $\mathfrak{C}'$–$\mathfrak{D}'$ bimodule. From the proof of Lemma 4.5, we see that $\mathcal{Q}$ are atoms of $\mathfrak{C}$ and $\mathcal{P}$ are atoms of $\mathfrak{D}$.

Following the proof of Theorem 2.7, we obtain families of projections $\mathcal{C} = \{C_j : j \in \mathcal{J}\}$ and $\mathcal{D} = \{D_j : j \in \mathcal{J}\}$ so that

$$\mathfrak{X} = \{T \in \mathcal{B}(\mathcal{H}) : C_j T D_j \in \mathfrak{X}_j \quad \text{for all} \quad j \in \mathcal{J}\}$$

where $\mathfrak{X}_j$ are $\mathfrak{C}$–$\mathfrak{D}$ bimodules in $C_j\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)D_j$ for $j \in \mathcal{J}$.

Furthermore, as in that proof, each $\mathfrak{X}_j$ is 1-hyperreflexive if and only if it is a nest bimodule. Corollary 2.8 shows that failure to be 1-hyperreflexive yields orthogonal projections $D_1, D_2, D_3 \in \mathfrak{D}$ and $C_1, C_2 \in \mathfrak{C}$ so that $(C_1 + C_2)\mathfrak{X}(D_1 + D_2 + D_3)$ has the form

$$\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \end{bmatrix}$$

where the $*$ entries are non-zero; or the analogous 3×2 form.

We claim that there are $i_1, i_2 \in \mathcal{I}$ so that $P_{i_k} \le D_k$ and $Q_{i_k} \le C_k$ for $k = 1, 2$. If not, then

$$(C_1 + C_2)\mathcal{S}(D_1 + D_2 + D_3) = (C_1 + C_2)\mathfrak{X}(D_1 + D_2 + D_3)$$

is not 1-hyperreflexive, contrary to hypothesis. By cutting down, we may suppose that $D_k = P_{i_k}$ and $C_k = Q_{i_k}$ for $k = 1, 2$. Thus

$$(C_1 + C_2)\mathcal{S}(D_1 + D_2 + D_3) = \begin{bmatrix} aT_{i_1} & 0 & 0 \\ 0 & bT_{i_2} & 0 \end{bmatrix}.$$

By Theorem 2.5, this is 1-hyperreflexive. And so Theorem 3.1 shows that a and b are dependent, say $b = ar$. But then this compression consists of multiples of the operator $\begin{bmatrix} T & \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix}$ where $T = \begin{bmatrix} T_{i_1} & 0 \\ 0 & T_{i_2} \end{bmatrix}$. However in this case, any vector $x = (D_1 + D_2)x$ satisfies $(C_1 + C_2)Qx = [Tx]$. Recall that T_{i_k} are injective on $D_i\mathcal{H}_1$ and have range dense in $C_k\mathcal{H}_2$. Hence $[Tx]$ is non-zero for all non-zero $x = (D_1 + D_2)x$. It is easy to see that if $x_k = D_k x_k$ are non-zero, then Q_{x_1} and $Q_{x_1+x_2}$ cannot commute. This contradicts the construction of the projections D_i by Lemma 4.4. Consequently, a and b are independent and the compression is not 1-hyperreflexive. As this is not possible, we deduce that $\mathfrak{X}_j$ is a nest bimodule.

Finally, observe that $C_j\mathcal{S}D_j$ is obtained from $\mathfrak{X}_j$ by the construction of Lemma 2.1. We have seen that $\mathfrak{X}_j$ is a nest bimodule and that $Q_i\mathcal{S}P_i = \mathbb{C}T_i$. Let $\mathcal{N}$, $\mathcal{M}$ be nests such that $\mathfrak{X}_j$ is a $\mathcal{T}(\mathcal{N})$–$\mathcal{T}(\mathcal{M})$ bimodule, and let $\theta : \mathcal{N} \to \mathcal{M}$, $\theta^* : \mathcal{M} \to \mathcal{N}$ be the functions so that $\mathfrak{X}_j = \mathfrak{X}(\theta)$ and $\mathfrak{X}_j^* = \mathfrak{X}(\theta^*)$. We may assume that $\mathcal{N}$ and $\mathcal{M}$ are minimal in the sense that $\mathcal{N} = \theta^*(\mathcal{M})$ and $\mathcal{M} = \theta(\mathcal{N})$. Define nests

$\mathcal{N}'$ and $\mathcal{M}'$ by

$$\mathcal{N}' = \mathcal{N} \cup \{N + P_i : N \in \mathcal{N},\ i \in \mathcal{I},\ N < N + P_i < N^+\},$$
$$\mathcal{M}' = \mathcal{M} \cup \{M^+ - Q_i : M \in \mathcal{M},\ i \in \mathcal{I},\ M < M^+ - Q_i < M^+\}$$

and let $\mathfrak{X}_j^0 = \{X \in \mathfrak{X}_j : Q_i X P_i = 0 \text{ for } i \in \mathcal{I}\}$. Then $(\mathfrak{X}_j, \mathfrak{X}_j^0)$ is a $\mathcal{T}(\mathcal{N}')$–$\mathcal{T}(\mathcal{M}')$ bimodule pair, and $\Delta(\mathfrak{X}_j, \mathfrak{X}_j^0) = \{(P_i, Q_i) : i \in \mathcal{I}\}$. It remains to verify that the subspaces $Q_i \mathcal{S} P_i = \mathbb{C} T_i$ are independent. To that end, it would suffice to show that $Q_i \mathcal{S} P_i$ belongs to $\mathcal{S}$ for each $i \in \mathcal{I}$. This follows from the reflexivity of $\mathcal{S}$. For if $P_i x = 0$, $T_i x = 0 \in \mathcal{S}x$. Suppose that $x_i := P_i x \neq 0$, say $x = x_i + y$. Then $Q_i Q_y$ is either 0 or Q_i while $Q_i Q_{x_i} = [T_i x_i]$ is exactly one dimensional. If $Q_i Q_y = 0$, then $Q_i Q_x = [T_i x_i]$; while if $Q_i Q_y = Q_i$, then $Q_i Q_x = Q_i$. In either case, $\mathcal{S}x$ contains $T_i x = T_i x_i$. By the reflexivity of $\mathcal{S}$, $T_i \in \mathcal{S}$ as desired. ∎

5. 1-Hyperreflexive Algebras

In this section, we apply Theorem 1.1 to the case of unital algebras.

Corollary 5.1. *A unital algebra $\mathfrak{A}$ is a 1-hyperreflexive if and only if either*

(1) *there is a nest $\mathcal{N}$ and a collection $\{A_i : i \in \mathcal{I}\}$ of atoms of $\mathcal{N}$ such that*

$$\mathfrak{A} = \{T \in \mathcal{T}(\mathcal{N}) : A_i T A_i \in \mathbb{C} A_i \text{ for all } i \in \mathcal{I}\}$$

or

(2) *there is a projection P so that $\mathfrak{A} = \mathcal{B}(P\mathcal{H}) \oplus \mathcal{B}(P^\perp \mathcal{H})$.*

Proof. The examples of (1) are unital algebras which are 1-hyperreflexive as subspaces by Lemma 2.1, and hence as algebras; while (2) is example A3. Both fall under the rubric of Theorem 1.1.

Conversely, consider the construction of Lemma 2.2 and suppose that there are two or more diagonal blocks involved. Then there are projections C_1, C_2, $C_3 = (C_1 + C_2)^\perp$, D_1, D_2 and $D_3 = (D_1 + D_2)^\perp$ so that $\mathfrak{A}$ contains

$$\{T \in \mathcal{B}(\mathcal{H}) : C_i T D_i = 0,\ i = 1, 2, 3\}$$

where, since C_3 and D_3 would consume all but the first two blocks, this includes all cases in which there are at least two such blocks if we allow one or both of C_3 and D_3 to be 0. So

$$\mathfrak{A} = \{T \in \mathcal{B}(\mathcal{H}) : C_i T D_i \in \mathfrak{X}_i,\ i = 1, 2, 3\}$$

where $\mathfrak{X}_i = C_i\mathfrak{A}D_i$. Moreover, $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are proper subspaces of the form of Lemma 2.1. $\mathfrak{X}_3$ is an arbitrary 1-hyperreflexive subspace.

Suppose that $(D_2 + D_3)(C_2 + C_3) \neq 0$. Then

$$\begin{aligned}\mathfrak{A}D_1 &\supset \mathfrak{A}D_1 + \mathfrak{A}(D_2 + D_3)\mathfrak{A}D_1\\ &\supset (C_2 + C_3)\mathcal{B}(\mathcal{H})D_1 + \mathfrak{A}(D_2 + D_3)(C_2 + C_3)\mathcal{B}(\mathcal{H})D_1\\ &\supset (C_2 + C_3)\mathcal{B}(\mathcal{H})D_1 + C_1\mathcal{B}(\mathcal{H})(D_2 + D_3)(C_2 + C_3)\mathcal{B}(\mathcal{H})D_1\end{aligned}$$

The WOT-closed span of $\mathcal{B}(\mathcal{H})(D_2 + D_3)(C_2 + C_3)\mathcal{B}(\mathcal{H})$ is all of $\mathcal{B}(\mathcal{H})$. As $\mathfrak{A}D_1$ is a WOT-closed subspace, we would conclude that $\mathfrak{A}D_1 = \mathcal{B}(\mathcal{H})D_1$. However $C_1\mathfrak{A}D_1 = \mathfrak{X}_1$ is proper, so $D_2 + D_3 \leq C_1$. Similarly $D_1+D_3 \leq C_2$. Consequently $D_3 = 0$. Therefore $D_2 = C_1$ and $D_1 = C_2$, and $C_3 = 0$.

