

CONJUGATE DYNAMICAL SYSTEMS ON C*-ALGEBRAS

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ABSTRACT. Let (A, α) and (B, β) be C*-dynamical systems where α and β are arbitrary *-endomorphisms. When α is injective or surjective, we show that the semicrossed products $A \times_{\alpha} \mathbb{Z}_+$ and $B \times_{\beta} \mathbb{Z}_+$ are isometrically isomorphic if and only if (A, α) and (B, β) are outer conjugate. This conclusion also holds in various other cases as well.

1. INTRODUCTION

If α is a *-endomorphism of a C*-algebra A , the semicrossed product $A \times_{\alpha} \mathbb{Z}_+$ is an operator algebra which encodes the dynamics in the sense that it is the universal operator algebra for covariant representations of (A, α) . We wish to determine to what extent the dynamical system can be recovered from the semicrossed product. It is easy to see that if two systems are outer conjugate, then the semicrossed products are completely isometrically isomorphic. We establish the converse when α is injective, and in various other cases, by showing that if two semicrossed products are isometrically isomorphic, then the dynamical systems are outer conjugate.

The use of nonself-adjoint operator algebras to encode C*-dynamics goes back to Arveson [1, 2] where ergodic actions on a space X were encoded in a concrete operator algebra. Under certain hypotheses, two such algebras were shown to be isomorphic if and only if the two (commutative) dynamical systems are conjugate. Peters [17] established that under much more general conditions, one obtains a universal operator algebra encoding the covariance relations. He extended the isomorphism results for the commutative case, although there were still

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conditions on the fixed point sets. This was further developed by Hadwin and Hoover [7]. Finally in [3], Davidson and Katsoulis showed that (in the commutative case), algebraic isomorphism of semicrossed products is a complete invariant for conjugacy. See [5] for a survey of these results.

The use of non-selfadjoint operator algebras is not just an artifact of convenience. They arise naturally when one tries to model the covariance relations. Moreover, the C^* -algebra crossed product loses information. Even when α is an automorphism, so that the usual C^* -algebra crossed product $A \rtimes_{\alpha} \mathbb{Z}$ is available, this algebra is not a complete invariant for the system. Hoare and Parry [8] give an example of an automorphic, commutative dynamical system (A, α) such that α and α^{-1} are not conjugate. However $A \rtimes_{\alpha} \mathbb{Z}$ and $A \rtimes_{\alpha^{-1}} \mathbb{Z}$ are always $*$ -isomorphic.

In the non-automorphic case, there is a variety of C^* -algebra crossed products by endomorphisms [16, 18, 6], introduced as possible generalizations of the crossed product. As in the automorphic case, these algebras do not generally allow recovery of the dynamics from the algebra.

Likewise there are several possible choices for the semicrossed product. See [12, 10] for a discussion. When the $*$ -endomorphism is injective, the various choices produce the ‘same’ operator algebra. The conclusion is basically that the original choice made by Peters is still the best option as one always obtains an isometric copy of the C^* -algebra in this semicrossed product.

There is not much literature showing that the semicrossed product is an invariant for a non-commutative dynamical system, even when α is an automorphism. One has to replace conjugacy by outer conjugacy, because outer conjugate systems yield completely isometric semicrossed products. See section 2. The first result in this direction was given by Muhly and Solel in [14, Theorem 4.1] where they show that if A is a separable C^* -algebra and α is an automorphism with a full Connes spectrum, then the semicrossed product is an isometric isomorphism invariant. Davidson and Katsoulis [4] extend this result for separable, simple C^* -algebras, also when α is an automorphism, by using nest representations and a result of Kishimoto [13] on universally weakly inner automorphisms.

In this paper, we show that this technology is unnecessary. We are able to show that isometric isomorphism of semicrossed products of automorphic dynamical systems is a complete invariant for conjugacy, without making any extra assumption on the C^* -algebras.

In fact we do more than that. We are able to prove the same for arbitrary unital C^* -algebras provided that we make the common assumption that the endomorphisms are injective or surjective. The general case, in which the endomorphisms are not surjective and have kernel, is delicate; and we have not resolved this case completely. To our knowledge, these are the first significant results in the literature for this situation.

The main results established here are:

Theorem 1.1. *Let (A, α) and (B, β) be unital C^* -algebra dynamical systems. Suppose that α is either injective or surjective. Then $A \times_\alpha \mathbb{Z}_+$ and $B \times_\beta \mathbb{Z}_+$ are isometrically isomorphic if and only if (A, α) and (B, β) are outer conjugate.*

We conjecture that no hypotheses on α are required. We gather considerable evidence towards this view, and establish the result in a number of other cases.

Theorem 1.2. *Let (A, α) and (B, β) be unital C^* -algebra dynamical systems. Suppose that one of the following holds:*

- *A has trivial centre, $Z(A) = \mathbb{C}1$. (e.g. A simple.)*
- *A is abelian.*
- *A is finite (no proper isometries).*
- *$\alpha(A)'$ is finite.*
- *$\alpha(R_\alpha) = R_\alpha$, where $R_\alpha = \overline{\bigcup_{k \geq 1} \ker(\alpha^k)}$.*
- *$\alpha(\text{ann}(R_\alpha)) \subseteq \text{ann}(R_\alpha)$.*

Then $A \times_\alpha \mathbb{Z}_+$ and $B \times_\beta \mathbb{Z}_+$ are isometrically isomorphic if and only if (A, α) and (B, β) are outer conjugate.

2. BACKGROUND

A C^* -dynamical system will be a unital C^* -algebra A together with a unital $*$ -endomorphism $\alpha \in \text{End}(A)$. A covariant representation of (A, α) is a pair (π, V) where π is a $*$ -representation of A on a Hilbert space \mathcal{H} and $V \in \mathcal{B}(\mathcal{H})$ is a contraction satisfying the covariance relation

$$\pi(a)V = V\pi\alpha(a) \quad \text{for all } a \in A.$$

Let \mathcal{P} be the space of formal polynomials of the form $\sum_{k=0}^n \mathbf{v}^k a_k$ where $a_k \in A$ and $n \geq 0$ with the natural vector space structure, and a multiplication given by the rule

$$a\mathbf{v} = \mathbf{v}\alpha(a) \quad \text{for } a \in A.$$

Then it is evident that a covariant pair (π, V) yields a representation of \mathcal{P} by

$$\pi \times V \left(\sum_{k=0}^n \mathbf{v}^k a_k \right) = \sum_{k=0}^n V^k \pi(a_k).$$

A norm is defined by

$$\|p\| = \sup_{(\pi, V) \text{ covariant}} \|\pi \times V(p)\|.$$

This quantity is finite because

$$\|p\| \leq \|p\|_1 := \sum_{k=0}^n \|a_k\|.$$

This likewise determines norms on $\mathcal{M}_n(\mathcal{P})$ for each $n \geq 1$.

The *semicrossed product* $A \times_\alpha \mathbb{Z}_+$ is the universal operator algebra for this family of representations. That is, it is the operator space completion of \mathcal{P} in this family of matrix norms. The characterizing property is that every covariant pair (π, V) yields a completely contractive representation $\pi \times V$ of $A \times_\alpha \mathbb{Z}_+$ extending the representation on \mathcal{P} .

