Nonself-adjoint operator algebras for dynamical systems

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ABSTRACT. This paper is a survey of our recent work on operator algebras associated to dynamical systems that lead to classification results for the systems in terms of algebraic invariants of the operator algebras.

1. Introduction

There is a long history of building operator algebras from dynamical systems going back to von Neumann's construction of a group von Neumann algebra. The use of nonself-adjoint operator algebras is more recent, but still goes back 40 years to Arveson's paper [2]. His algebra was closed in the weak operator topology. We will mostly be interested in norm closed (generally separable) algebras, but these algebras are the same vintage, beginning with work of Arveson and Josephson [3].

For the most part, we consider a dynamical system to be a locally compact Hausdorff space X together with one or more proper maps σ_i, $1 \leq i \leq n$, of X into itself, which we write as (X, σ). A proper map σ of X into itself induces an endomorphism $\alpha(f) = f \circ \sigma$ of $C_0(X)$. So more generally, we could consider an arbitrary semigroup of endomorphisms of an arbitrary algebra. This is too general a setting, but we will consider some variants of this kind. In particular, we will consider a C* dynamical system $(\mathfrak{A}, \alpha)$, where $\mathfrak{A}$ is a C*-algebra with a single automorphism α.

Two systems (X, σ) and (Y, τ) are said to be *conjugate* provided that there is a homeomorphism γ of X onto Y so that $\tau_i \gamma = \gamma \sigma_i$ for $1 \leq i \leq n$. Similar notions may defined for endomorphisms of other operator algebras as well. The main question is whether the operator algebra encodes enough information intrinsically so that the dynamical system can be recovered up to conjugacy or some similar relation between dynamical systems.

Our operator algebras can be considered in the abstract, in the spirit of the Blecher–Ruan–Sinclair Theorem [7]. Generally they are defined by a universal property dealing with representations. So the question of an sufficient family of such representations is moot. See [38, 6] for more on this viewpoint.

2. Semicrossed Products

Arveson [2] and Arveson–Josephson [3] were the first to consider nonself-adjoint operator algebras constructed from dynamical systems. They considered a homeomorphism σ of a compact Hausdorff space X which preserves a probability measure μ satisfying $\mu(O) > 0$ for every non-empty open set $O \subset X$. Given such a system, one can define a representation of $C(X)$ on $L^2(\mu)$ as multiplication operators, and define a unitary $Uf = f \circ \sigma$. Let $\mathcal{A}(X, \sigma)$ denote the norm-closed subalgebra of $\mathcal{B}(L^2(\mu))$ generated by U and the multiplication operators. Their main result is that, under a further technical condition, that the algebraic structure of the algebra determines the dynamical system:

THEOREM 2.1 (Arveson–Josephson [3]). *Suppose that for $i = 1, 2$, σ_i is a homeomorphism of a compact Hausdorff space X_i which preserves a probability measure μ_i satisfying $\mu_i(O) > 0$ for every non-empty open set O of X_i. Moreover suppose that the set of periodic points has measure 0. Then the following are equivalent:*

(i) *(X_1, σ_1) and (X_2, σ_2) are conjugate.*
(ii) *$\mathcal{A}(X_1, \sigma_1)$ and $\mathcal{A}(X_2, \sigma_2)$ are isometrically isomorphic.*
(iii) *$\mathcal{A}(X_1, \sigma_1)$ and $\mathcal{A}(X_2, \sigma_2)$ are algebraically isomorphic.*

In 1985, Peters [39] introduced an abstract construction of an operator algebra associated to the system (X, σ). He did not require the map σ to be a homeomorphism, only that it be a proper map so that it induces an endomorphism $\alpha(f) = f \circ \sigma$ of $C_0(X)$. Moreover, he does not require an invariant measure.

Suppose that X is compact. The idea is to construct a universal operator algebra which contains $C(X)$ as a C*-subalgebra, that is generated by $C(X)$ and a single isometry $\mathfrak{s}$ which encodes the dynamics via the *covariance relation*

$$f\mathfrak{s} = \mathfrak{s}(f \circ \sigma) \quad \text{for all} \quad f \in C_0(X).$$

Indeed, consider a covariant representation of this system as a pair (ρ, S) where ρ is a $*$-representation of $C(X)$ and S is an isometry satisfying $\rho(f)S = S\rho(f \circ \sigma)$ for all $f \in C(X)$. It is not difficult to show that there is a unique operator algebra, denoted $C(X) \times_\alpha \mathbb{Z}^+$, satisfying these properties together with the additional property that for every covariant representation (ρ, S), there is a completely contractive representation π of $\mathfrak{A} \times_\alpha \mathbb{Z}^+$ such that $\pi|_{C(X)} = \rho$ and $\pi(\mathfrak{s}) = S$. This is called the *semi-crossed product algebra*.

In the non-compact case, we do not include $\mathfrak{s}$ in the algebra. $C_0(X) \times_\alpha \mathbb{Z}^+$ is the universal algebra generated by $C_0(X)$ and the elements $\mathfrak{s}f$ for $f \in C_0(X)$. The isometry $\mathfrak{s}$ still belongs to the multiplier algebra however, and the covariance relations make sense if we multiply on the right by a function $g \in C_0(X)$. It has the property that for every covariant representation (ρ, S), there is a completely contractive representation π of $\mathfrak{A} \times_\alpha \mathbb{Z}^+$ such that $\pi|_{C(X)} = \rho$ and $\pi(\mathfrak{s}f) = S\rho(f)$ for every $f \in C_0(X)$.