If $\mathfrak{X}_1 = \mathfrak{X}_2 = 0$, this yields $\mathfrak{A} = \mathcal{B}(C_1\mathcal{H}) \oplus \mathcal{B}(C_2\mathcal{H})$ which is case (2). Otherwise, since $\mathfrak{A}$ is a WOT-closed algebra containing this as a subalgebra, $\mathfrak{X}_1 = \mathcal{B}(C_2\mathcal{H}, C_1\mathcal{H})$ or 0, and $\mathfrak{X}_2 = \mathcal{B}(C_1\mathcal{H}, C_2\mathcal{H})$ or 0. The three possibilities are all nest algebras.

Also, there is the case in which there is one block $\mathfrak{X}_1 \subset C_1\mathcal{B}(\mathcal{H})D_1$, but at least one of C_1, D_1 is a proper projection. If both C_2 and D_2 are non-zero, the same argument shows that $\mathfrak{A} = \mathcal{B}(\mathcal{H})$. If $C_2 = 0$ and $D_2 \neq 0$, then we may suppose that D_2 is the largest subspace such that $\mathfrak{A}D_2 = \mathcal{B}(\mathcal{H})D_2$. Then observe that if $D_2\mathfrak{A}D_1 \neq 0$, there are vectors $x = D_1x$ such that $\mathfrak{A}x = \mathcal{H}$, contrary to our assumption on D_2. Thus $\mathfrak{A}D_1 = D_1\mathcal{H}$. As $\mathfrak{X}_1 = \mathfrak{A}D_1$ has the form of Lemma 2.1, it is clear that $\mathfrak{A} = \mathfrak{X}_1 + \mathcal{B}(\mathcal{H})D_2$ also has this form. A similar analysis holds if $D_2 = 0$. Hence we may now assume that the 1-hyperreflexive unital algebra $\mathfrak{A}$ consists of a single block obtained using Lemma 2.1.

We have reduced the problem to the situation where there are nests $\mathcal{M}$ and $\mathcal{N}$ and a bimodule pair $(\mathfrak{X}, \mathfrak{X}_0)$ with atoms $\{(A_i, B_i) : i \in \mathcal{I}\}$ and operators $X_i \in \mathcal{B}(B_i\mathcal{H}, A_i\mathcal{H})$ so that

$$\mathfrak{A} = \{T \in \mathfrak{X} : A_iTB_i \in \mathbb{C}X_i,\ i \in \mathcal{I}\}$$

and

$$\mathfrak{X}_0 = \{T \in \mathfrak{X} : A_iTB_i = 0,\ i \in \mathcal{I}\}.$$

Let M_i and N_i be the elements of $\mathcal{M}$ and $\mathcal{N}$ respectively such that $A_i = M_i^+ - M_i$ and $B_i = N_i^+ - N_i$ for $i \in \mathcal{I}$.

Let θ and θ_0 be the left continuous order preserving maps of $\mathcal{N}$ into $\mathcal{M}$ such that $\mathfrak{X} = \mathfrak{X}(\theta)$ and $\mathfrak{X}_0 = \mathfrak{X}(\theta_0)$. Recall that $\theta_0(N) = \theta(N)$ unless $N = N_i^+$ for some $i \in \mathcal{I}$, in which case $\theta_0(N_i^+) = M_i \supset \theta(N_i)$ and $\theta(N_i^+) = M_i^+$. We may assume that each X_i is injective on $B_i\mathcal{H}$ with range dense in $A_i\mathcal{H}$. In particular, this ensures that $\overline{\mathfrak{A}N} = \theta(N)$ for all $N \in \mathcal{N}$.

Observe that if $N_i^+ \subset M_i$, then

$$M_i \supset \theta(N_i) = \overline{\mathfrak{A}N_i} = \overline{\mathfrak{A}^2 N_i} = \overline{\mathfrak{A}M_i} \supset \overline{\mathfrak{A}N_i^+} = M_i^+,$$

which is absurd. If $x = B_i x \notin M_i$, then because $\mathfrak{A}$ is unital,

$$\overline{\mathfrak{A}x} = M_i \vee \mathbb{C}X_i x \supset M_i \vee \mathbb{C}x.$$

Thus $\overline{\mathfrak{A}x} = M_i \vee \mathbb{C}x$. By lower semicontinuity, this identity persists for all $x = B_i x$. Since X_i is injective, this means that $M_i \cap B_i\mathcal{H} = \{0\}$. Now

$$M_i \vee A_i\mathcal{H} = M_i^+ = \overline{\mathfrak{A}N_i^+} = \bigvee_{x=B_i x} \mathfrak{A}x = M_i \vee B_i\mathcal{H}.$$

That is, $M_i^+ = M_i \vee B_i\mathcal{H}$.

Suppose that $M \in \theta(\mathcal{N})$, and let N be the largest element of $\mathcal{N}$ with $\theta(N) = M$, which exists since θ is left continuous. Also let N' denote the smallest element of $\mathcal{N}$ containing M. Then since $\mathfrak{A}$ is unital, $N \subset \overline{\mathfrak{A}N} = M \subset N'$. So if $N' = N$, we see that $M = N$. Otherwise

$$M = \overline{\mathfrak{A}N} = \overline{\mathfrak{A}^2 N} = \overline{\mathfrak{A}M} \supset \overline{\mathfrak{X}_0 M} = \mathfrak{X}_0 N' = \theta_0(N').$$

Since $N' > N$, $M' := \theta(N') > M$ and so

$$M = \theta_0(N') \geq \theta_0(N')^- = (M')^-.$$

Therefore $M = (M')^-$. It follows that $N' = N_i^+$ for some $i \in \mathcal{I}$; and hence $N = N_i$, $M = M_i$ and $M' = M_i^+$. But $N_i \subset M_i$, and by the previous paragraph, $M_i \cap B_i\mathcal{H} = \{0\}$. Since $M_i \subset N_i^+$, this means that $M_i = N_i$ and in fact $N' = N_i$ after all.

Consequently we deduce that whenever N is the largest element of $\mathcal{N}$ with $\theta(N) = M \in \mathcal{M}$, then $M = N$. This includes every N_i, $i \in \mathcal{I}$ because $\theta(N_i^+) > \theta(N_i)$. Again the analysis of the atoms yields

$$\theta(N_i^+) = M_i \vee B_i\mathcal{H} = N_i \vee B_i\mathcal{H} = N_i^+.$$

Since $\theta(N) \geq N$, we see that each N_i^+ is also the largest element of $\mathcal{N}$ with $\theta(N) = N_i^+$.

Let $\mathcal{N}_0 = \{N \in \mathcal{N} : \theta(N) = N\}$. We claim that $\mathcal{N}_0$ is complete. Indeed, if not, there is an element N in the completion which is a *monotone* limit of elements $N_\alpha \in \mathcal{N}_0$. If it is a limit from below, then by left continuity

$$\theta(N) = \sup \theta(N_\alpha) = \sup N_\alpha = N.$$

And if it is a limit from the above,

$$N \leq \theta(N) \leq \inf \theta(N_\alpha) = \inf N_\alpha = N.$$

Moreover $\mathcal{N}_0$ contains $\theta(\mathcal{N})$; whence $\mathcal{N}_0 = \theta(\mathcal{N})$.

Now we have $\mathfrak{X} = \mathcal{T}(\mathcal{N}_0)$ and $\mathfrak{X}_0$ is the ideal

$$\{T \in \mathcal{T}(\mathcal{N}_0) : A_i T A_i = 0,\ i \in \mathcal{I}\}.$$

Since $\mathfrak{A}$ is unital, it follows that each $X_i = A_i$ for $i \in \mathcal{I}$. So $\mathfrak{A}$ has the form of (1). ∎

6. Complete Hyperreflexivity

As mentioned in the introduction, we make the following natural definition. Here for $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{S} \mathbin{\bar{\otimes}} \mathcal{B}(\mathcal{K})$ the WOT-closure of the spatial tensor product in $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

Definition 6.1. A WOT-closed subalgebra (or subspace) $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is *completely hyperreflexive* if $\mathcal{S} \mathbin{\bar{\otimes}} \mathcal{B}(\mathcal{K})$ is hyperreflexive for a separable Hilbert space $\mathcal{K}$. Let $\kappa^c_{\mathcal{S}}$ denote the hyperreflexivity constant of $\mathcal{S} \mathbin{\bar{\otimes}} \mathcal{B}(\mathcal{K})$.