Covariant representations always exist. Indeed, let π be any representation of A on \mathcal{H} . Let $\mathcal{H}^{(\infty)} = \mathcal{H} \otimes \ell^2$ be the infinite ampliation of \mathcal{H} . Define a representation $\tilde{\pi}$ on $\mathcal{H}^{(\infty)}$ by

$$\tilde{\pi}(a) = \text{diag}(\pi \alpha^k(a) : k \geq 0).$$

Let S be the unilateral shift on ℓ^2 and set $V = I \otimes S$. Then it is straightforward to check that $(\tilde{\pi}, V)$ is a covariant representation, known as the *orbit representation induced by π* . Peters [17] shows that if π is a faithful representation of A , then $\tilde{\pi} \times V$ is an isometric representation of $A \times_\alpha \mathbb{Z}_+$. A different proof is contained in [10] based on a gauge invariance uniqueness theorem, that additionally provides that $\tilde{\pi} \times V$ is completely isometric.

One could also define covariant representations that insist that V be an isometry. By a dilation result of Muhly and Solel [15], every contractive covariant pair dilates to an isometric covariant pair. Hence there is no difference in the universal operator algebra obtained. Moreover, the element \mathbf{v} in $A \times_\alpha \mathbb{Z}_+$ is an isometry in its C^* -envelope.

If (π, V) is a covariant representation and $z \in \mathbb{T} = \{z : |z| = 1\}$, then (π, zV) is also a covariant representation. It follows from the universal property that there is a completely isometric automorphism γ_z of $A \times_\alpha \mathbb{Z}_+$ such that $\gamma_z(a) = a$ for $a \in A$ and $\gamma_z(\mathbf{v}) = z\mathbf{v}$. The

map $z \rightarrow \gamma_z$ is a group homomorphism which is point-norm continuous. This enables us to define Fourier coefficients by

$$\mathbf{v}^n E_n(X) = \int_{\mathbb{T}} \gamma_z(X) \bar{z}^n dm \quad \text{for } n \geq 0,$$

where m is normalized Lebesgue measure on \mathbb{T} . These are evidently completely contractive maps of $A \times_{\alpha} \mathbb{Z}_+$ into A . In particular, E_0 is an expectation on $A \times_{\alpha} \mathbb{Z}_+$ onto A (i.e. a completely contractive idempotent map) which is also a homomorphism. The standard proof of Féjer's Theorem shows that

$$\Sigma_n(X) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \mathbf{v}^k E_k(X)$$

are completely contractive maps, and that $\Sigma_n(X)$ converges to X in norm for all $X \in A \times_{\alpha} \mathbb{Z}_+$. In particular, the map taking X to its Fourier series is injective.

Two dynamical systems (A, α) and (B, β) are said to be *outer conjugate* if there is a $*$ -isomorphism γ of A onto B and a unitary $v \in A$ such that

$$\alpha(a) = \text{ad}_v \gamma^{-1} \beta \gamma(a) = v(\gamma^{-1} \beta \gamma(a)) v^* \quad \text{for all } a \in A.$$

Setting $w = \gamma(v)^*$, it is easy to deduce that

$$\beta(b) = \text{ad}_w \gamma \alpha \gamma^{-1}(b) = w(\gamma \alpha \gamma^{-1}(b)) w^* \quad \text{for all } b \in B.$$

In this case the semicrossed products $A \times_{\alpha} \mathbb{Z}_+$ and $B \times_{\beta} \mathbb{Z}_+$ are completely isometrically isomorphic. To see this, write elements of $B \times_{\beta} \mathbb{Z}_+$ as Fourier series in a variable \mathbf{w} and coefficients in B . Observe that for $a \in A$,

$$\gamma(a)(\mathbf{w}w) = \mathbf{w} \beta \gamma(a) w = \mathbf{w} w (\gamma \alpha \gamma^{-1} \gamma(a)) w^* w = (\mathbf{w}w) \gamma \alpha(a).$$

Thus $(\gamma, \mathbf{w}w)$ is a covariant representation of (A, α) . Therefore there is a completely contractive homomorphism φ of $A \times_{\alpha} \mathbb{Z}_+$ into $B \times_{\beta} \mathbb{Z}_+$ so that $\varphi|_A = \gamma$ and $\varphi(\mathbf{v}) = \mathbf{w}w$. Similarly there is a map ψ of $B \times_{\beta} \mathbb{Z}_+$ into $A \times_{\alpha} \mathbb{Z}_+$ such that $\psi|_B = \gamma^{-1}$ and $\psi(\mathbf{w}) = \mathbf{v}v$. It is easy to check that $\psi = \varphi^{-1}$, and thus φ is a unital completely isometric isomorphism.

In this paper, we are concerned with the converse. Suppose that $A \times_{\alpha} \mathbb{Z}_+$ and $B \times_{\beta} \mathbb{Z}_+$ are (completely) isometrically isomorphic. Are (A, α) and (B, β) outer conjugate? A positive answer for the injective and surjective cases is the content of Theorem 1.1. Theorem 1.2 includes extensions to many situations where α is not injective. This requires a careful analysis of the sequence of ideals $\ker(\alpha^k)$ and their closed union, R_{α} .

3. ALGEBRAIC MANIPULATIONS

Suppose that φ is an isomorphism of $A \times_\alpha \mathbb{Z}_+$ onto $B \times_\beta \mathbb{Z}_+$. The universal property shows [10] that φ is completely isometric if and only if it is isometric. We assume from now on that φ is a fixed (completely) isometric isomorphism. A standard argument, using the fact that A is the largest C^* -algebra contained in $A \times_\alpha \mathbb{Z}_+$ and that an isometric isomorphism of a C^* -algebra is a $*$ -isomorphism, shows that $\gamma = \varphi|_A$ is a $*$ -isomorphism of A onto B . Conversely, if $\varphi|_A$ is a $*$ -isomorphism of A onto B and $\|\varphi(\mathbf{v})\| = 1 = \|\varphi^{-1}(\mathbf{w})\|$, then φ is completely isometric.

In this section, we develop a variety of algebraic results that are quite ring theoretic in nature. The only use of the isometric hypothesis is the consequence mentioned in the previous paragraph that the restriction to the C^* -algebra A is a $*$ -isomorphism onto B .

Consider the image of the generator \mathbf{v} of $A \times_\alpha \mathbb{Z}_+$ under φ . Using the Fourier decomposition, we may write

$$\varphi(\mathbf{v}) = b_0 + \mathbf{w}b_1 + \mathbf{w}^2Y \quad \text{and} \quad \varphi^{-1}(\mathbf{w}) = a_0 + \mathbf{v}a_1 + \mathbf{v}^2X,$$

where $Y \in B \times_\beta \mathbb{Z}_+$ and $X \in A \times_\alpha \mathbb{Z}_+$. Note that the elements of the form \mathbf{w}^2Y form a closed ideal of $B \times_\beta \mathbb{Z}_+$ consisting of those elements Z such that $E_0(Z) = E_1(Z) = 0$.