Every dynamical system (X, σ) has a natural family of covariant representations called *orbit representations*. For any point $x \in X$, build a representation π_x on ℓ^2 by

$$\pi_x(f) = \text{diag}(f(x), f(\sigma(x)), f(\sigma^2(x)), \dots)$$

and $\pi_x(\mathfrak{s}) = S_+$, the unilateral shift. Peters shows that $\Pi = \sum_{x \in X} \pi_x$ is a faithful completely isometric representation of the semicrossed product $C_0(X) \times_\alpha \mathbb{Z}^+$. So it has a rather concrete spatial representation.

Peters was able to significantly extend the Arveson–Josephson Theorem. He still assumes that X is compact, but works with an arbitrary continuous map σ of X into itself, subject to the assumption that there are no fixed points.

THEOREM 2.2 (Peters [39]). *Let (X_i, σ_i) be compact dynamical systems with no fixed points. Then the following are equivalent:*

(i) *(X_1, σ_1) and (X_2, σ_2) are conjugate.*
(ii) *$C(X_1) \times_{\sigma_1} \mathbb{Z}^+$ and $C(X_2) \times_{\sigma_2} \mathbb{Z}^+$ are completely isometrically isomorphic.*
(iii) *$C(X_1) \times_{\sigma_1} \mathbb{Z}^+$ and $C(X_2) \times_{\sigma_2} \mathbb{Z}^+$ are algebraically isomorphic.*

In 1988, Hadwin and Hoover [20] considered a much more general class of *conjugacy algebras* which contain $C(X)$ and an element $\mathfrak{s}$ satisfying the covariance relations. They do not even require the algebras to be closed. Their methods weakened the condition on fixed points to saying: $\{x \in X_1 : \sigma_1^2(x) = \sigma_1(x) \neq x\}$ *has no interior.* With this hypothesis, they reach the same conclusion. Moreover, they show that the algebraic isomorphism of any single conjugacy algebra for the system (X_1, σ_1) with any conjugacy algebra for the system (X_2, σ_2) is enough to imply conjugacy of the systems, and hence completely isometric isomorphism of the semicrossed products.

Another result along these lines is due to Power [43]. He loosens the condition on X to be locally compact, but requires the map σ to be a homeomorphism. The key advance is that he recovers the Arveson–Josephson conclusions without any hypothesis on fixed points.

In 2006, the authors removed all extraneous hypotheses on the dynamical system to obtain:

THEOREM 2.3 (Davidson–Katsoulis [12]). *Let X_i be locally compact Hausdorff spaces, and let σ_i be a proper continuous map of X_i into itself. Then the following are equivalent:*

(i) *(X_1, σ_1) and (X_2, σ_2) are conjugate.*
(ii) *$C(X_1) \times_{\sigma_1} \mathbb{Z}^+$ and $C(X_2) \times_{\sigma_2} \mathbb{Z}^+$ are completely isometrically isomorphic.*
(iii) *$C(X_1) \times_{\sigma_1} \mathbb{Z}^+$ and $C(X_2) \times_{\sigma_2} \mathbb{Z}^+$ are algebraically isomorphic.*

We will discuss some of the ideas that go into the proof. Hadwin and Hoover study the characters of $C(X) \times_\sigma \mathbb{Z}^+$. The restriction of a character θ to $C_0(X)$ will be a point evaluation map δ_x for some $x \in X$. The rest of θ is determined by $\theta(\mathfrak{s}) = z$. As θ is contractive, one has $|z| \leq 1$. Call this character $\theta_{x,z}$. It is not difficult to show that $\theta_{x,0}$ always exists, and that this is the only possibility if $\sigma(x) \neq x$. When $\sigma(x) = x$, then $\theta_{x,z}$ exists for all $z \in \overline{\mathbb{D}}$. We write $\mathfrak{M}$ for the character space, and $\mathfrak{M}_x$ for those characters which restrict to δ_x on $C_0(X)$.

We introduce the notion of an *analytic set* in the character space. This is the image of a continuous non-constant map F of a domain $\Omega \subset \mathbb{C}$ into $\mathfrak{M}$ such that $f_a(w) = F(w)(a)$ is analytic on Ω for all $a \in C(X) \times_\sigma \mathbb{Z}^+$. A subset of $\mathfrak{M}$ is a *maximal analytic set* if it is an analytic set which is not properly contained in any other. A crucial observation is that $\{\theta_{x,z} : z \in \mathbb{D}\}$ are maximal analytic sets for every fixed point $x = \sigma(x)$, and there are no others.

The other important device is the notion of a nest representation. A representation of an algebra into $\mathcal{B}(\mathcal{H})$ is said to be a *nest representation* provided that the lattice of invariant subspaces of its range is a nest (complete chain) of subspaces. Nest representations and their kernels were originally introduced by Lamoureux

[29] as a generalization for primitive ideals. The second author, in joint work with Peters [25] and Kribs [22], applied the concept of a nest representation to the classification theory for non-selfadjoint operator algebras, including graph algebras [22] and limit algebras [25]. (See also [46].) For our purposes, nest representations into the 2×2 upper triangular matrices are sufficient.