Unlike an arbitary compression considered in Theorem 2.5, it is a very different situation when the compression remains in the subspace. This lemma also appears in [17] with a different proof.

Lemma 6.2. *Let $\mathcal{S}$ be a subspace of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Suppose that P and Q are projections such that $Q\mathcal{S}P \subset \mathcal{S}$. Then considering $Q\mathcal{S}P$ as a subspace of $\mathcal{B}(P\mathcal{H}_1, Q\mathcal{H}_2)$, we obtain the inequality $\kappa_{Q\mathcal{S}P} \le \kappa_{\mathcal{S}}$.*

Proof. Let $T \in \mathcal{B}(P\mathcal{H}_1, Q\mathcal{H}_2)$. Then $\operatorname{dist}(T, Q\mathcal{S}P) = \operatorname{dist}(T, \mathcal{S})$ and

$$\begin{aligned}\beta_{\mathcal{S}}(T) &= \sup_{\|x\|=1} \inf_{S \in \mathcal{S}} \|Tx - Sx\| \le \sup_{\|x\|=1} \inf_{S \in Q\mathcal{S}P} \|Tx - Sx\| \\ &= \sup_{\substack{\|x\|=1,\\ x=Px}} \inf_{S \in Q\mathcal{S}P} \|Tx - Sx\| = \beta_{Q\mathcal{S}P}(T).\end{aligned}$$

So

$$\begin{aligned}\kappa_{Q\mathcal{S}P} &= \sup\{1/\beta_{Q\mathcal{S}P}(T) : T = QTP,\ \operatorname{dist}(T, Q\mathcal{S}P) = 1\} \\ &\le \sup\{1/\beta_{\mathcal{S}}(T) : T = QTP,\ \operatorname{dist}(T, \mathcal{S}) = 1\} \le \kappa_{\mathcal{S}}.\end{aligned}$$ ∎

Proposition 6.3. *If $\mathcal{S}$ is a WOT-closed subspace, then the hyperreflexivity constants for $\mathcal{S} \otimes \mathfrak{M}_n$ are increasing, and*

$$\kappa^c_{\mathcal{S}} := \lim_{n\to\infty} \kappa_{\mathcal{S}\otimes\mathfrak{M}_n}.$$

Proof. Fix an orthonormal basis $\{e_n\}_{n\ge1}$ for $\mathcal{K}$. For $n \ge 1$, let P_n be the orthogonal projection $I_{\mathcal{H}} \otimes Q_n \in \mathcal{B}(\mathcal{H} \otimes K)$ where $Q_n \in \mathcal{B}(\mathcal{K})$ is the orthogonal projection onto $\operatorname{span}\{e_1, e_2, \dots, e_n\}$. We identify $\mathcal{S} \otimes \mathfrak{M}_n$ and $P_n(\mathcal{S} \mathbin{\bar{\otimes}} \mathcal{B}(\mathcal{K}))$. Lemma 6.2 applies to show that

$$\kappa_{\mathcal{S}\otimes\mathfrak{M}_n} \le \kappa_{\mathcal{S}\otimes\mathfrak{M}_{n+1}} \le \kappa^c_{\mathcal{S}}.$$

Denote the limit by $\kappa = \lim_{n\to\infty} \kappa_{\mathcal{S}\otimes\mathfrak{M}_n}$. Fix $T \in \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ and for $n \geq 1$, let $T_n = P_n T P_n$. Then

$$\begin{aligned}
\beta_{\mathcal{S}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})}(T) &\geq \sup_{\|x\|=\|P_n x\|=1} \inf_{S\in\mathcal{S}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})} \|(T-S)x\| \\
&\geq \sup_{\|x\|=\|P_n x\|=1} \inf_{S\in\mathcal{S}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})} \|P_n(T-S)P_n x\| \\
&= \sup_{\|x\|=\|P_n x\|=1} \inf_{S\in\mathcal{S}\otimes\mathfrak{M}_n} \|(T_n-S)x\| \\
&\geq \kappa^{-1}\operatorname{dist}(T_n, \mathcal{S}\otimes\mathfrak{M}_n).
\end{aligned}$$

It is easy to verify from the lower semicontinuity of the norm in the strong operator topology that

$$\lim_{n\to\infty} \operatorname{dist}(T_n, \mathcal{S}\otimes\mathfrak{M}_n) = \operatorname{dist}(T, \mathcal{S}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})).$$

Thus taking a supremum over all $n \geq 1$ in the previous expression yields

$$\beta_{\mathcal{S}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})}(T) \geq \kappa^{-1}\operatorname{dist}(T, \mathcal{S}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})).$$

So $\kappa_{\mathcal{S}}^c \leq \kappa \leq \kappa_{\mathcal{S}}^c$; whence equality holds. ∎

A long-standing question posed in [9] is:

Question 6.4. Is every hyperreflexive subspace completely hyperreflexive?

Proposition 6.5. *The algebra $\mathcal{D} \subset \mathfrak{M}_3$ consisting of matrices of the form $\begin{bmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \beta\end{bmatrix}$ has distance constant at least $2/\sqrt{3}$.*

Proof. Consider the matrix

$$T = \begin{bmatrix} 0 & 0 & \sqrt{2} \\ -\sqrt{2} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Observe that

$$\begin{aligned}
\left\| \begin{bmatrix} \alpha & 0 & \sqrt{2} \\ -\sqrt{2} & \beta-1 & 0 \\ 0 & 0 & \beta+1 \end{bmatrix} \right\| &\geq \max\left\{ \left\| \begin{bmatrix} -\sqrt{2} & \beta-1 \end{bmatrix} \right\|, \left\| \begin{bmatrix} \sqrt{2} \\ \beta+1 \end{bmatrix} \right\| \right\} \\
&= \max\left\{ \left(2+|\beta\pm 1|^2\right)^{1/2} \right\}.
\end{aligned}$$

The minimum over all α, β is $\sqrt{3}$, and this is attained when $\alpha = \beta = 0$.

Next we note that the proper invariant subspaces have the form

$$\operatorname{span}\{e_1\},\ \operatorname{span}\{e_1\}^\perp, \operatorname{span}\{v_s\}, \quad \text{and} \quad \operatorname{span}\{v_s\}^\perp$$

where $v_s = (0, c, s)^t$, $|s| \le 1$, and $c = \sqrt{1 - |s|^2}$. If V is a 1-dimensional invariant subspace containing a unit vector v, we can compute

$$\|P_V^\perp T P_V\| = \|P_V^\perp T v\| = \|Tv - \langle Tv, v\rangle v\|.$$

While if P is a 2-dimensional invariant subspace orthogonal to a unit vector v, we can instead compute

$$\|P_V^\perp T P_V\| = \|T^* v - \langle T^* v, v\rangle v\|.$$

For $\mathrm{span}\{e_1\}$ and $\mathrm{span}\{e_1\}^\perp$, we obtain $\sqrt{2}$.

Consider $V = \mathrm{span}\{v_s\}$. Then

$$\begin{aligned}\|P_V^\perp T P_V\| &= \left\| \begin{bmatrix} \sqrt{2}s \\ -c \\ s \end{bmatrix} - \big(|s|^2 - c^2\big) \begin{bmatrix} 0 \\ c \\ s \end{bmatrix} \right\| = \left\| \begin{bmatrix} \sqrt{2}s \\ -2s^2 c \\ 2sc^2 \end{bmatrix} \right\| \\ &= \sqrt{2|s|^2 + 4|s|^2(1 - |s|^2)} \le \frac{3}{2}.\end{aligned}$$

This bound is attained when $s = \sqrt{3}/2$. Thus $\beta_{\mathcal{D}}(T) = 3/2$.

Therefore $\kappa_{\mathcal{D}} \ge \frac{\sqrt{3}}{3/2} = \frac{2}{\sqrt{3}}$. ∎

Corollary 6.6. *If $\mathcal{S}$ is a subspace of $\mathfrak{M}_n$ or $\mathcal{B}(\mathcal{H})$ of dimension at least 2, then $\mathcal{S} \otimes \mathbb{C}I_n$ is not 1-hyperreflexive for any $n \ge 3$ or $n = \infty$.*

Proof. Choose unit vectors x_1, x_2 and y_1, y_2 so that the functionals $\psi_i(A) = \langle Ax_i, y_i\rangle$ on $\mathcal{S}$ are linearly independent. In $\mathcal{H} \otimes l^2_n$, let P be the projection onto $\mathrm{span}\{x_1 \otimes e_1, x_2 \otimes e_2, x_2 \otimes e_3\}$ and let Q be the projection onto $\mathrm{span}\{y_1 \otimes e_1, y_2 \otimes e_2, y_2 \otimes e_3\}$. Consider the compression $Q\mathcal{S}P$. This is evidently the algebra $\mathcal{D}$ of Proposition 6.5. Since $\mathcal{D}$ does not have distance constant 1, Theorem 2.5 shows that $\mathcal{S} \otimes \mathbb{C}I_n$ also does not have distance constant 1. ∎

Corollary 6.7. *The algebra $\mathbb{C}I$ of scalar matrices is completely hyperreflexive. However $1 = \kappa_{\mathbb{C}I} < \kappa^c_{\mathbb{C}I}$.*

Getting an explicit lower bound greater than 1 takes a lot more work. As far as we know, the bound that we can get is not very good.