Lemma 3.1. *For all $a \in A$ we obtain:*

- (i) $\gamma(a)b_0 = b_0\gamma\alpha(a)$ and $\beta\gamma(a)b_1 = b_1\gamma\alpha(a)$.
- (ii) $b_0b_0^*$ lies in the centre $Z(B)$; $b_0b_0^*$ and $b_1^*b_1$ commute with $\gamma\alpha(A)$; and $b_1b_1^*$ commutes with $\beta(B)$.

The symmetrical relations hold for a_0 and a_1 . Moreover,

- (iii) $a_1\gamma^{-1}(b_1)$ commutes with $\alpha(A)$.

Proof. Rewrite the covariance relations for (A, α) under φ , for $a \in A$, as

$$\begin{aligned} \varphi(a\mathbf{v}) &= \gamma(a)b_0 + \gamma(a)\mathbf{w}b_1 + \gamma(a)\mathbf{w}^2Y \\ &= \gamma(a)b_0 + \mathbf{w}\beta\gamma(a)b_1 + \mathbf{w}^2\beta^2\gamma(a)Y \\ &= \varphi(\mathbf{v}\alpha(a)) = b_0\gamma\alpha(a) + \mathbf{w}b_1\gamma\alpha(a) + \mathbf{w}^2Y\gamma\alpha(a). \end{aligned}$$

Therefore, equating the Fourier coefficients of the two expressions, we obtain that

$$\gamma(a)b_0 = b_0\gamma\alpha(a) \quad \text{and} \quad \beta\gamma(a)b_1 = b_1\gamma\alpha(a) \quad \text{for all } a \in A.$$

This yields (i). Since γ is an isomorphism of A onto B , we obtain the useful variant

$$bb_0 = b_0(\gamma\alpha\gamma^{-1})(b) \quad \text{and} \quad \beta(b)b_1 = b_1(\gamma\alpha\gamma^{-1})(b) \quad \text{for all } b \in B.$$

Since A is a C^* -algebra, conjugation of the first identity yields

$$b_0^* \gamma(a) = \gamma \alpha(a) b_0^* \quad \text{for all } a \in A.$$

For $b \in B$, let $a = \gamma^{-1}(b)$. We obtain

$$(b_0 b_0^*) b = b_0 b_0^* \gamma(a) = b_0 \gamma \alpha(a) b_0^* = \gamma(a) b_0 b_0^* = b(b_0 b_0^*).$$

This establishes the first part of (ii). The other statements are established similarly. By symmetry, we get the analogous relations for a_0 and a_1 .

Statement (iii) is also similar. Let $a \in A$. Then

$$\begin{aligned} a_1 \gamma^{-1}(b_1) \alpha(a) &= a_1 \gamma^{-1}(b_1 \gamma \alpha(a)) \\ &= a_1 \gamma^{-1}(\beta \gamma(a) b_1) = \alpha(a) a_1 \gamma^{-1}(b_1). \end{aligned} \quad \blacksquare$$

Lemma 3.2. *In the notation above, a_1 and b_1 are right invertible.*

Proof. It suffices to prove the result for b_1 . Observe that since φ is surjective, there is a polynomial $p = \sum_{k=0}^n \mathbf{v}^k x_k$, with $x_k \in A$, so that $\|\varphi(p) - \mathbf{w}\| < 1/2$. Compute

$$\begin{aligned} E_1(\varphi(p)) &= E_1 \sum_{k=0}^n \varphi(\mathbf{v})^k \gamma(x_k) \\ &= \sum_{k=0}^n E_1((b_0 + \mathbf{w} b_1 + \mathbf{w}^2 Y)^k \gamma(x_k)) \\ &= \sum_{k=0}^n \sum_{i=0}^{k-1} E_1(b_0^i \mathbf{w} b_1 b_0^{k-1-i} \gamma(x_k)) \\ &= \sum_{k=0}^n \sum_{i=0}^{k-1} \beta(b_0^i) b_1 b_0^{k-1-i} \gamma(x_k) \\ &= b_1 \sum_{k=0}^n \sum_{i=0}^{k-1} \gamma \alpha \gamma^{-1}(b_0^i) b_0^{k-1-i} \gamma(x_k) = b_1 c. \end{aligned}$$

Therefore

$$\|b_1 c - 1\| = \|E_1(\varphi(p) - \mathbf{w})\| < 1/2.$$

It follows that $b_1 c$ is invertible, and hence b_1 is right invertible. \blacksquare

Corollary 3.3. *If φ is an isometric isomorphism of $A \times_\alpha \mathbb{Z}_+$ onto $B \times_\beta \mathbb{Z}_+$, then $\gamma(\ker \alpha) = \ker \beta$.*

Proof. If $a \in \ker \alpha$, then

$$\beta \gamma(a) b_1 = b_1 \gamma \alpha(a) = 0.$$

Since b_1 has a right inverse, $\gamma(a) \in \ker \beta$. Thus $\gamma(\ker \alpha) \subset \ker \beta$. Similarly, $\gamma^{-1}(\ker \beta) \subset \ker \alpha$. Therefore we have equality. ■

We now establish a key general result.

Key Lemma 3.4. *If b_1 is invertible, then (A, α) and (B, β) are outer conjugate.*

Proof. By Lemma 3.1(ii), $b_1^* b_1$ commutes with $\gamma\alpha(A)$. Thus $|b_1| = (b_1^* b_1)^{1/2}$ also commutes with $\gamma\alpha(A)$. Let $b_1 = w|b_1|$ be the polar decomposition. Since b_1 is invertible by hypothesis, w is unitary. Compute

$$\begin{aligned} (w \gamma\alpha(a) w^*) b_1 &= w \gamma\alpha(a) |b_1| = w |b_1| \gamma\alpha(a) \\ &= b_1 \gamma\alpha(a) = \beta\gamma(a) b_1. \end{aligned}$$

Therefore $w \gamma\alpha(a) w^* = \beta\gamma(a)$. Substituting $b = \gamma(a)$ yields

$$\beta(b) = (\text{ad}_w \gamma\alpha\gamma^{-1})(b) \quad \text{for all } b \in B.$$

This establishes outer conjugacy. ■

The following is a special case in which we obtain invertibility.

Lemma 3.5.

- (i) *If $a_0 = 0$ or $b_0 = 0$, then $a_0 = 0 = b_0$, b_1 and a_1 are unitary, and $a_1 \gamma^{-1}(b_1) = 1$.*
- (ii) *If $X = 0$ or $Y = 0$, then $X = 0 = Y$, b_1 and a_1 are unitary, and $a_1 \gamma^{-1}(b_1) = 1$.*

Consequently, $\varphi(\mathbf{v}) = \mathbf{w}b_1$, $\varphi^{-1}(\mathbf{w}) = \mathbf{v}a_1$ and $\gamma(a_1) = b_1^*$, in either case.