In our case, such a 2×2 nest representation of $C(X) \times_\sigma \mathbb{Z}^+$ will have diagonal entries which are characters $\theta_{x,z}$ and $\theta_{y,w}$, and a $1, 2$ entry which is linearly independent of the diagonal (in order that there be no other invariant subspace). The dynamical system is recovered by showing that this implies that $\sigma(x) = y$. A complication occurs when $\sigma(y) = y$. This is the case that forced Hadwin and Hoover to include an additional hypothesis. We circumvent that by considering an analytic family of nest representations.

3. Function Algebra Systems

Peter's construction was actually developed in greater generality. If $\mathfrak{A}$ is any operator algebra and α is a completely contractive endomorphism, one defines the semicrossed product $\mathfrak{A} \times_\alpha \mathbb{Z}^+$ in an analogous manner. This is the universal operator algebra which is generated by a completely isometric copy of $\mathfrak{A}$ and an isometry $\mathfrak{s}$ satisfying the covariance relation

$$a\mathfrak{s} = \mathfrak{s}\alpha(a) \quad \text{for all} \quad a \in \mathfrak{A}.$$

Given any covariant representation (ρ, S) consisting of a completely contractive representation ρ of $\mathfrak{A}$ and an isometry S, there is a completely contractive representation π of $\mathfrak{A} \times_\alpha \mathbb{Z}^+$ such that $\pi|_{\mathfrak{A}} = \rho$ and $\pi(\mathfrak{s}) = S$. Note that when $\mathfrak{A}$ is a C*-algebra, the completely contractive representations are precisely the *-representations.

There is a natural analogue of the orbit representations for this semicrossed product. Starting with a completely contractive representation ρ of $\mathfrak{A}$ on $\mathcal{H}$, form a representation on $\ell^2(\mathcal{H})$ by setting

$$\pi_\rho(f) = \text{diag}(\rho(a), \rho(\alpha(a)), \rho(\alpha^2(a)), \dots) \text{ and } \pi_\rho(\mathfrak{s}) = S_+ \otimes I_{\mathcal{H}}.$$

Again the direct sum of these representations yields a faithful completely isometric representation of $\mathfrak{A} \times_\alpha \mathbb{Z}^+$.

To illustrate how the ideas of the previous section can be applied in another setting, we consider certain function algebras. Let G be a Cauchy domain in $\mathbb{C}$, namely a bounded open subset such that ∂G consists of a finite union of disjoint Jordan curves. Let $K = \overline{G}$. Then $A(K)$ is the algebra of continuous functions on K which are analytic on G. Suppose that $\sigma \in A(K)$ is a non-constant function such that $\sigma(K) \subset K$. Then $\alpha(f) = f \circ \sigma$ yields a completely contractive endomorphism of $A(K)$. Thus we can form the semicrossed product $A(K) \times_\sigma \mathbb{Z}_+$.

We say that two such maps $\sigma_i : K_i \to K_i$ are *analytically conjugate* if there is a biholomorphic map of G_1 onto G_2 such that $\sigma_2\gamma = \gamma\sigma_1$. (Any biholomorphic map between bounded domains of $\mathbb{C}$ extends to a homeomorphism of the closures.)

EXAMPLE 3.1 (Buske–Peters [9]). Let σ be an elliptic Mobius map of $\overline{\mathbb{D}}$ onto itself (i.e. σ has a single fixed point, and it lies in the open disk $\mathbb{D}$). Then σ is analytically conjugate to to a rotation $\eta(z) = e^{2\pi\theta i}z$. They show that $A(\mathbb{D}) \times_\sigma \mathbb{Z}_+$ is isometrically isomorphic to the subalgebra $\text{Alg}\{U, V\}$ of the rotation C*-algebra $\mathcal{A}_\theta$. It follows that there is an automorphism interchanging U and V which has the

effect of flipping the automorphism σ to σ^{-1}. So $A(\mathbb{D}) \times_\sigma \mathbb{Z}_+$ and $A(\mathbb{D}) \times_{\sigma^{-1}} \mathbb{Z}_+$ are completely isometrically isomorphic. In particular, the dynamics cannot be completely recovered from the semicrossed product in this case.

It turns out that this is the only thing which can complicate matters.