Proposition 6.8. *The complete distance constant κ for the algebra $\mathbb{C}I$ is at least* 1.03.

Proof. Let $\mathcal{S} = \mathbb{C}I \otimes \mathfrak{M}_{2,1} \subset \mathcal{B}(\mathcal{H}) \otimes \mathfrak{M}_{2,1}$. By Lemma 6.2, $\kappa \ge \kappa_{\mathcal{S}}$.

Let $\alpha = \sin(\pi/8)$ and $\beta = \cos(\pi/8)$. Let $\mathcal{K}$ be a two-dimensional subspace of $\mathcal{H}$ with orthonormal basis $\{e_1, e_2\}$. Define $T_1, T_2 \in \mathcal{B}(K)$

by

$$T_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \alpha & -\beta \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \alpha & -\beta \\ \alpha & -\beta \end{bmatrix}$$

and

$$T_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \beta & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \beta & \alpha \end{bmatrix}.$$

Consider $\mathcal{H} \oplus \mathcal{H}$ as $\mathcal{H} \otimes \mathbb{C}^2$ and let $\{u_1, u_2\}$ be the standard basis for $\mathbb{C}^2$. Decompose $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$ and

$$\mathcal{H} \oplus \mathcal{H} = (\mathcal{K} \otimes u_1) \oplus (\mathcal{K}^\perp \otimes u_1) \oplus (\mathcal{K} \otimes u_2) \oplus (\mathcal{K}^\perp \otimes u_2)$$

Define $T \in \mathcal{B}(\mathcal{H}) \otimes \mathfrak{M}_{2,1} = \mathcal{B}(\mathcal{H}, \mathcal{H} \oplus \mathcal{H})$ by

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \\ T_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $f_1 = \frac{1}{\sqrt{2}}(e_1 + e_2) \otimes u_1$ and $f_2 = e_2 \otimes u_2$. Let P be the orthogonal projection onto $\mathcal{K}$ and let Q be the orthogonal projection onto $\operatorname{span}\{f_1, f_2\}$, which is the range of T. With respect to the bases $\{e_1, e_2\}$ and $\{f_1, f_2\}$,

$$QTP = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \quad \text{and} \quad Q\mathcal{S}P = \left\{ \begin{bmatrix} a & a \\ 0 & b \end{bmatrix} : a, b \in \mathbb{C} \right\}.$$

Observe that QTP is unitary. The proof of Theorem 3.1 shows that $\operatorname{dist}(QTP, Q\mathcal{S}P) = 1 = \|T\|$. As in the proof of Theorem 2.5, it follows that $\operatorname{dist}(T, \mathcal{S}) = 1$. Thus $\beta_{\mathcal{S}}(T)^{-1} \le \kappa_{\mathcal{S}} \le \kappa$.

Let $v = xe_1 + ye_2 + z$ be a unit vector in $\mathcal{H}$ where $x, y \in \mathbb{C}$ and $z \in \mathcal{K}^\perp$. Then $\mathcal{S}v = v \otimes \mathbb{C}^2$, and a computation shows that

$$\begin{aligned} \|P_{\mathcal{S}v}^\perp T v\|^2 &= \|Tv\|^2 - |\langle Tv, v \otimes u_1 \rangle|^2 - |\langle Tv, v \otimes u_2 \rangle|^2 \\ &= |x|^2 + |y|^2 - \tfrac{1}{2}|x+y|^2 |\alpha x - \beta y|^2 - |y|^2 |\beta x + \alpha y|^2. \end{aligned}$$

If we expand this expression and consider what happens when $|x|$, $|y|$ and $\|z\|$ are fixed, then the only variable terms form a quadratic in $\rho = 2\operatorname{Re}(x\overline{y})$ whose leading term is $\frac{1}{2}\alpha\beta\rho^2$. Since $\alpha\beta > 0$, this function is maximized over the interval $\rho \in [-2|xy|, 2|xy|]$ at an endpoint, so we may assume that x and y are real.

Let

$$\psi(x, y) = \tfrac{1}{2}(x+y)^2(\alpha x - \beta y)^2 + y^2(\beta x + \alpha y)^2,$$

so that the expression above becomes

$$\|P_{\mathcal{S}v}^\perp T v\|^2 = x^2 + y^2 - \psi(x, y).$$

Let $k = \inf_{x^2+y^2=1} \psi(x,y)$. Numerical experiments reveal that $k > 0.058$. Since $\psi(rx,ry) = r^4\psi(x,y)$ and $k < 1$,

$$\begin{aligned}\beta_{\mathcal{S}}(T)^2 &= \sup_{\substack{0\le r\le 1\\ x^2+y^2=r^2}} \|P_{\mathcal{S}v}^{\perp} Tv\|^2\\ &= \sup_{0\le r\le 1} \big(r^2 - \inf_{x^2+y^2=r^2} \psi(x,y)\big)\\ &= \sup_{0\le r\le 1} r^2 - kr^4 = 1-k < 0.942.\end{aligned}$$

Thus $\kappa \ge (1-k)^{-1/2} > 1.03$. ∎

Now $\mathbb{C}I \otimes \mathcal{B}(\mathcal{H})$ is a type I von Neumann algebra. So it is hyperreflexive by Christensen's Theorem [4] (see [6, Theorem 9.6]) with constant at most 4. We will show that the constant is at most 2.

Proposition 6.9. *The distance constant for* $\mathbb{C}I \otimes \mathcal{B}(\mathcal{H})$ *is at most* 2. *The distance constant for* $\mathbb{C}I_2 \otimes \mathcal{B}(\mathcal{H})$ *is at most* $\frac{3}{2}$.

Proof. The idea is to average over a group of unitaries which have two point spectrum. Consider the eight element group $\mathcal{G}$ consisting of matrices

$$\begin{bmatrix}\pm 1 & 0\\ 0 & \pm 1\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}0 & \pm 1\\ \pm 1 & 0\end{bmatrix}.$$

Observe that $\mathcal{G}$ has trivial commutant. Hence $\mathcal{G} \otimes I$ has commutant $\mathbb{C}I_2 \otimes \mathcal{B}(\mathcal{H})$. Also note that the spectrum of an element of $\mathcal{G} \otimes I$ is one of $\{1\}$, $\{-1\}$, $\{\pm 1\}$ or $\{\pm i\}$. So the elements can be written as $\pm I_2$, $2P - I_2$ or $i(2Q - I_2)$ where $P \otimes I$ and $Q \otimes I$ are projections in $\mathfrak{M}_2 \otimes \mathbb{C}I$, and hence in $\operatorname{Lat}(\mathbb{C}I_2 \otimes \mathcal{B}(\mathcal{H}))$. Indeed P is one of $E_{11} \otimes I$, $E_{22} \otimes I$ or $\begin{bmatrix}\frac{1}{2}I & \pm\frac{1}{2}I\\ \pm\frac{1}{2}I & \frac{1}{2}I\end{bmatrix}$ and Q is one of $\begin{bmatrix}\frac{1}{2}I & \pm\frac{i}{2}I\\ \mp\frac{i}{2}I & \frac{1}{2}I\end{bmatrix}$.

Define an expectation onto $\mathbb{C}I_2 \otimes \mathcal{B}(\mathcal{H})$ by

$$\Phi(T) = \frac{1}{8}\sum_{G\in\mathcal{G}\otimes I_2} GTG^* = \frac{1}{4}T + \frac{1}{8}\sum_{G\neq \pm I_4} GTG^*$$

It is easy to check that $\Phi(T)$ commutes with $\mathcal{G} \otimes I$, and hence lies in $\mathbb{C}I_2 \otimes \mathcal{B}(\mathcal{H})$. Hence

$$\operatorname{dist}(T, \mathbb{C}I_2 \otimes \mathcal{B}(\mathcal{H})) \le \|T - \Phi(T)\| \le \frac{3}{4}\max \|TG - GT\|.$$

However

$$\begin{aligned}\|T(2P-1) - (2P-I)T\| &= 2\|P^{\perp}TP - PTP^{\perp}\|\\ &= 2\max\{\|P^{\perp}TP\|, \|PTP^{\perp}\|\}.\end{aligned}$$

Similarly we obtain the same for $G = i(2Q - I)$. Hence it follows that

$$\operatorname{dist}(T, \mathbb{C}I_2 \otimes \mathfrak{M}_2) \leq \frac{3}{2} \max_{P \in \operatorname{Lat}(\mathbb{C}I_2 \otimes \mathfrak{M}_2)} \|P^\perp T P\|.$$