Proof. By symmetry we can assume that $a_0 = 0$ or $Y = 0$. Let us write $\varphi^{-1}(Y) = c_0 + \mathbf{v}c_1 + \mathbf{v}^2 Z$. First we compute

$$\begin{aligned} \mathbf{v} &= \varphi^{-1}(\varphi(\mathbf{v})) = \varphi^{-1}(b_0 + \mathbf{w}b_1 + \mathbf{w}^2 Y) \\ &= \gamma^{-1}(b_0) + (a_0 + \mathbf{v}a_1 + \mathbf{v}^2 X) \gamma^{-1}(b_1) \\ &\quad + (a_0 + \mathbf{v}a_1 + \mathbf{v}^2 X)^2 (c_0 + \mathbf{v}c_1 + \mathbf{v}^2 Z) \\ &= (\gamma^{-1}(b_0) + a_0 \gamma^{-1}(b_1) + a_0^2 c_0) \\ &\quad + \mathbf{v} (a_1 \gamma^{-1}(b_1) + \alpha(a_0) a_1 c_0 + a_1 a_0 c_0 + \alpha(a_0)^2 c_1) + \mathbf{v}^2 Z' \end{aligned}$$

Therefore

$$0 = \gamma^{-1}(b_0) + a_0 \gamma^{-1}(b_1) + a_0^2 c_0,$$

and

$$1 = a_1 \gamma^{-1}(b_1) + \alpha(a_0) a_1 c_0 + a_1 a_0 c_0 + \alpha(a_0)^2 c_1.$$

If $a_0 = 0$, then it follows that $b_0 = 0$, and so $a_1\gamma^{-1}(b_1) = 1$. If $Y = 0$, then $c_0 = c_1 = 0$; and therefore $a_1\gamma^{-1}(b_1) = 1$ again. In either case $\gamma^{-1}(b_1)$ is left invertible. Thus b_1 is invertible, and $a_1 = \gamma^{-1}(b_1^{-1})$ is also invertible. Since a_1 and b_1 are contractions, it follows that they must be unitary.

Also when $Y = 0$, the identity for $\mathbf{v} = \varphi^{-1}(\varphi(\mathbf{v}))$ simplifies and yields $\mathbf{v}^2 X \gamma^{-1}(b_1) = 0$. Since b_1 is right invertible, $X = 0$ as well. By symmetry, it suffices to check just these cases.

Now choose a faithful representation π of B and consider the orbit representation $(\tilde{\pi}, I \otimes S)$ that it induces. Applying this to $\varphi(\mathbf{v})$ yields a matrix operator with the first column equal to $(b_0, b_1, b_2, \dots)^t$ where $\mathbf{v}^2 Y = \sum_{n \geq 2} \mathbf{v}^n b_n$. This column has norm at most one, and b_1 is unitary. Hence $b_n = 0$ for all $n \neq 1$. That is, $b_0 = 0$ and $Y = 0$, and thus $\varphi(\mathbf{v}) = \mathbf{w}b_1$. Similarly $\varphi^{-1}(\mathbf{w}) = \mathbf{v}a_1$. ■

The following remark provides invertibility in cases when a_0 , and as a consequence X , are not zero.

Remark 3.6. *If $\|a_0\| < \delta := \frac{2}{3}\sqrt{3} - 1 \simeq 0.1547$, then b_1 and a_1 are invertible.*

We know that $\|a_1\| \leq \|\mathbf{w}\| = 1$ and $\|Y\| \leq 1 + \|b_0\| + \|b_1\| \leq 3$. Since $\varphi^{-1}(Y) = c_0 + \mathbf{v}c_1 + \mathbf{v}Z$, we see that $\|c_0\| \leq 3$ and $\|c_1\| \leq 3$. So we can plug these estimates into the identity obtained in the proof of the previous lemma to obtain

$$\begin{aligned} \|1 - a_1\gamma^{-1}(b_1)\| &= \|\alpha(a_0)a_1c_0 + a_1a_0c_0 + \alpha(a_0)^2c_1\| \\ &\leq 6\|a_0\| + 3\|a_0\|^2 < 6\delta + 3\delta^2 = 1. \end{aligned}$$

It follows that $a_1\gamma^{-1}(b_1)$ is invertible. Therefore $\gamma^{-1}(b_1)$ is left invertible; and thus b_1 is invertible. Moreover, if c is the inverse of $a_1\gamma^{-1}(b_1)$, then $b_1^{-1} = \gamma(ca_1)$. Hence $\gamma^{-1}(b_1)c$ is the inverse of a_1 .

When (A, α) is not injective, we need careful analysis of the family of kernels $\ker(\alpha^n)$ for $n \geq 1$ and their closed union $R_\alpha = \overline{\cup_n \ker \alpha^n}$, known as the *radical ideal*. It is straightforward to prove that $a \in R_\alpha$ if and only if $\lim_n \alpha^n(a) = 0$.

Lemma 3.7. *If $A \times_\alpha \mathbb{Z}_+$ is isometrically isomorphic to $B \times_\beta \mathbb{Z}_+$, then there are right invertible elements $b_n \in B$ such that*

$$\beta^n \gamma(a) b_n = b_n \gamma \alpha^n(a) \quad \text{for all } a \in A.$$

Proof. We will prove by induction that $b_n = \beta^{n-1}(b_1) \cdots b_1$. Note that $b_{n+1} = \beta(b_n)b_1$. Since b_1 is right invertible by Lemma 3.2, it follows that each b_n is also right invertible.

It has already been shown that

$$\beta\gamma(a)b_1 = b_1\gamma\alpha(a) \quad \text{for all } a \in A.$$

Suppose that

$$\beta^n\gamma(a)b_n = b_n\gamma\alpha^n(a) \quad \text{for all } a \in A.$$

Then

$$\begin{aligned} \beta^{n+1}\gamma(a)b_{n+1} &= \beta(\beta^n\gamma(a))\beta(b_n)b_1 = \beta(\beta^n\gamma(a)b_n)b_1 \\ &= \beta(b_n\gamma\alpha^n(a))b_1 = \beta(b_n)\beta\gamma(\alpha^n(a))b_1 \\ &= \beta(b_n)b_1\gamma\alpha(\alpha^n(a)) = b_{n+1}\gamma\alpha^{n+1}(a). \end{aligned} \quad \blacksquare$$

Corollary 3.8. *If φ is an isometric isomorphism of $A \times_\alpha \mathbb{Z}_+$ onto $B \times_\beta \mathbb{Z}_+$, then $\gamma(\ker(\alpha^n)) = \ker(\beta^n)$ for $n \geq 1$, and $\gamma(R_\alpha) = R_\beta$.*

Proof. Corollary 3.3 showed that $\gamma(\ker \alpha) = \ker \beta$. Using Lemma 3.7, the same argument shows that $\gamma(\ker \alpha^n) = \ker \beta^n$ for each $n \geq 1$. Therefore $\gamma(R_\alpha) = R_\beta$. \blacksquare

If J is an ideal of a C*-algebra B , then the *annihilator* of J is

$$\text{ann}(J) = \{b \in B : bJ = 0\} = \{b \in B : Jb = 0\}.$$

It is a consequence of the fact that J has a bounded approximate identity that the left and right annihilators coincide, and hence $\text{ann}(J)$ is a (closed) ideal of B .

Lemma 3.9. *With notation as before, $a_0 \in \text{ann}(R_\alpha)$ and $b_0 \in \text{ann}(R_\beta)$.*

Proof. By symmetry, it suffices to establish the result for b_0 . Recall Lemma 3.1(i):

$$\gamma(a)b_0 = b_0\gamma\alpha(a) \quad \text{for all } a \in A.$$

If $x \in \ker \beta$, then $x = \gamma(a)$ for some $a \in \ker \alpha$ by Corollary 3.8. Thus

$$xb_0 = \gamma(a)b_0 = b_0\gamma\alpha(a) = 0.$$

Hence $b_0 \in \text{ann}(\ker \beta)$.