THEOREM 3.2 (Davidson–Katsoulis [**12**]). *For $i = 1, 2$, let G_i be Cauchy domains in $\mathbb{C}$ with $K_i = \overline{G_i}$. Then the following are equivalent:*

(i) *$A(\mathbb{K}_1) \times_{\sigma_1} \mathbb{Z}_+$ and $A(\mathbb{K}_2) \times_{\sigma_2} \mathbb{Z}_+$ are algebraically isomorphic.*
(ii) *$A(\mathbb{K}_1) \times_{\sigma_1} \mathbb{Z}_+$ and $A(\mathbb{K}_2) \times_{\sigma_2} \mathbb{Z}_+$ are completely isometrically isomorphic.*
(iii) (i) *σ_1 and σ_2 are analytically conjugate, or*
 (ii) *$\mathbb{K}_i$ are simply connected, σ_i are homeomorphisms with a unique fixed point which lies in the interior G_i, and σ_2 is analytically conjugate to σ_1^{-1}.*

When one applies the same analysis to $A(K)$, one finds an additional maximal analytic set in the character space, namely $\{\theta_{x,0} : x \in G\}$. This causes no difficulties when G is not simply connected because it must be identified with the corresponding non-simply connected analytic set of the other algebra. However when G is conformally equivalent to the disk, there can be some interplay—and this is precisely what occurs in the Buske–Peters example. Only the elliptic case is complicated, as otherwise the fixed point(s) are on the boundary, and topological considerations show that the new analytic disk is uniquely determined. In the elliptic case, one needs to work harder to show that an algebraic isomorphism is automatically continuous. This makes use of ideas from a paper of Sinclair [**45**].

Finally, we mention that the case of $\sigma = \mathrm{id}$ actually requires a special result from function theory. In this case, $A(K) \times_\sigma \mathbb{Z}_+ \simeq A(K \times \overline{\mathbb{D}})$. We use the fact that any biholomorphic map of one product space onto another actually decomposes as a product map [**30**].

4. C*-dynamical Systems

In this section, we will consider the (nonself-adjoint) semicrossed product constructed from a pair $(\mathfrak{A}, \alpha)$ where $\mathfrak{A}$ is a C*-algebra and α is a $*$-automorphism. In this case, $\mathfrak{A} \times_\alpha \mathbb{Z}_+$ is the subalgebra of the crossed product C*-algebera $\mathfrak{A} \times_\alpha \mathbb{Z}$ generated by $\mathfrak{A}$ and the unitary U implementing α.

Two C*-dynamical systems $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are *conjugate* if there is a $*$-isomorphism γ of $\mathfrak{A}$ onto $\mathfrak{B}$ so that $\beta\gamma = \gamma\alpha$. In the non-commutative setting, this is too strong because one can modify α by an inner automorphism of $\mathfrak{A}$ without affecting the algebra structure of the crossed product. Therefore we say that these systems are *outer conjugate* if there is a unitary element $u \in \mathfrak{A}$ so that $\beta\gamma = \gamma \,\mathrm{ad}_u\, \alpha$.

Automorphisms of C*-algebras have been widely studied. One useful tool is the *Connes spectrum*. Let $\mathcal{H}^\alpha(\mathfrak{A})$ denote the set of all hereditary sub-C*-algebras of $\mathfrak{A}$ which are α-invariant. Define a subset of the circle $\mathbb{T}$ by

$$\Gamma(\alpha) = \cap_{\mathfrak{B} \in \mathcal{H}^\alpha(\mathfrak{A})} \sigma(\alpha|_\mathfrak{B}).$$

This is in some sense analogous to the Weyl spectrum of a bounded operator. The following result of Olesen and Pedersen [**37**] characterizes the important special case of full Connes spectrum, $\Gamma(\alpha) = \mathbb{T}$.

We write $\hat{\mathfrak{A}}$ denote the spectrum of $\mathfrak{A}$, and let $\hat{\alpha}$ be the induced action of α on $\hat{\mathfrak{A}}$ by $\hat{\alpha}([\pi]) = [\pi\alpha]$.

THEOREM 4.1 (Olesen–Pedersen [**37**]). *Let $\mathfrak{A}$ be a separable C*-algebra, and fix an automorphism α of $\mathfrak{A}$. Then the following are equivalent:*

(i)   *α has full Connes spectrum, $\Gamma(\alpha) = \mathbb{T}$.*
(ii) *there is a dense subset Δ of the spectrum $\hat{\mathfrak{A}}$ which is $\hat{\alpha}$-invariant on which $\hat{\alpha}$ acts freely.*
(iii) *$\{[\pi] \in \hat{\mathfrak{A}} : \hat{\alpha}^n([\pi]) = [\pi]\}$ has no interior for all $n \geq 1$.*

This was used by Muhly and Solel to show the following:

THEOREM 4.2 (Muhly–Solel [**36**]). *Suppose that $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are C*-dynamical systems such that $\Gamma(\alpha) = \mathbb{T}$. Then $\mathfrak{A} \times_\alpha \mathbb{Z}_+$ and $\mathfrak{B} \times_\beta \mathbb{Z}_+$ are completely isometrically isomorphic if and only if $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are outer conjugate.*

We say that an automorphism α is *universally weakly inner* with respect to irreducible representations if for every irreducible representation π of $\mathfrak{A}$, there exists a unitary $W \in \pi(A)''$ so that $\pi\alpha(A) = W\pi(A)W^*$. Kishimoto [**27**] proves the remarkable result that if $\mathfrak{A}$ is a simple separable C*-algebra, then every universally weakly inner automorphism is actually inner.

We were able to avoid the condition on Connes spectrum by modifying the arguments used in the commutative case.