To handle $\mathbb{C}I \otimes \mathcal{B}(\mathcal{H})$, form the group

$$\mathcal{G}_n = \mathcal{G}^{(n)} \otimes I = (\mathcal{G} \otimes \mathcal{G} \otimes \cdots \otimes \mathcal{G}) \otimes I$$

as a subgroup of unitaries in $\mathfrak{M}_{2^n} \otimes \mathbb{C}I$. Observe that the commutant is $\mathbb{C}I_{2^n} \otimes \mathcal{B}(\mathcal{H})$. Also each element is the tensor product of elements with 2 point spectrum. So this property is preserved. Averaging over $\mathcal{G}_n$ is an expectation Φ_n onto $\mathbb{C}I_{2^n} \otimes \mathcal{B}(\mathcal{H})$. As above, we obtain

$$\begin{aligned}\operatorname{dist}(T, \mathbb{C}I_{2^n} \otimes \mathcal{B}(\mathcal{H})) &\leq \|T - \Phi_n(T)\| \leq \max_{G \in \mathcal{G}_n} \|TG - GT\| \\ &\leq 2 \max_{P \in \operatorname{Lat}(\mathbb{C}I_2 \otimes \mathfrak{M}_2)} \|P^\perp T P\|.\end{aligned}$$

Now one obtains the same estimate for $\mathbb{C}I \otimes \mathcal{B}(\mathcal{H})$ by a routine approximation argument. ∎

If $\mathfrak{A}$ is any unital WOT-closed algebra then $\mathfrak{A} \mathbin{\bar{\otimes}} \mathcal{B}(\mathcal{H})$ contains two isometries with orthogonal ranges. Hence we may apply Bercovici's Theorem [2] to conclude that $\kappa_{\mathfrak{A} \bar{\otimes} \mathcal{B}(\mathcal{H})} \leq 3$. Here is a more elementary argument that improves on it.

Remark 6.10. The estimate of a constant 2 cannot be improved by using an expectation. This is because $\|T - \Phi(T)\|$ can be close to $2 \operatorname{dist}(T, \mathbb{C}I \otimes \mathcal{B}(\mathcal{H}))$. Indeed, forgetting about the tensor product with $\mathcal{B}(\mathcal{H})$, consider $T = \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^*$ where $\mathbf{1}_n$ is the vector with n ones. So T is a projection, and its distance to the scalars is $\frac{1}{2}$. However $\Phi(T) = \frac{1}{n} I_n$ and $\|T - \Phi(T)\| = 1 - \frac{1}{n} = (2 - \frac{2}{n}) \operatorname{dist}(T, \mathbb{C}I_n)$. (As well, we know that the off-diagonal projection $\mathrm{id} - \Phi$ has norm $2 - \frac{2}{n}$ on $\mathfrak{M}_n$ [3].)

However the expectation has an advantage. Let $\mathfrak{A}$ be an arbitrary weak-$*$ closed subspace and consider $\mathbb{C}I \otimes \mathfrak{A}$. Observe that for each unitary $G \in \mathcal{B}(\mathcal{H}) \otimes I$,

$$\operatorname{dist}(GTG^*, \mathbb{C}I \otimes \mathfrak{A}) = \operatorname{dist}(T, G^*(\mathbb{C}I \otimes \mathfrak{A})G) = \operatorname{dist}(T, \mathbb{C}I \otimes \mathfrak{A}).$$

So

$$\operatorname{dist}(\Phi(T), \mathbb{C}I \otimes \mathfrak{A}) \leq \operatorname{dist}(T, \mathbb{C}I \otimes \mathfrak{A}).$$

Similarly, if $P \in \operatorname{Lat}(\mathbb{C}I \otimes \mathfrak{A})$, then G^*PG is also in $\operatorname{Lat}(\mathbb{C}I \otimes \mathfrak{A})$. Thus $\beta_{\mathbb{C}I \otimes \mathfrak{A}}(GTG^*) = \beta_{\mathbb{C}I \otimes \mathfrak{A}}(T)$. Again averaging yields

$$\beta_{\mathbb{C}I \otimes \mathfrak{A}}(\Phi(T)) \leq \beta_{\mathbb{C}I \otimes \mathfrak{A}}(T).$$

Now $\Phi(T) = I \otimes T_0$. It is a well-known argument due to Arveson that if $\operatorname{dist}(T_0, \mathfrak{A}) = r$, then there is a weak-$*$ continuous functional ψ of norm one on $\mathcal{B}(\mathcal{H})$ annihilating $\mathfrak{A}$ so that $\psi(T_0) \approx r$. The corresponding trace class operator is put into polar decomposition as $\psi = \sum_{n\geq 1} s_n e_n f_n^*$ where $\{e_n\}$ and $\{f_n\}$ are orthonormal and $\sum_{n\geq 1} s_n = 1$. Then $x = \sum_{n\geq 1}^{\oplus} \sqrt{s_n} e_n$ and $y = \sum_{n\geq 1}^{\oplus} \sqrt{s_n} f_n$ are unit vectors in $l^2 \otimes \mathcal{H}$ such that

$$\psi(X) = \langle I \otimes Xx, y \rangle$$

for all $X \in \mathcal{B}(\mathcal{H})$. The subspace $\overline{\mathbb{C}I \otimes \mathfrak{A}x}$ is invariant for $\mathbb{C}I \otimes \mathfrak{A}$ and orthogonal to y. Hence we conclude that

$$\beta_{\mathbb{C}I\otimes\mathfrak{A}}(\Phi(T)) = \operatorname{dist}(\Phi(T), \mathbb{C}I \otimes \mathfrak{A}) = r.$$

Therefore

$$\operatorname{dist}(T, \mathbb{C}I \otimes \mathfrak{A}) \leq \|T - \Phi(T)\| + \beta_{\mathbb{C}I\otimes\mathfrak{A}}(\Phi(T)) \leq 3\beta_{\mathbb{C}I\otimes\mathfrak{A}}(T).$$

This yields the constant 3 in a more elementary way than by applying Bercovici's Theorem.

Lemma 6.11. *For $0 \neq T \in \mathcal{B}(\mathcal{H})$, where $\dim \mathcal{H} \geq 3$, the subspace $\mathbb{C}T$ has complete hyperreflexivity constant one if and only if $\operatorname{rank} T = 1$.*

Proof. If T is rank one, then $T = syx^*$ for unit vectors x and y and nonzero scalar s. Thus $\mathbb{C}T$ is a nest bimodule for the nests $\mathcal{M} = \{0, \mathbb{C}x, \mathcal{H}\}$ and $\mathcal{N} = \{0, \mathbb{C}y, \mathcal{H}\}$. Hence $\mathbb{C}T \otimes \mathcal{B}(\mathcal{H})$ is also a nest bimodule, and thus has distance constant one.

If the rank of T is at least two, one cannot put $\mathbb{C}T \otimes \mathcal{B}(\mathcal{H})$ into the form of Theorem 1.1. Indeed one can find three orthonormal vectors x_1, x_2, x_3 so that $Tx_i = s_i y_i$ where y_1, y_2, y_3 are orthonormal and $s_1 s_2 \neq 0$. The compression of $\mathbb{C}T \otimes \mathcal{B}(\mathcal{H})$ to the domain $\operatorname{span}\{x_1, x_2, x_3\} \otimes \mathcal{H}$ and range $\operatorname{span}\{y_1, y_2, y_3\} \otimes \mathcal{H}$ contains all elements of the form $s_1 A \oplus s_2 A \oplus s_3 A$ for $A \in \mathcal{B}(\mathcal{H})$. If this were 1-hyperreflexive, then the form of Theorem 1.1 would have a single block. However the subspace is neither a nest bimodule nor is it one dimensional. So this compression is not 1-hyperreflexive. By Theorem 2.5, $\mathbb{C}T \otimes \mathcal{B}(\mathcal{H})$ is also not 1-hyperreflexive. ∎

Proposition 6.12. *A subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ has complete distance constant one if and only if there are partitions of the identity of $\mathcal{H}_2$ and $\mathcal{H}_1$ respectively: $\mathcal{C} = \{C_j : j \in \mathcal{J}\}$ and $\mathcal{D} = \{D_j : j \in \mathcal{J}\}$, and for each $j \in \mathcal{J}$, there are nest bimodules $\mathfrak{X}_j$ of $C_j \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) D_j$ so that*

$$\mathcal{S} := \{T \in \mathcal{B}(\mathcal{H}) : C_j T D_j \in \mathfrak{X}_j \quad \textit{for all} \quad j \in \mathcal{J}\}$$

Proof. Since $\mathcal{S}$ must be 1-hyperreflexive, Theorem 1.1 yields the desired form for $\mathcal{S}$ except that the subspaces $\mathfrak{X}_j$ could have atoms of the bimodules replaced by 1-dimensional subspaces $\mathbb{C}T_{jk}$. The compression to this subspace must still be completely 1-hyperreflexive. So by the preceeding lemma, each T_{jk} must be rank one. But then, it is easy to see that $\mathfrak{X}_j$ is a nest bimodule. ∎

The following easy result of Ionascu [15, Prop. 1.3] will be useful.