Assume $b_0 \in \text{ann}(\ker(\beta^n))$. Take $x \in \ker(\beta^{n+1})$. Since $\gamma(\ker(\alpha^{n+1})) = \ker(\beta^{n+1})$ by Corollary 3.8, there is an $a \in \ker(\alpha^{n+1})$ so that $x = \gamma(a)$. Moreover since $\alpha(a) \in \ker(\alpha^n)$, we get

$$xb_0 = \gamma(a)b_0 = b_0\gamma\alpha(a) = 0$$

by the inductive hypothesis. Hence $b_0 \in \text{ann}(\ker(\beta^{n+1}))$.

Taking the union for $n \geq 1$ yields $b_0 \in \text{ann}(R_\beta)$. \blacksquare

4. MAIN THEOREM 1.1

4.1. The surjective case. First we assume that α is surjective.

Theorem 4.1. *Let (A, α) and (B, β) be unital C^* -algebra dynamical systems. Suppose that $A \rtimes_{\alpha} \mathbb{Z}_+$ and $B \rtimes_{\beta} \mathbb{Z}_+$ are isometrically isomorphic. If α is surjective, then (A, α) and (B, β) are outer conjugate.*

Proof. It suffices to show that b_1 is invertible. If $b_1 c = 1$, then since $\gamma \alpha \gamma^{-1}$ is surjective, there is an element $b \in B$ so that $\gamma \alpha \gamma^{-1}(b) = c$. Therefore

$$\beta(b) b_1 = b_1 \gamma \alpha \gamma^{-1}(b) = b_1 c = 1.$$

So b_1 has a left and a right inverse; and hence it is invertible. Therefore (A, α) and (B, β) are outer conjugate by the Key Lemma 3.4. \blacksquare

The following corollary is an immediate consequence which we record for further use and reference.

Corollary 4.2. *Let (A, α) and (B, β) be unital C^* -algebra dynamical systems. Suppose that $A \rtimes_{\alpha} \mathbb{Z}_+$ and $B \rtimes_{\beta} \mathbb{Z}_+$ are isometrically isomorphic. If α is an automorphism, then (A, α) and (B, β) are outer conjugate.*

Remark 4.3. It follows that if $A \rtimes_{\alpha} \mathbb{Z}_+$ and $B \rtimes_{\beta} \mathbb{Z}_+$ are isometrically isomorphic, then α is an automorphism if and only if β is an automorphism; and α is surjective if and only if β is surjective.

4.2. The injective case. In this subsection, we assume that α is injective. In this case, there is a standard method to imbed (A, α) into an automorphic system.

Let (A, α) be an injective dynamical system. Define the direct limit dynamical system $(A_{\infty}, \alpha_{\infty})$ by

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \cdots \longrightarrow A_{\infty} \\ \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha_{\infty} \\ A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \cdots \longrightarrow A_{\infty} \end{array}$$

Then A is contained injectively in A_{∞} , say via a map ι . The limit map α_{∞} is an automorphism.

Let the unitary generator of the crossed product $A_{\infty} \rtimes_{\alpha_{\infty}} \mathbb{Z}$ be \tilde{v} . It is shown in [11, 10] that the semicrossed product $A \rtimes_{\alpha} \mathbb{Z}_+$ is unitaly completely isometrically isomorphic to the subalgebra generated by $\iota(A)$ and \tilde{v} .

Note that $ad_{\tilde{\mathbf{v}}}^*$ defines an injective endomorphism of $A \times_{\alpha} \mathbb{Z}_+$. Therefore we can construct the direct sequence

$$A \times_{\alpha} \mathbb{Z}_+ \xrightarrow{ad_{\tilde{\mathbf{v}}}^*} A \times_{\alpha} \mathbb{Z}_+ \xrightarrow{ad_{\tilde{\mathbf{v}}}^*} A \times_{\alpha} \mathbb{Z}_+ \xrightarrow{ad_{\tilde{\mathbf{v}}}^*} \dots$$

Observe that $ad_{\tilde{\mathbf{v}}}^*$ induces (concretely) the injective endomorphism α . Also, $ad_{\tilde{\mathbf{v}}}^*(\tilde{\mathbf{v}}) = \tilde{\mathbf{v}}$. Hence it follows that the direct limit of the above sequence is $A_{\infty} \times_{\alpha_{\infty}} \mathbb{Z}_+$. Let \tilde{i} denote the injection of $A \times_{\alpha} \mathbb{Z}_+$ into $A_{\infty} \times_{\alpha_{\infty}} \mathbb{Z}_+$ extending ι . Observe that $\tilde{i}(\mathbf{v}) = \tilde{\mathbf{v}}$.

Now assume that φ is an isometric isomorphism of $A \times_{\alpha} \mathbb{Z}_+$ onto $B \times_{\beta} \mathbb{Z}_+$. Then β is also injective by Corollary 3.3. So we can perform the same direct limit construction for (B, β) to obtain an automorphic limit system $(B_{\infty}, \beta_{\infty})$. We denote the unitary generator of the crossed product $B_{\infty} \rtimes_{\beta_{\infty}} \mathbb{Z}$ by $\tilde{\mathbf{w}}$. Let j be the injection of B into B_{∞} ; and let \tilde{j} denote the injection of $B \times_{\beta} \mathbb{Z}_+$ into $B_{\infty} \times_{\beta_{\infty}} \mathbb{Z}_+$.

Construct the following diagram

$$\begin{array}{ccccccc} A \times_{\alpha} \mathbb{Z}_+ & \xrightarrow{ad_{\tilde{\mathbf{v}}}^*} & A \times_{\alpha} \mathbb{Z}_+ & \xrightarrow{ad_{\tilde{\mathbf{v}}}^*} & A \times_{\alpha} \mathbb{Z}_+ & \xrightarrow{ad_{\tilde{\mathbf{v}}}^*} & \dots \longrightarrow A_{\infty} \times_{\alpha_{\infty}} \mathbb{Z}_+ \\ \downarrow \varphi_0 & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_{\infty} \\ B \times_{\beta} \mathbb{Z}_+ & \xrightarrow{ad_{\tilde{\mathbf{w}}}^*} & B \times_{\beta} \mathbb{Z}_+ & \xrightarrow{ad_{\tilde{\mathbf{w}}}^*} & B \times_{\beta} \mathbb{Z}_+ & \xrightarrow{ad_{\tilde{\mathbf{w}}}^*} & \dots \longrightarrow B_{\infty} \times_{\beta_{\infty}} \mathbb{Z}_+ \end{array}$$

where $\varphi_n := ad_{\tilde{\mathbf{w}}}^* \circ \varphi_{n-1} \circ ad_{\tilde{\mathbf{v}}}$ for $n \in \mathbb{Z}_+$. The diagram clearly commutes. Therefore the limit map φ_{∞} is a completely isometric isomorphism of $A_{\infty} \times_{\alpha_{\infty}} \mathbb{Z}_+$ onto $B_{\infty} \times_{\beta_{\infty}} \mathbb{Z}_+$.