THEOREM 4.3 (Davidson–Katsoulis [**14**]). *Let $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ be C*-dynamical systems, and assume that the semicrossed products $\mathfrak{A} \times_\alpha \mathbb{Z}^+$ and $\mathfrak{B} \times_\beta \mathbb{Z}^+$ are completely isometrically isomorphic. Then there is a C*-isomorphism $\gamma : \mathfrak{A} \to \mathfrak{B}$ so that $\alpha \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma$ is universally weakly inner with respect to irreducible representations.*

Then using Kishimoto's result, we obtain that

COROLLARY 4.4. *If $\mathfrak{A}$ is a separable simple C*-algebra, then $\mathfrak{A} \times_\alpha \mathbb{Z}_+$ and $\mathfrak{B} \times_\beta \mathbb{Z}_+$ are isometrically isomorphic if and only if $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are outer conjugate.*

Our methods yield a new proof of the Muhly–Solel theorem. They suggest that isometric isomorphism of the semicrossed product may imply outer conjugacy under quite general hypotheses.

5. Multivariable dynamical systems

A multivariable dynamical system is a locally compact Hausdorff space X together with a collection $\sigma_1, \ldots, \sigma_n$ of proper continuous maps of X into itself. We will seek an appropriate analogue of the semicrossed product. To this end, we seek an operator algebra which contains $C_0(X)$ and operators $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ satisfying the covariance relations

$$f\mathfrak{s}_i = \mathfrak{s}_i(f \circ \sigma_i) \quad \text{for} \quad 1 \leq i \leq n \text{ and } f \in C_0(X).$$

Again, in the non-compact case, we do not include the $\mathfrak{s}_i$ in the algebra, but do include the elements $\mathfrak{s}_i g$.

Since we do not require any relations between the maps, the natural semigroup that arises is the free semigroup $\mathbb{F}_n^+$ of all words in an alphabet of n letters (including the empty word). If $w = i_1 \ldots i_k$ is an element of $\mathbb{F}_n^+$, we write $\mathfrak{s}_w = \mathfrak{s}_{i_1} \ldots \mathfrak{s}_{i_k}$. Similarly, we write $\sigma_w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}$.

The issue of norming the elements $\mathfrak{s}_i$ leads to a couple of natural choices that yield different algebras. The simplest condition is just to insist that each $\|\mathfrak{s}_i\| \leq 1$. The universal algebra subject to this constraint will be called the semicrossed product, denoted $C_0(X) \times_\sigma \mathbb{F}_n^+$. This has the universal property that given $(\rho, S_1, \ldots, S_n)$, where ρ is a $*$-representation of $C_0(X)$ on a Hilbert space $\mathcal{H}$, and $S_1, \ldots, S_n$ are n contractions on $\mathcal{H}$ satisfying the covariance relations, then there is a completely contractive representation π of $C_0(X) \times_\sigma \mathbb{F}_n^+$ such that $\pi|_{C_0(X)} = \rho$ and $\pi(\mathfrak{s}_i f) = S_i \rho(f)$ for $1 \leq i \leq n$ and $f \in C_0(X)$.

The other reasonable option is to require that $S = \begin{bmatrix} S_1 & \ldots & S_n \end{bmatrix}$ be a row contraction (as an operator from $\mathcal{H}^{(n)}$ to $\mathcal{H}$). We call the universal algebra obtained in this manner the *tensor algebra* $\mathcal{A}(X, \sigma)$.

You may notice that we did not require the operators to be isometries. However such a requirement would make no difference. The reason is that there is a dilation theorem showing that any contractive covariant representation of (X, σ) dilates to one in which each S_i is an isometry; and each row contractive covariant representation dilates to a row isometric covariant representation. One obvious advantage of this formulation is that we now know more about the (completely contractive) representations of these algebras.

The tensor algebra turns out to be more tractable in general. One reason is that there is a natural analogue of the orbit representations. Fix $x \in X$ and build a representation on Fock space $\ell^2(\mathbb{F}_n^+)$, with orthonormal basis $\{\xi_w : w \in \mathbb{F}_n^+\}$, by setting

$$\pi_x(f)\xi_w = f(\sigma_w(x))\xi_w$$

$$\pi_x(\mathfrak{s}_i)\xi_w = \xi_{iw} \qquad \text{for } w \in \mathbb{F}_n^+.$$

As in the one variable case, the direct sum of all orbit representations yields a faithful, completely isometric representation of $\mathcal{A}(X, \sigma)$. The semicrossed product does not appear to have a nice family of norming representations that can be explicitly described.

In either case, the universal property leads to the existence of gauge automorphisms, namely automorphisms γ_z for $z \in \mathbb{T}$ of our algebra with $\gamma_z|_{C_0(X)} = \text{id}$ and $\gamma_z(\mathfrak{s}_i f) = z\mathfrak{s}_i f$. In the standard manner, integration over the unit circle yields a completely contractive expectation onto $C_0(X)$. This leads to an automatic continuity result that any isomorphism of $\mathcal{A}(X, \sigma)$ onto $\mathcal{A}(Y, \tau)$ or of $C_0(X) \times_\sigma \mathbb{F}_n^+$ onto $C_0(Y) \times_\tau \mathbb{F}_n^+$ is automatically norm continuous.