Lemma 6.13. *Suppose that $\mathcal{S}$ and $\mathcal{T}$ are subspaces and that X is an invertible operator such that $\mathcal{S}X = \mathcal{T}$. Then one subspace is hyperreflexive if and only if the other is; and the constants are related by*

$$\kappa_{\mathcal{T}} \le \kappa_{\mathcal{S}} \|X\|\,\|X^{-1}\| \quad \textit{and} \quad \kappa_{\mathcal{S}} \le \kappa_{\mathcal{T}} \|X\|\,\|X^{-1}\|.$$

It is well-known that if $\mathfrak{D}$ is a von Neumann algebra with abelian commutant, then it is hyperreflexive with constant at most 2. As well, Rosenoer [25] showed with a more sophisticated argument that this is also true for abelian von Neumann algebras. However we need a variant which may include a zero summand. As we have seen, adding a zero summand will generally increase the distance constant. For example, $\mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})$ has constant 1 while $\mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H}) \oplus 0$ does not. So a modification of the proof is required.

Proposition 6.14. *Let $\{e_i : i \in \mathcal{I}\}$ and $\{f_i : i \in \mathcal{I}\}$ be orthonormal sets Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Let*

$$\mathcal{D} = \textsc{wot}\,\mathrm{span}\{f_i e_i^* \otimes \mathcal{B}(\mathcal{K}) : i \in \mathcal{I}\} \subset \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{K}, \mathcal{H}_2 \otimes \mathcal{K}).$$

Then $\mathcal{D}$ is (completely) hyperreflexive with constant $\kappa_{\mathcal{D}} \le 2$.

Moreover, suppose that $\mathcal{S} \subset \mathcal{D}$. Let Φ be the contractive expectation of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{K}, \mathcal{H}_2 \otimes \mathcal{K})$ onto $\mathcal{D}$. Then

$$\beta_{\mathcal{S}}(\Phi(T)) \le \beta_{\mathcal{S}}(T) \quad \textit{for all} \quad T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{K}, \mathcal{H}_2 \otimes \mathcal{K}).$$

Proof. For each subset X of $\mathcal{I}$, let $E(X)$ be the projection onto $\mathrm{span}\{e_i : i \in X\} \otimes \mathcal{K}$; and similarly let $F(X)$ be the projection onto $\mathrm{span}\{f_i : i \in X\} \otimes \mathcal{K}$. Let $E_i = E(\{i\})$ and $F_i = F(\{i\})$. Define an expectation Φ onto $\mathcal{D}$ by putting the standard product measure μ on $2^{\mathcal{I}}$ and integrating:

$$\Phi(T) = \int_{2^{\mathcal{I}}} (F(X) - F(X^c)) T (E(X) - E(X^c))\, d\mu(X)$$

for $T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Clearly this is a completely contractive map. Observe that $\Phi(T) = F(\mathcal{I})\Phi(T)E(\mathcal{I}) = \Phi(F(\mathcal{I})TE(\mathcal{I}))$. Moreover for every element $i \in \mathcal{I}$, $F_i \Phi(T) = \Phi(T) E_i$. From this, it easily follows that

$$\Phi(T) = \text{SOT--}\sum_{i \in \mathcal{I}} F_i \Phi(T) E_i,$$

and thus it belongs to $\mathcal{D}$. On the other hand, if $D \in \mathcal{D}$, then

$$D = (F(X) - F(X^c)) D (E(X) - E(X^c))$$

for every X and hence $\Phi(D) = D$. So Φ is an expectation onto $\mathcal{D}$.

We use the expectation to compute

$$\begin{aligned} \operatorname{dist}(T, \mathcal{D}) &\le \|T - \Phi(T)\| \\ &\le \sup_{X \subset \mathcal{I}} \|T - (F(X) - F(X^c)) T (E(X) - E(X^c))\|. \end{aligned}$$

The proof will be completed by bounding this by $2\beta_{\mathcal{D}}(T)$. This is where the proof is a bit trickier than the von Neumann algebra case.

We may write T as a 3×3 matrix with respect to the decompositions of the domain and range into

$$\mathcal{H}_1 \otimes \mathcal{K} = \operatorname{Ran} E(X) \oplus \operatorname{Ran} E(X^c) \oplus \operatorname{Ran} E(\mathcal{I})^\perp$$

and

$$\mathcal{H}_2 \otimes \mathcal{K} = \operatorname{Ran} F(X) \oplus \operatorname{Ran} F(X^c) \oplus \operatorname{Ran} F(\mathcal{I})^\perp$$

as $T = \left[T_{ij}\right]_{i,j=1}^3$. Decompose $T - (F(X) - F(X^c)) T (E(X) - E(X^c))$ as

$$\begin{aligned} \begin{bmatrix} 0 & 2T_{12} & T_{13} \\ 2T_{21} & 0 & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 0 & T_{12} & T_{13} \\ T_{21} & 0 & 0 \\ 0 & T_{32} & T_{33} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & T_{12} & 0 \\ T_{21} & 0 & T_{23} \\ T_{31} & 0 & T_{33} \end{bmatrix} \\ &\quad + \frac{1}{2} \begin{bmatrix} 0 & T_{12} & 0 \\ T_{21} & 0 & T_{23} \\ 0 & T_{32} & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & T_{12} & T_{13} \\ T_{21} & 0 & 0 \\ T_{31} & 0 & 0 \end{bmatrix} \end{aligned}$$

Each term is bounded by $\frac{1}{2}\beta_{\mathcal{D}}(T)$. For example, with

$$P = E(X)^\perp = E(X^c) + E(\mathcal{I})^\perp,$$

$\operatorname{Ran} \mathcal{D}P = \operatorname{Ran} F(X^c)$; and $\operatorname{Ran} \mathcal{D}E(X) = \operatorname{Ran} F(X)$. Hence

$$\begin{aligned} \beta_{\mathcal{D}}(T) &\ge \max\left\{\|F(X^c)^\perp T P\|, \|F(X)^\perp T E(X)\|\right\} \\ &\ge \max\left\{\|F(X^c)^\perp T E(X)^\perp\|, \|F(X^c) T E(X)\|\right\} \\ &= \|F(X^c)^\perp T E(X)^\perp + F(X^c) T E(X)\| \\ &= \left\| \begin{bmatrix} 0 & T_{12} & T_{13} \\ T_{21} & 0 & 0 \\ 0 & T_{32} & T_{33} \end{bmatrix} \right\| \end{aligned}$$

The other three terms are handled similarly.

For the last claim, let $\mathcal{S}$ be any subspace of $\mathcal{D}$. For each unit vector $x \in \mathcal{H}_1 \otimes \mathcal{K}$, let Q_x be the projection onto $\overline{\mathcal{S}x}$. Since

$$\mathcal{S} = (F(X) - F(X^c))\mathcal{S}(E(X) - E(X^c)),$$

the vector $x' = (E(X) - E(X^c))x$ has the same range, i.e. $Q_{x'} = Q_x$; and $\|x'\| \le 1$. So

$$\begin{aligned}
\beta_{\mathcal{S}}(\Phi(T)) &= \sup_{\|x\|=1} \|Q_x^\perp \Phi(T)x\| \\
&\le \sup_{\|x\|=1} \sup_{X \subset \mathcal{I}} \|Q_x^\perp (F(X) - F(X^c))T(E(X) - E(X^c))x\| \\
&= \sup_{\|x\|=1} \sup_{X \subset \mathcal{I}} \|(F(X) - F(X^c))Q_{x'}^\perp T x'\| \\
&\le \sup_{\|x'\| \le 1} \|Q_{x'}^\perp T x'\| = \beta_{\mathcal{S}}(T).
\end{aligned}$$

∎

Theorem 6.15. *If $0 \neq T \in \mathcal{B}(\mathcal{H})$, the subspace $\mathbb{C}T$ has complete hyperreflexivity constant at most 4.*

Proof. Use the polar decomposition to write $T = UP$ where P is positive. We may suppose that the spectrum of P is a countable set with 0 as the only limit point. Indeed, suppose that we have established the result for this case. Without loss of generality, $\|T\| = 1$. Given any $0 < r < 1$, write $P_r = \sum_{n \ge 0} r^n E_P(r^{n+1}, r^n]$ where E_P is the spectral measure for P; and $T_r = UP_r$. Then $rP_r \le P \le P_r$ and there is an invertible operator S_r in $W^*(P)$ so that $P = S_r P_r$ and $rI \le S_r \le I$. Then

$$\mathbb{C}T \otimes \mathcal{B}(\mathcal{K}) = \big(\mathbb{C}T_r \otimes \mathcal{B}(\mathcal{K})\big)(S_r \otimes I).$$

Thus the two spaces have distance constants related by a constant bounded by $\|S_r\|\,\|S_r^{-1}\| = r^{-1}$. If each $\mathbb{C}T_r$ has complete hyperreflexivity constant bounded by 4, then it follows by letting r tend to 1 that so does $\mathbb{C}T$.