Moreover, one can see that the isomorphism γ_{∞} of A_{∞} onto B_{∞} extends γ . Also the generator $\tilde{\mathbf{v}} = \tilde{i}(\mathbf{v})$ and $\tilde{j}(\mathbf{w}) = \tilde{\mathbf{w}}$. Therefore

$$\varphi_{\infty}(\tilde{\mathbf{v}}) = \tilde{j}(\varphi(\mathbf{w})) = j(b_0) + \tilde{\mathbf{w}}j(b_1) + \tilde{\mathbf{w}}^2j(Y).$$

This yields a proof of our theorem in the injective case.

Theorem 4.4. *Let (A, α) and (B, β) be unital C^* -algebra dynamical systems. Suppose that $A \times_{\alpha} \mathbb{Z}_+$ and $B \times_{\beta} \mathbb{Z}_+$ are isometrically isomorphic. If α is injective, then (A, α) and (B, β) are outer conjugate.*

Proof. The preceding argument shows that the two automorphic systems $(A_{\infty}, \alpha_{\infty})$ and $(B_{\infty}, \beta_{\infty})$ have isometrically isomorphic semicrossed products. Therefore by Corollary 4.2, these two dynamical systems are outer conjugate. Indeed, the proof shows more precisely that

$$\beta_{\infty}(b) = (ad_w \gamma_{\infty} \alpha_{\infty} \gamma_{\infty}^{-1})(b) \quad \text{for all } b \in B_{\infty}$$

where w is the unitary in the polar decomposition of $j(b_1)$. In particular, $j(b_1)$ is invertible in B_{∞} , and hence is invertible in $j(B)$. Therefore w belongs to $j(B)$. Restricting to b in B yields the desired outer conjugacy of (A, α) and (B, β) . \blacksquare

5. MAIN THEOREM 1.2

5.1. Automatic Outer Conjugacy. In this subsection we show how outer conjugacy is obtained in a number of cases, other than those described in Section 4, because of a structure of the C^* -algebras.

First of all we can obtain the following strengthening of the main result of [4], which required α and β to be automorphisms.

Corollary 5.1. *Let A be a simple C^* -algebra; and let (A, α) and (B, β) be unital C^* -algebra dynamical systems. Suppose that $A \times_\alpha \mathbb{Z}_+$ and $B \times_\beta \mathbb{Z}_+$ are isometrically isomorphic. Then (A, α) and (B, β) are outer conjugate.*

Proof. Since A is simple, α is injective and Theorem 4.4 applies. ■

We can improve this to the case of trivial centre.

Corollary 5.2. *If the centre of A is trivial ($Z(A) = \mathbb{C}1$), then the two systems are outer conjugate.*

Proof. By Lemma 3.1(ii), $b_0 b_0^* \in Z(B)$. If $b_0 = 0$, then b_1 is unitary by Lemma 3.5(i). If $b_0 \neq 0$, then $b_0 b_0^*$ is a non-zero scalar in $\text{ann}(R_\beta)$ by Lemma 3.9. Hence $R_\beta = \{0\}$. Therefore β is injective. So b_1 is invertible by Theorem 4.4. In either case, the two systems are outer conjugate by the Key Lemma 3.4. ■

When A is commutative, Lemma 3.2 shows that b_1 is right invertible and thus invertible. Thus Lemma 3.4 applies. A stronger result is known in the commutative case [3], namely: if two semicrossed products of commutative C^* -algebras are *algebraically isomorphic*, then the dynamical systems are conjugate. We record the isometric case as another corollary.

Proposition 5.3. *Let A be an abelian C^* -algebra; and let (A, α) and (B, β) be unital C^* -algebra dynamical systems. Suppose that $A \times_\alpha \mathbb{Z}_+$ and $B \times_\beta \mathbb{Z}_+$ are isometrically isomorphic. Then (A, α) and (B, β) are conjugate.*

Indeed, when A is any finite C^* -algebra (i.e. no proper isometries), right invertibility implies invertibility. So we have:

Proposition 5.4. *Let A be a finite unital C^* -algebra; and let (A, α) and (B, β) be unital C^* -algebra dynamical systems. Suppose that the semicrossed products $A \times_\alpha \mathbb{Z}_+$ and $B \times_\beta \mathbb{Z}_+$ are isometrically isomorphic. Then (A, α) and (B, β) are outer conjugate.*

Since it suffices that b_1 lie in a finite subalgebra of B to reach this conclusion, we obtain two results of this type.

Proposition 5.5. *Let (A, α) and (B, β) be unital C^* -algebra dynamical systems. Suppose that $A \times_\alpha \mathbb{Z}_+$ and $B \times_\beta \mathbb{Z}_+$ are isometrically isomorphic, and that $b_1 = E_1\varphi(\mathbf{v})$ belongs to the centre $Z(B)$. Then (A, α) and (B, β) are outer conjugate.*

Proposition 5.6. *Let (A, α) and (B, β) be unital C^* -algebra dynamical systems, and suppose that $\alpha(A)'$, the commutant of $\alpha(A)$ in A , is a finite C^* -algebra. Then if $A \times_\alpha \mathbb{Z}_+$ and $B \times_\beta \mathbb{Z}_+$ are isometrically isomorphic, it follows that (A, α) and (B, β) are outer conjugate.*

Proof. This result relies on Lemma 3.1(iii) that $a_1\gamma^{-1}(b_1) \in \alpha(A)'$. By Lemma 3.2, we know that $a_1\gamma^{-1}(b_1)$ is right invertible. By hypothesis, it lies in a finite algebra, and therefore it is invertible. So b_1 is invertible. Hence the two systems are outer conjugate by the Key Lemma 3.4. ■

An immediate consequence of Lemma 3.5 and the Key Lemma 3.4 is the following.

Proposition 5.7. *Let (A, α) and (B, β) be unital C^* -algebra dynamical systems. Suppose that $A \times_\alpha \mathbb{Z}_+$ and $B \times_\beta \mathbb{Z}_+$ are isometrically isomorphic. If $b_0 := E_0\varphi(\mathbf{v}) = 0$ or if $E_n\varphi(\mathbf{v}) = 0$ for all $n \geq 2$ (i.e. $Y = 0$), then (A, α) and (B, β) are outer conjugate.*

5.2. Analysis of the Kernels. We complete the proof of Theorem 1.2. First we need a general result about quotients of semicrossed products.

Lemma 5.8. *Let I be an ideal of A such that $\alpha(I) \subset I$, and let $\dot{\alpha}$ denote the induced endomorphism of A/I . Let I_α denote the ideal of $A \times_\alpha \mathbb{Z}_+$ generated by I , namely*

$$I_\alpha = \{X \in A \times_\alpha \mathbb{Z}_+ : E_n(X) \in I \text{ for all } n \geq 0\}.$$

Then $A/I \times_{\dot{\alpha}} \mathbb{Z}_+$ is completely isometrically isomorphic to $A \times_\alpha \mathbb{Z}_+ / I_\alpha$.