There are no labels on our maps, or on the isometries $\mathfrak{s}_i$. So an isomorphism can permute the maps arbitrarily. It is less obvious, but still true, that in some circumstances, one can change from one permutation to another. This leads to our definition of *piecewise conjugate* systems. Say that (X, σ) is piecewise conjugate to (Y, τ) if there is a homeomorphism $\gamma : X \to Y$ and an open cover $\{\mathcal{O}_\alpha : \alpha \in S_n\}$ of X so that

$$\tau_i \gamma|_{\mathcal{O}_\alpha} = \gamma \sigma_{\alpha(i)}|_{\mathcal{O}_\alpha} \quad \text{for} \quad \alpha \in S_n.$$

To appreciate this notion, consider two maps σ_1 and σ_2 that map $[0,1]$ into itself and coincide on an interval (a, b). Then one can construct τ_1 which agrees with σ_1 on $[0, b)$ and with σ_2 on $(a, 1]$; and similarly τ_2 agrees with σ_2 on $[0, b)$ and with σ_1 on $(a, 1]$. Then $([0,1], \sigma_1, \sigma_2)$ and $([0,1], \tau_1, \tau_2)$ are piecewise conjugate. On the other hand, if the two maps only agree at a point $\{a\}$, then one can still define the maps τ_1 and τ_2 as above, but the new system will not be piecewise conjugate

because there is no neighbourhood of a on which we can match up the two pairs of functions.

This appears to be a new notion in dynamics. There is a parallel with the *full group* introduced by Dye [**18**] in his analysis of group actions on von Neumann algebras. One begins with a group $\{\alpha_g : g \in G\}$ of measure preserving automorphisms of a measure space $\mathfrak{M}$. Dye considers all automorphisms α which are pieced together by a countable partition of the space into measureable sets P_g such that $\alpha_g^{-1}\alpha$ is the identity on $P_g\mathfrak{M}$. So the notion of mixing and matching maps occurs here, but in a rather different context.

The main result of our paper [**13**] is the following:

THEOREM 5.1 (Davidson–Katsoulis [**13**]). *Let (X, σ) and (Y, τ) be two multivariable dynamical systems. If there is an algebra isomorphism of $\mathcal{A}(X, \sigma)$ onto $\mathcal{A}(Y, \tau)$ or of $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ onto $\mathrm{C}_0(Y) \times_\tau \mathbb{F}_n^+$, then (X, σ) and (Y, τ) are piecewise conjugate.*

The proof follows the ideas of the $n = 1$ case, but non-trivial complications arise. In particular, one must be able to count the number of maps in the system which send a point x to a point y. The key is again an analytic structure on the set of nest representations. The ability to count the number of maps relies on the well-known, but non-trivial, fact from several complex variable theory [**19**] that the zero set of an analytic function mapping $\mathbb{C}^k$ into $\mathbb{C}^l$, for $l < k$, has no isolated points.

In the case $n = 1$, the converse direction was trivial. But here there are difficult issues about how to intertwine the isometries to form the new ones. This appears to be possible in the case of the tensor algebra. We have no idea how this could be accomplished in the semicrossed product case. A partial converse is the following:

THEOREM 5.2 (Davidson–Katsoulis). *Suppose that at least one of the following holds:*

- *$n \le 3$, or*
- *X has covering dimension at most 1, or*
- *$\{x : |\sigma(x)| < n\}$ has no interior.*

Then the following are equivalent:

(i) *(X, σ) and (Y, τ) are piecewise conjugate.*
(ii) *$\mathcal{A}(X, \sigma)$ and $\mathcal{A}(Y, \tau)$ are algebraically isomorphic.*
(iii) *$\mathcal{A}(X, \sigma)$ and $\mathcal{A}(Y, \tau)$ are completely isometrically isomorphic.*

We conjecture that the converse holds in complete generality. It was reduced to a technical conjecture about the existence of a nice map from the polytope with vertices indexed by S_n into the unitary group $\mathcal{U}_n$ which takes the vertices to the corresponding permutation matrix, and satisfies some strict compatibility conditions on the various faces. Chris Ramsey, a student at the University of Waterloo, has been making progress on this conjecture [**44**]. He has extended the converse to the case $n = 4$.

It would be interesting to study the ideal structure of these algebras. Peters [**40**] has made progress on this in the case $n = 1$. A natural test question is to determine when the algebra is semisimple, and more generally, to identify the radical. This has been accomplished in the case $n = 1$ by Donsig, Katavolos and Manoussos [**16**], with earlier work by Muhly [**31**]. We can answer the question about semisimplicity, but have little to say about the radical.

Call an open subset $U \subset X$ a (u, v)–*wandering set* if

$$\sigma_{uwv}^{-1}(U) \cap U = \varnothing \quad \text{for all} \quad w \in \mathbb{F}_n^+.$$

A generalized wandering set is a (u, v)-wandering set for some pair (u, v). If there are no wandering sets, then necessarily each σ_i is surjective.

Wandering sets have a parallel notion of recurrence. Say that $x \in X$ is (u, v)–*recurrent* if for every open set $U \ni x$, there is some $w \in \mathbb{F}_n^+$ so that $\sigma_{uwv}(x) \in U$. In the metrizable case, there are no non-empty generalized wandering sets if and only if the (u, v)–recurrent points are dense for all pairs (u, v). This is in turn equivalent to the surjectivity of each σ_i and the density of the $(\varnothing, v)$–recurrent points for each $v \in \mathbb{F}_n^+$.