Since P has discrete spectrum, it is diagonalizable. So select an orthonormal basis $\{e_n : n \ge 0\}$ so that $Pe_n = \tau_n e_n$ where $\tau_0 = 1 \ge \tau_n$ for all $n \ge 1$. Let $f_n = Ue_n$. Let $\mathcal{D}$ be the WOT-closed subspace of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ spanned by $\{f_n e_n^* \otimes \mathcal{B}(\mathcal{K}) : n \ge 0\}$. Then by Proposition 6.14, $\mathcal{D}$ has complete hyperreflexivity constant at most 2. Let $\mathcal{S} = \mathbb{C}T \otimes \mathcal{B}(\mathcal{K})$. Moreover if Φ is the expectation onto $\mathcal{D}$ constructed in Proposition 6.14, we obtain that

$$\beta_{\mathcal{S}}(\Phi(T)) \le \beta_{\mathcal{S}}(T) \quad \text{for all} \quad T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{K}, \mathcal{H}_2 \otimes \mathcal{K}).$$

We now consider a relative distance constant for $\mathcal{S}$ within $\mathcal{D}$. Note that

$$\mathcal{S} = \mathbb{C}T \otimes \mathcal{B}(\mathcal{K}) = \Big\{ \sum_{n\geq 0} f_n e_n^* \otimes \tau_n A : A \in \mathcal{B}(\mathcal{K}) \Big\}.$$

An element of the predual $\mathcal{D}_*$ is given by a sequence $\varphi = (\varphi_n)$ where φ_n is a weak-$*$ continuous functional on $\mathbb{C} f_n e_n^* \otimes \mathcal{B}(\mathcal{K})$ for each $n \geq 0$ and $\|\varphi\| = \sum_{n\geq 0} \|\varphi_n\| < \infty$. As usual, we identify each φ_n with an element of the space $\mathfrak{S}_1$ of trace class operators on $\mathcal{K}$. The pre-annihilator $\mathcal{S}_\perp$ intersects $\mathcal{D}_*$ in the set $\mathcal{A}$ of functionals satisfying $\sum_{n\geq 0} \tau_n \varphi_n = 0$ considered as an absolutely convergent sum in $\mathfrak{S}_1$.

We claim that $\varphi \in \mathcal{A}$ may be decomposed as a sum of functionals in $\mathcal{A}$ which have rank at most one in each entry, and have norms summing to at most $2\|\varphi\|$. Indeed, for each $n \geq 1$, decompose φ_n using polar decomposition into a sum of rank one functionals ρ_{nj} so that $\|\varphi\| = \sum_j \|\rho_{nj}\|$. Define a functional $\psi_{nj} = (\psi_{nji})_{i\geq 0}$ by

$$\psi_{njn} = \rho_{nj}, \quad \psi_{nj0} = -\tau_n \rho_{nj} \quad \text{and} \quad \psi_{nji} = 0 \text{ otherwise.}$$

Then it is clear that $\psi_{nj} \in \mathcal{A}$. Moreover

$$\sum_{n\geq 1} \sum_j \|\psi_{nj}\| = \sum_{n\geq 1} \sum_j (1+\tau_n)\|\rho_{ij}\| \leq 2 \sum_{n\geq 1} \|\varphi_n\| \leq 2\|\varphi\|.$$

So the sum $\sum_{n\geq 1} \sum_j \psi_{nj}$ converges to an element of $\mathcal{A}$. It is clear that for $n \geq 1$, the nth component of the sum is just φ_n. If the zeroth component is ψ, then we have

$$\psi + \sum_{n\geq 1} \tau_n \varphi_n = 0 = \varphi_0 + \sum_{n\geq 1} \tau_n \varphi_n.$$

Hence $\psi = \varphi_0$ and this sum is precisely φ.

Now we use the fact that if $\varphi = (\varphi_n)_{n\geq 0}$ in $\mathcal{D}_*$ has the property that each φ_n is rank one, then there is a rank one functional of the same norm on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ which agrees with φ on $\mathcal{D}$. Indeed, we may choose vectors $x_n, y_n \in \mathcal{K}$ so that $\varphi_n = y_n x_n^*$ and $\|x_n\|_2 = \|y_n\|_2 = \|\varphi_n\|^{1/2}$. Let $x = \sum_{n\geq 0} e_n \otimes x_n$ and $y = \sum_{n\geq 0} f_n \otimes y_n$. Then

$$\|x\|^2 = \|y\|^2 = \sum_{n\geq 0} \|\varphi_n\| = \|\varphi\|.$$

So $\psi = yx^*$ has $\|\psi\| = \|\varphi\|$. Finally it is evident that the restriction of ψ to $\mathcal{D}$ is equal to φ.

From the predual formulation of hyperreflexivity, we can conclude that for all $D \in \mathcal{D}$,

$$\beta_{\mathcal{S}}(D) = \sup \big\{ |\varphi(D)| : \varphi = (\varphi_n) \in \mathcal{A},\ \|\varphi\| = 1,\ \operatorname{rank} \varphi_n \leq 1 \text{ for all } n \big\}.$$

The calculation above shows that the convex hull of these rank one functionals contains the ball in $\mathcal{A}$ of radius 1/2. Hence

$$\operatorname{dist}(D, \mathcal{S}) \le 2\beta_{\mathcal{S}}(D).$$

Now consider $T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Then

$$\begin{aligned}\operatorname{dist}(T, \mathcal{S}) &\le \|T - \Phi(T)\| + \operatorname{dist}(\Phi(T), \mathcal{S}) \\ &\le 2\beta_{\mathcal{D}}(T) + 2\beta_{\mathcal{S}}(\Phi(T)) \le 4\beta_{\mathcal{S}}(T).\end{aligned}$$ ∎

Remark 6.16. We can do better in this analysis if T has rank 2. In this case, there is no loss of generality in taking $T_s = \operatorname{diag}(1, s, 0, 0, \dots)$ where $0 < s \le 1$. Then for an infinite dimension Hilbert space $\mathcal{K}$, the space $\mathcal{S} = \mathbb{C}T_s \otimes \mathcal{B}(\mathcal{K})$ is unitarily equivalent to the subset of $\mathfrak{M}_3(\mathcal{B}(\mathcal{K}))$ given by

$$\left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & sA & 0 \\ 0 & 0 & 0 \end{bmatrix} : A \in \mathcal{B}(\mathcal{K}) \right\}.$$

There are now two improvements in the argument of Theorem 6.15. The first is that the expectation Φ onto $\mathfrak{D}_3 \otimes \mathcal{B}(\mathcal{K})$ yields a better estimate because one averages over the finite group of diagonal matrices with ± 1 as entries. As in the proof of Proposition 6.9, there is an economy because two of the 8 group elements are $\pm I$. So one obtains an upper bound of $\frac{3}{2}\beta_{\mathfrak{D}_3 \otimes \mathcal{B}(\mathcal{K})}(X)$ for $\operatorname{dist}(X, \mathfrak{D}_3 \otimes \mathcal{B}(\mathcal{K}))$. Actually, the distance constant for $\mathfrak{D}_3 \otimes \mathcal{B}(\mathcal{K})$ is known to be exactly $\sqrt{3/2}$ [9]. However the expectation has the advantage that $\beta_{\mathcal{S}}(\Phi(X)) \le \beta_{\mathcal{S}}(X)$, which we do not know for the closest point.

The second improvement is that $\mathcal{S}$ has a relative distance constant of 1 within $\mathfrak{D}_3 \otimes \mathcal{B}(\mathcal{K})$. This can also be seen from the proof of the previous theorem. Indeed, $\mathcal{S}_\perp \cap \big(\mathfrak{D}_3 \otimes \mathcal{B}(\mathcal{K})\big)_*$ consists of $\varphi = (\varphi_n)_{n \ge 0}$ such that $\varphi_{00} = -s\varphi_{11}$. Decompose each φ_n for $n \ge 1$ into a sum of rank one elements ρ_{nj} so that $\|\varphi_n\| = \sum_j \|\rho_{nj}\|$. Then set $\psi_{1j} = (-s\rho_{1j}, \rho_{1j}, 0, 0, \dots)$; and set ψ_{nj} to have ρ_{nj} in the nth entry and 0 elsewhere. Then each ψ_{nj} belongs to $\mathcal{S}_\perp \cap \big(\mathfrak{D}_3 \otimes \mathcal{B}(\mathcal{K})\big)_*$ and is rank at most one in each entry. Moreover the norms sum exactly to $\|\varphi\|$. The proof is completed as above. So one obtains a distance constant of at most $1.5 + 1 = 2.5$.

The distance constant fails to be continuous except in rare cases [23, 15]. The example in the remark above displays this in a striking way that we have not seen before. Observe that Proposition 6.13 shows that the constant $\kappa^c_{\mathbb{C}T_s}$ is a continuous function of s for $s > 0$ [15].

Proposition 6.17. *Let* $T_s = \operatorname{diag}(1, s, 0, 0, 0 \dots)$ *for* $0 \le s \le 1$. *Then* $\sqrt{2} \le \lim_{s\to 0^+} \kappa^c_{\mathbb{C}T_s} \le \frac{5}{2}$ *while* $\kappa^c_{\mathbb{C}T_0} = 1$.