Proof. Let q be the quotient map of A onto A/I ; and let $\dot{\mathbf{v}}$ denote the generator of $A/I \times_{\dot{\alpha}} \mathbb{Z}_+$. Every covariant representation (π, V) of $(A/I, \dot{\alpha})$ gives rise to a covariant representation $(\pi q, V)$ of (A, α) . Therefore there is a canonical completely contractive map $\dot{q} = q \times \dot{\mathbf{v}}$ of $A \times_\alpha \mathbb{Z}_+$ into $A/I \times_{\dot{\alpha}} \mathbb{Z}_+$ such that $\dot{q}|_A = q$ and $\dot{q}(\mathbf{v}) = \dot{\mathbf{v}}$.

It is easy to verify that $\ker \dot{q} = I_\alpha$. Therefore there is an injective completely contractive map σ of $A \times_\alpha \mathbb{Z}_+ / I_\alpha$ into $A/R_\alpha \times_{\dot{\alpha}} \mathbb{Z}_+$ such that $\sigma|_{A/R_\alpha} = \text{id}$ and $\sigma(\tilde{\mathbf{v}}) = \dot{\mathbf{v}}$, where $\tilde{\mathbf{v}} = \mathbf{v} + I_\alpha$.

Conversely, suppose that $A \times_\alpha \mathbb{Z}_+ / I_\alpha$ is represented completely isometrically on a Hilbert space \mathcal{H} by $\hat{\pi}$. It is completely determined by

the restriction to A/I , which is a $*$ -isomorphism π , and the contraction $V = \hat{\pi}(\dot{\mathbf{v}})$. It is evident that (π, V) yields a covariant representation of $(A/I, \dot{\alpha})$. Therefore by the universal property, $\pi \times V$ is a completely contractive map of $A/I \times_{\dot{\alpha}} \mathbb{Z}_+$ into $\hat{\pi}(A \times_{\alpha} \mathbb{Z}_+/I_{\alpha})$. When composed with $\hat{\pi}^{-1}$, one obtains a completely contractive map τ of $A/I \times_{\dot{\alpha}} \mathbb{Z}_+$ into $A \times_{\alpha} \mathbb{Z}_+/I_{\alpha}$ which restricts to the identity on A/R_{α} and takes $\dot{\mathbf{v}}$ to $\tilde{\mathbf{v}}$. Clearly $\tau = \sigma^{-1}$. Therefore σ is a completely isometric isomorphism. \blacksquare

Now we identify how an isometric isomorphism passes to quotients.

Lemma 5.9. *Suppose that φ is a completely isometric isomorphism of $A \times_{\alpha} \mathbb{Z}_+$ onto $B \times_{\beta} \mathbb{Z}_+$. Let I be an ideal of A and $J = \gamma(I)$ an ideal in B such that $\alpha(I) \subset I$ and $\beta(J) \subset J$. Let $\dot{\alpha}$ denote the induced endomorphism of A/I , and let I_{α} be the ideal of $A \times_{\alpha} \mathbb{Z}_+$ generated by I . Similarly, define $\dot{\beta}$ and J_{β} . Then I_{α} and J_{β} are completely isometrically isomorphic via the restriction of φ , and φ induces a completely isometric map $\dot{\varphi}$ of $A/I \times_{\dot{\alpha}} \mathbb{Z}_+$ onto $B/J \times_{\dot{\beta}} \mathbb{Z}_+$.*

Proof. First observe that $\varphi(I_{\alpha}) = J_{\beta}$. Indeed, a dense subset of I_{α} is given by polynomials $p = \sum_{k=0}^n \mathbf{v}^k r_k$ where $r_k \in I$. So $\varphi(p) = \sum_{k=0}^n \varphi(\mathbf{v})^k \gamma(r_k)$ lies in J_{β} . Similarly, $\varphi^{-1}(J_{\beta}) \subset I_{\alpha}$. Therefore we get the desired equality.

Combining this with the previous lemma, we obtain the following diagram:

$$\begin{array}{ccc} A \times_{\alpha} \mathbb{Z}_+ & \xrightarrow{\varphi} & B \times_{\beta} \mathbb{Z}_+ \\ \downarrow \tilde{q} & & \downarrow \tilde{q} \\ A \times_{\alpha} \mathbb{Z}_+/I_{\alpha} & \xrightarrow{\tilde{\varphi}} & B \times_{\beta} \mathbb{Z}_+/J_{\beta} \\ \downarrow \simeq & & \downarrow \simeq \\ A/I \times_{\dot{\alpha}} \mathbb{Z}_+ & \xrightarrow{\dot{\varphi}} & B/J \times_{\dot{\beta}} \mathbb{Z}_+ \end{array}$$

Since the vertical maps labelled \tilde{q} are complete quotient maps, and $\varphi(I_{\alpha}) = J_{\beta}$, the map $\tilde{\varphi}$ is also completely isometric. Combining this with the complete isometries of Lemma 5.8 yields that $\dot{\varphi}$ is a complete isometry. \blacksquare

Define $\dot{\alpha} \in \text{End}(A/R_{\alpha})$ by $\dot{\alpha}(a + R_{\alpha}) = \alpha(a) + R_{\alpha}$. It is easy to see that the dynamical system $(A/R_{\alpha}, \dot{\alpha})$ is injective. Let \mathcal{R}_{α} denote the ideal of $A \times_{\alpha} \mathbb{Z}_+$ generated by R_{α} , i.e.

$$\mathcal{R}_{\alpha} = \{X \in A \times_{\alpha} \mathbb{Z}_+ : E_n(X) \in R_{\alpha} \text{ for all } n \geq 0\}.$$

Recall that $\varphi(\mathbf{v}) = b_0 + \mathbf{w}b_1 + \mathbf{w}^2Y$. Let q_∞ be the quotient map $q_\infty : B \rightarrow B/R_\beta$. The previous two lemmas combine to yield the first two parts of this proposition.

Proposition 5.10. *With the above notation,*

- (i) $A/R_\alpha \times_{\dot{\alpha}} \mathbb{Z}_+$ is completely isometrically isomorphic to $A \times_\alpha \mathbb{Z}_+ / \mathcal{R}_\alpha$.
- (ii) The isomorphism φ induces a completely isometric isomorphism $\dot{\varphi}$ of $A/R_\alpha \times_{\dot{\alpha}} \mathbb{Z}_+$ onto $B/R_\beta \times_{\dot{\beta}} \mathbb{Z}_+$.
- (iii) The systems $(A/R_\alpha, \dot{\alpha})$ and $(B/R_\beta, \dot{\beta})$ are outer conjugate. In particular, $q_\infty(b_1)$ is invertible in B/R_β .

Proof. The first statement is immediate from Lemma 5.8. The second statement follows from Corollary 3.8 and Lemma 5.9.

Observe that $\alpha^{-1}(R_\alpha) \subset R_\alpha$ because $x \in \alpha^{-1}(R_\alpha)$ implies that $\alpha(x) \in R_\alpha$. Hence $\lim_n \alpha^{n+1}(x) = 0$, showing that x is in R_α . Therefore the dynamical system $(A/R_\alpha, \dot{\alpha})$ is injective. It now follows from Theorem 4.4 that $(A/R_\alpha, \dot{\alpha})$ and $(B/R_\beta, \dot{\beta})$ are outer conjugate. Moreover the proof shows that the coefficient of $\dot{\mathbf{w}}$ in $\dot{\varphi}(\dot{\mathbf{v}})$, namely $q_\infty(b_1)$, is invertible. ■

Now using the fact that R_β is the union of an increasing sequence of ideals, we improve the previous result to a quotient by the kernel of a finite power of α . Let q_n be the quotient map $q_n : B \rightarrow B/\ker(\beta^n)$ for $n \geq 1$.