THEOREM 5.3 (Davidson–Katsoulis). *The following are equivalent:*

  (i) $\mathcal{A}(X, \sigma)$ *is semisimple.*
  (ii) $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ *is semisimple.*
  (iii) *There are no non-empty generalized wandering sets.*

6. C*-envelopes

Arveson's seminal paper [**4**] proposes that to study a nonself-adjoint operator algebra, there should be a canonical minimal C*-algebra that contains it (completely isometrically). This C*-algebra, $\mathrm{C}_{\mathrm{env}}^*(\mathcal{A})$, is the proposed analogue of the Shilov boundary in the function algebra case, and is called the *C*-envelope* of $\mathcal{A}$. Let j_0 be the completely isometric imbedding of $\mathcal{A}$ into $\mathrm{C}_{\mathrm{env}}^*(\mathcal{A})$. Then $\mathrm{C}_{\mathrm{env}}^*(\mathcal{A})$ is determined by the universal property that whenever j is a completely isometric isomorphism of $\mathcal{A}$ into another C*-algebra $\mathrm{C}^*(j(\mathcal{A}))$, there exists a $*$-homomorphism π of $\mathrm{C}^*(j(\mathcal{A}))$ onto $\mathrm{C}_{\mathrm{env}}^*(\mathcal{A})$ such that $\pi j = j_0$.

Unlike the other universal constructions mentioned in this paper, it is not at all apparent that the C*-envelope exists. Arveson constructed it for a large family of examples, but left the existence in general as a conjecture. This was verified a decade later by Hamana [**21**]. A new proof was found a few years ago by Dritschel and McCullough [**17**]. Their proof is based on the notion of a *maximal dilation*. A representation ρ of $\mathcal{A}$ on $\mathcal{H}$ is maximal if any dilation of ρ to a completely contractive representation π on a larger space $\mathcal{K}$ (meaning that $\rho(A) = P_{\mathcal{H}}\pi(A)|_{\mathcal{H}}$) has the form $\pi = \rho \oplus \pi'$ on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$. It is not particularly difficult to show that any completely contractive representation can be dilated to a maximal one. The point is that maximal representations extend to $*$-representations of the enveloping C*-algebra of $\mathcal{A}$ and factor through the C*-envelope. Some of these ideas were already known due to work of Muhly and Solel [**33**]. This new proof provides a tangible way to get hold of the C*-envelope. One starts with a completely isometric representation ρ, dilates it to a maximal representation π, and $\mathrm{C}_{\mathrm{env}}^*(\mathcal{A}) = \mathrm{C}^*(\pi(\mathcal{A}))$.

In [**13**], we provide two views of the C*-envelope of the tensor algebra $\mathcal{A}(X, \sigma)$. The first is a rather abstract approach. Pimsner [**42**] developed a construction of a C*-algebra from a C*-correspondence, a Hilbert C*-module with a compatible left action, now known as the Cuntz–Pimsner algebra of the correspondence. Muhly and Solel [**34, 35**] developed an extensive theory of an associated nonself-adjoint *tensor algebra* of a C*-correspondence. They show that when the left action is faithful, the C*-envelope of the tensor algebra is the Cuntz–Pimsner C*-algebra of the C*-correspondence. Katsura [**26**] extended this theory, defining the Cuntz–Pimsner algebra for more general left actions which need not be faithful. The

second author and Kribs [**24**] used Katsura's work to generalize the Muhly–Solel theorem to describe the C*-envelope of the tensor algebra in full generality.

We show explicitly [**13**] that the tensor algebra of a multivariable dynamical system is the tensor algebra of a naturally associated C*-correspondence. Consequently, by the results in the previous paragraph, we have a description of the C*-envelope as a Cuntz–Pimsner algebra. Unfortunately, because this algebra is a quotient of the Cuntz–Toeplitz algebra by Katsura's ideal, this is not a very concrete description. We were looking for something more explicit.

The first step, carried out in [**13**], is to describe the maximal dilations of the orbit representations. Notice that if $x = \sigma_i(y)$, then the orbit representation π_x can be obtained as the restriction of π_y to an invariant subspace. Hence π_x dilates to π_y. One can repeat this procedure, and when the system is surjective, construct an infinite chain of orbit representations, each being a dilation of the previous one. The inductive limit of this procedure yields a family of maximal representations. When the system is not surjective, this procedure stops if we arrive at a point y which is not in the range of any map. It turns out that the orbit representation of such a point is also maximal. Since the direct sum of all orbit representations is completely isometric on the tensor algebra, it follows that the direct sum of all of these maximal dilations is also completely isometric. Hence the C*-envelope is given as the algebra generated by this large representation. This still is not very explicit, so we seek to develop this some more.