Proof. The upper bound of 2.5 was just established. And $\kappa^c_{\mathbb{C}T_0} = 1$ follows from Lemma 6.11. As above, we consider $\mathcal{S}_s = \mathbb{C}T_s \otimes \mathcal{B}(\mathcal{K})$ as the set of operators of the form $\begin{bmatrix} X & 0 & 0 \\ 0 & sX & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for $X \in \mathcal{B}(\mathcal{K})$.

We observe as in Theorem 6.15 that $(\mathcal{S}_s)_\perp$ consists of all trace class operators $\varphi = \big[\varphi_{ij}\big]_{i,j=1}^3$ such that $\varphi_{11} = -s\varphi_{22}$. Let e_1 and e_2 be two orthonormal vectors and f any unit vector, and consider

$$\psi = \begin{bmatrix} \frac{-s}{s+\sqrt{2}} f e_2^* & 0 & 0 \\ \frac{1}{s+\sqrt{2}} f e_1^* & \frac{1}{s+\sqrt{2}} f e_2^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -e_2 f^* & \frac{1}{\sqrt{2}} e_1 f^* & 0 \\ 0 & \frac{1}{\sqrt{2}} e_2 f^* & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that $\psi \in (\mathcal{S}_s)_\perp$ and $\|\psi\|_1 = 1$; and that $\|A\| = 1 = \psi(A)$. So $\operatorname{dist}(A, \mathcal{S}_s) = \|A\|$. Hence $\kappa_{\mathcal{S}_s} \ge \beta_{\mathcal{S}_s}(A)^{-1}$.

The predual formulation of the constant shows that $\beta_{\mathcal{S}_s}(A)$ is obtained as $\sup|\varphi(A)|$ taken over all rank one elements of the unit ball of $(\mathcal{S}_s)_\perp$. Note that the compression of any such functional to the upper left 2×2 corner still lies in $(\mathcal{S}_s)_\perp$ and still has rank one. As A is supported in this 2×2 corner, the set of compressions will yield the same supremum. This reduces the problem to 2×2 matrices. That is,

$$A = \begin{bmatrix} -e_2 f^* & \frac{1}{\sqrt{2}} e_1 f^* \\ 0 & \frac{1}{\sqrt{2}} e_2 f^* \end{bmatrix} \text{ and } (\mathcal{S}_s)_\perp = \left\{ \begin{bmatrix} -s\varphi_{22} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} : \varphi_{ij} \in \mathfrak{S}_1(\mathcal{K}) \right\}.$$

Here the description of rank one elements of $(\mathcal{S}_s)_\perp$ is particularly easy. There are three families:

$$\begin{bmatrix} 0 & 0 \\ xy^* & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & xy^* \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -saxy^* & bxy^* \\ cxy^* & axy^* \end{bmatrix}$$

where $\|x\| = \|y\| \le 1$ and $a, b, c \in \mathbb{C}$ satisfy

$$sa^2 + bc = 0 \quad \text{and} \quad (1 + s^2)|a|^2 + |b|^2 + |c|^2 \le 1.$$

Indeed, a rank one that has 0 in the $2, 2$ entry must be supported in either the $2, 1$ entry or the $1, 2$ entry. When $\varphi_{22} = axy^*$ for $\|x\| = \|y\| = 1$, this forces $\varphi_{11} = -saxy^*$. Then to make φ rank one, the other two entries must also be multiples of xy^*. The condition $sa^2 + bc = 0$ is a determinant condition equivalent to being rank one. Then the trace norm equals the Hilbert-Schmidt norm, so it is easily calculated and one obtains $(1 + s^2)|a|^2 + |b|^2 + |c|^2 \le 1$.

Evaluating the first two classes on A yields $1/\sqrt{2}$ and 0 respectively. So consider the third class. Then from Cauchy-Schwarz,

$$\begin{aligned}\sup|\varphi(A)| &= \sup\left|(s+\tfrac{1}{\sqrt{2}})a\langle e_2,y\rangle\langle x,f\rangle + \frac{a}{\sqrt{2}}\langle e_2,y\rangle\langle x,f\rangle\right| \\ &= \sup\left((s+\tfrac{1}{\sqrt{2}})^2|a|^2+\tfrac{1}{2}|c|^2\right)^{1/2} \\ &= \sup\sqrt{\frac{(2s^2+2\sqrt{2}s+1)|a|^2+|c|^2}{2}}\end{aligned}$$

The constraints yield $b=-sa^2/c$ and

$$1 \geq (1+s^2)|a|^2+|b|^2+|c|^2 = \frac{(1+s^2)|a|^2|c|^2+s^2|a|^4+|c|^4}{|c|^2}.$$

This problem may be solved by Lagrange multipliers. Since we are interested in the limit of this supremum as s tends to 0^+, we may observe that this will be the solution to the extremal problem

$$\sup\left\{\sqrt{\frac{x^2+y^2}{2}} : 0<(x^2+y^2)y^2\leq y^2\right\} = \frac{1}{\sqrt{2}}. \qquad \blacksquare$$

References

[1] W.B. Arveson, *Interpolation problems in nest algebras*, J. Func. Anal. **3** (1975), 208–233.

[2] H. Bercovici, *Hyper-reflexivity and the factorization of linear functionals*, J. Func. Anal. **158** (1998), 242–252.

[3] R. Bhatia, M.D. Choi and C. Davis, *Comparing a matrix to its off-diagonal part*, The Gohberg anniversary collection, Vol. I (Calgary, AB, 1988), 151–164, Oper. Theory Adv. Appl. **40**, Birkhduser, Basel, 1989.

[4] E. Christensen, *Perturbations of operator algebras II*, Indiana Univ. Math. J. **26** (1977), 891–904.

[5] E. Christensen, *Extensions of derivations*, J. Func. Anal. **27** (1978), 234–247.

[6] K.R. Davidson, *Nest Algebras*, Pitman Research Notes in Mathematics Series **191**, Longman Scientific and Technical Pub. Co., London, New York, 1988.

[7] K.R. Davidson, *The distance to the analytic Toeplitz operators*, Illinois J. Math. **31** (1987), 265–273.

[8] K.R. Davidson, E. Katsoulis and D.R. Pitts, *The structure of free semigroup algebras*, J. reine angew. Math. (Crelle) **533** (2001), 99–125.

[9] K.R. Davidson and M.S. Ordower, *Some exact distance constants*, Lin. Alg. Appl. **208/209** (1994), 37–55.

[10] K.R. Davidson and D.R. Pitts, *Invariant subspaces and hyperreflexivity for free semigroup algebras*, Proc. London Math. Soc. **78** (1999), 401–430.

[11] K.R. Davidson and S.C. Power, *Failure of the distance formula*, J. London Math. Soc. (2) **32** (1984), 157–165.

[12] J. Erdos and S. Power, *Weakly closed ideals of nest algebras*, J. Operator Thy. **7** (1982), 219–235.

[13] D. Hadwin, *A general view of reflexivity*, Trans. Amer. Math. Soc. **344** (1994), 325–360.

[14] D. Hadwin, *Compressions, graphs, and hyperreflexivity*, J. Func. Anal. **145** (1997), 1–23.

[15] I. Ionascu, *The continuity of the constant of hyperreflexivity*, J. Operator Theory **32** (1994), 353–379.

[16] F. Jaeck and S.C. Power, *Hyper-reflexivity of free semigroupoid algebras*, preprint, 2004.

[17] J. Kraus and D. Larson, *Reflexivity and distance formulae*, Proc. London Math. Soc. **53** (1986), 340–356.

[18] E.C. Lance, *Cohomology and perturbations of nest algebras*, Proc. London Math. Soc. (3) **423** (1981), 334–356.

[19] D.R. Larson, *Annihilators of operator algebras*, Topics in Modern Operator Theory **6**, pp. 119–130, Birkhauser Verlag, Basel, 1982.

[20] Loginov, A.I. and Sulman, G.S., *Hereditary and intermediate reflexivity of W*-algebras*, (Russian) Izv. Akad. Nauk. SSSR Ser. Mat. **39** (1975), 1260–1273; English transl., Math. USSR-Izv. **9** (1975), 1189–1201.

[21] B. Magajna, *On the distance to finite-dimensional subspaces in operator algebras.* J. London Math. Soc. (2) **47** (1993), 516–532.

[22] V. Müller and M. Ptak, *Hyperreflexivity of finite dimensional subspaces*, J. Func. Anal. **218** (2005), 395–408.

[23] F. Pop and D.T. Vuza, *Continuity properties of the distance constant function*, J. Operator Theory **22** (1989), 73–84.

[24] S.C. Power, *The distance to upper triangular operators*, Math. Proc. Camb. Phil. Soc. **88** (1980), 327–329.

[25] S. Rosenoer, *Distance estimates for von Neumann algebras*, Proc. Amer. Math. Soc. **86** (1982), 248–252.

[26] J. Stampfli, *The norm of a derivation*, Pacific J. Math. **33** (1970), 737–747.

Pure Math. Dept., U. Waterloo, Waterloo, ON N2L–3G1, CANADA
E-mail address: `krdavids@uwaterloo.ca`
E-mail address: `rlevene@uwaterloo.ca`