In the case $n = 1$ when σ is surjective and X is compact, this was accomplished by Peters [**41**]. The idea is to take the projective limit $\tilde{X}$ of the system (X, σ)

$$X \xleftarrow{\ \sigma\ } X \xleftarrow{\ \sigma\ } X \xleftarrow{\ \sigma\ } \ldots \longleftarrow \tilde{X}.$$

There is canonical projection p of Y onto X, and a map $\tilde{\sigma}$ of Y onto itself such that $p\tilde{\sigma} = \sigma p$. Moreover, $\tilde{\sigma}$ is always a homeomorphism. Consequently, one can form the C*-crossed product $C(\tilde{X}) \times_{\tilde{\sigma}} \mathbb{Z}$. There is a natural injection of $C(X) \times_\sigma \mathbb{Z}^+$ into this algebra by sending f to $f \circ p$ and sending $\mathfrak{s}$ to the canonical unitary of the crossed product. Peters shows that this map is a complete isometry, and that the image generates the crossed product as a C*-algebra. Then with a bit more work, one obtains:

THEOREM 6.1 (Peters [**41**]). *Let X be a compact Hausdorff space, and let σ be a surjective continuous map of X onto itself. Construct $(\tilde{X}, \tilde{\sigma})$ as above. Then*

$$C^*_{env}(\mathcal{A}(X, \sigma)) \simeq C(\tilde{X}) \times_{\tilde{\sigma}} \mathbb{Z}.$$

The first author and Jean Roydor [**15**] extended Peters' construction to the multivariable setting. First assume that (X, σ) is surjective in the sense that $X = \bigcup_{i=1}^n \sigma_i(X)$. One can again construct a projective limit system. An infinite tail is an infinite sequence $\mathbf{i} \in \mathbf{n}^{\mathbb{N}}$, where $\mathbf{n} = \{1, \ldots, n\}$; say $\mathbf{i} = (i_0, i_1, \ldots)$. One considers the set $\tilde{X}$ of all pairs $(\mathbf{i}, \mathbf{x}) \in \mathbf{n}^{\mathbb{N}} \times X^{\mathbb{N}}$ such that $\sigma_{i_s}(x_{s+1}) = x_s$. There is again a natural map $p(\mathbf{i}, \mathbf{x}) = x_0$ of $\tilde{X}$ onto X and maps $\tilde{\sigma}_j(\mathbf{i}, \mathbf{x}) = \big((j, \mathbf{i}), (\sigma_j(x_0), \mathbf{x})\big)$ that satisfy $p\tilde{\sigma}_j = \sigma_j p$ for $1 \le j \le n$. These maps are no longer homeomorphisms. However the range $\tilde{X}_j$ of $\tilde{\sigma}_j$ consists of all points $(\mathbf{i}, \mathbf{x})$ such that $i_0 = j$. These are pairwise disjoint clopen sets, and $\tilde{\sigma}_j$ is a homeomorphism of $\tilde{X}$ onto $\tilde{X}_j$. The inverse map τ given by $\tau|_{\tilde{X}_j} = \tilde{\sigma}_j^{-1}$ is a local homeomorphism. The tensor algebra $\mathcal{A}(X, \sigma)$ imbeds completely isometrically into $\mathcal{A}(\tilde{X}, \tilde{\sigma})$, and they have the same C*-envelope.

This leads to a more concrete description of the C*-envelope, because the new system is much simpler to handle. One description is that this is the groupoid C*-algebra of the system $(\tilde{X}, \tau)$ in the sense of Deaconu [10]. Another is that it is the crossed product of a certain inductive limit $\mathfrak{B}$ of homogeneous C*-algebras by an endomorphism α.

THEOREM 6.2 (Davidson–Roydor [15]). *Let X be a locally compact Hausdorff space, and let $\sigma_1, \ldots, \sigma_n$ be proper maps of X into itself such that $X = \bigcup_{i=1}^n \sigma_i(X)$. Construct $(\tilde{X}, \tilde{\sigma})$ as above. Then*

$$\mathrm{C}_{env}^*(\mathcal{A}(X, \sigma)) \simeq \mathrm{C}_{env}^*(\mathcal{A}(\tilde{X}, \tilde{\sigma})) \simeq \mathrm{C}^*(\tilde{X}, \tau) \simeq \mathfrak{B} \times_\alpha \mathbb{Z}^+.$$

When (X, σ) is not surjective, there is a well-known technique from graph algebras of "adding a tail". Let $U = X \setminus \bigcup_{i=1}^n \sigma(X)$. Form $X^T = X \cup T$ where $T = \{(u, k) : u \in \overline{U},\ k < 0\}$. Extend σ_i to maps σ_i^T by setting

$$\sigma_i^T(u, k) = (u, k+1) \text{ for } k < -1, \quad \text{and} \quad \sigma_i^T(u, -1) = u.$$

It is shown that the natural imbedding of $\mathcal{A}(X, \sigma)$ into (X^T, σ^T) is a completely isometric isomorphism. Moreover, the C*-envelope of $\mathcal{A}(X, \sigma)$ is a full corner of $\mathrm{C}_{env}^*(\mathcal{A}(X^T, \sigma^T))$.

One consequence is a characterization of when the C*-envelope is simple. When X is compact, the system (X, σ) is called *minimal* if there are no proper closed σ-invariant subsets of X.

THEOREM 6.3 (Davidson–Roydor [15]). *Let (X, σ) be a compact multivariable dynamical system. Then $\mathrm{C}_{env}^*(\mathcal{A}(X, \sigma))$ is simple if and only if (X, σ) is minimal.*

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