Corollary 5.11. *Suppose that φ is a completely isometric isomorphism of $A \times_\alpha \mathbb{Z}_+$ onto $B \times_\beta \mathbb{Z}_+$. Then there is an integer n_0 so that $q_{n_0}(b_i)$ is invertible in $B/\ker \beta^{n_0}$. Therefore $(A/\ker \alpha^{n_0}, \tilde{\alpha})$ and $(B/\ker \beta^{n_0}, \tilde{\beta})$ are outer conjugate.*

Proof. Since $q_\infty(b_1)$ is invertible and R_α is the closed union of $\ker(\alpha^n)$ for $n \geq 1$, it follows that there is an integer n_0 so that $b_1 + \ker(\beta^{n_0})$ is invertible in $B/\ker \beta^{n_0}$. Therefore $(A/\ker \alpha^{n_0}, \tilde{\alpha})$ and $(B/\ker \beta^{n_0}, \tilde{\beta})$ are outer conjugate by Lemma 5.9 and the Key Lemma 3.4. ■

For $k \geq 0$, let $A_k = A/\ker(\alpha^k)$ and $J_k = \ker(\alpha^{k+1})/\ker(\alpha^k)$. Choose a faithful non-degenerate representation π_k of J_k . This induces a representation of A_k on the same space, which we also call π_k . Also let σ_n denote a faithful non-degenerate representation of A_n . Then

$$\rho = \sum_{k=0}^{n_0-1} \oplus \pi_k q_k \oplus \sigma_{n_0} q_{n_0}$$

yields a representation of A . It follows from basic C*-algebra theory that ρ is faithful.

Lemma 5.12. *Consider the following properties:*

- (1) $\alpha(R_\alpha) = R_\alpha$.
- (2) $\alpha(\ker(\alpha^{k+1})) = \ker(\alpha^k)$ for all $k \geq 0$.
- (3) $\alpha(\text{ann}(R_\alpha)) \subseteq \text{ann}(R_\alpha)$.
- (4) $\alpha^k(a_0) \in \text{ann}(R_\alpha)$ for all $k \geq 0$.
- (5) $\alpha(a_0) \in \text{ann}(R_\alpha)$.
- (6) $\alpha^k(a_0) \in \text{ann}(\ker(\alpha))$ for all $k \geq 0$.
- (7) $\alpha(a_0) \ker(\alpha^{k+1}) \subset \ker(\alpha^k)$ for all $k \geq 0$.
- (8) $\pi_k q_k(a_0) = \pi_k q_k \alpha(a_0) = 0$ for all $k \geq 0$.

Then

$$(1) \iff (2) \implies (3) \implies (4) \begin{matrix} \implies (5) \\ \implies (6) \end{matrix} \begin{matrix} \implies (7) \\ \implies (8) \end{matrix} \iff (8).$$

Proof. Clearly $\alpha(\ker(\alpha^{k+1}))$ is contained in $\ker(\alpha^k)$. If they are not equal, take any $x \in \ker(\alpha^k)$ which is not in $\alpha(\ker(\alpha^{k+1}))$. Then x is not in the range of α . So $\alpha(R_\alpha)$ does not contain x . Therefore (1) implies (2). Conversely, it is clear that if (2) holds, then $\alpha(R_\alpha)$ contains $\ker(\alpha^k)$ for all $k \geq 0$. Since the image is a C*-algebra, it is closed and thus is all of R_α . So (2) implies (1).

Clearly (1) implies (3) because then

$$\alpha(\text{ann}(R_\alpha)) \subseteq \text{ann}(\alpha(R_\alpha)) = \text{ann}(R_\alpha).$$

By Lemma 3.9, $a_0 \in \text{ann}(R_\alpha)$. Thus (3) implies that $\alpha^k(a_0)$ lies in $\text{ann}(\alpha^k(R_\alpha))$ for all $k \geq 0$. So (4) follows. Clearly (4) implies (5) and (6); and (5) trivially implies (7).

Assume (6), and suppose that $x \in \ker(\alpha^{k+1})$. Then $\alpha^k(x) \in \ker(\alpha)$. So

$$\alpha^k(\alpha(a_0)x) = \alpha^{k+1}(a_0)\alpha^k(x) = 0.$$

Therefore $\alpha(a_0)x \in \ker(\alpha^k)$; whence (6) implies (7).

Since $a_0 \in \text{ann}(R_\alpha)$, we always have $\pi_k q_k(a_0) = 0$. Assume that (7) holds. The representation π_k is non-degenerate on J_k . Thus $\pi_k(J_k) = \pi_k q_k(\ker(\alpha^{k+1}))$ has dense range in \mathcal{H}_{π_k} . By (7), $\pi_k q_k \alpha(a_0)x = 0$ for all $x \in \ker(\alpha^{k+1})$. So $\pi_k q_k \alpha(a_0) = 0$. Hence (7) implies (8).

Conversely, if (8) holds, then $\pi_k q_k(\alpha(a_0)x) = 0$ for all $x \in \ker(\alpha^{k+1})$. Since π_k is faithful on J_k , it follows that $q_k(\alpha(a_0)x) = 0$; namely, $\alpha(a_0)x$ lies in $\ker(\alpha^k)$. So (7) follows. ■

Lemma 5.13. *Assume that $\alpha(a_0) \ker(\alpha^{k+1}) \subset \ker(\alpha^k)$ for all $k \geq 0$. Then $\pi_k q_k(a_1)$ is invertible for all $k \geq 0$.*

Proof. By Lemma 5.12, we have that $\pi_k q_k(a_0) = \pi_k q_k \alpha(a_0) = 0$ for all $k \geq 0$. Lemma 3.5 shows that

$$1 = a_1 \gamma^{-1}(b_1) + \alpha(a_0) a_1 c_0 + a_1 a_0 c_0 + \alpha(a_0)^2 c_1.$$

Apply $\pi_k q_k$ to obtain

$$\pi_k q_k(1) = \pi_k q_k(a_1 \gamma^{-1}(b_1)).$$

Thus $\pi_k q_k \gamma^{-1}(b_1)$ is left invertible, and so invertible by Lemma 3.2. Now as in Lemma 3.5, it follows that $\pi_k q_k \gamma^{-1}(b_1)$ and $\pi_k q_k(a_1)$ are unitary. \blacksquare

We obtain the following partial result in the non-injective case.

Proposition 5.14. *Suppose that φ is an isometric isomorphism of the semicrossed product $A \times_\alpha \mathbb{Z}_+$ onto $B \times_\beta \mathbb{Z}_+$. If any of the conditions of Lemma 5.12 holds, particularly*

$$(1) \alpha(R_\alpha) = R_\alpha \quad \text{or} \quad (3) \alpha(\text{ann}(R_\alpha)) \subseteq \text{ann}(R_\alpha),$$

then (A, α) and (B, β) are outer conjugate.

Proof. By the previous two lemmas, we obtain that $\pi_k q_k(a_1)$ is invertible for all $k \geq 0$. By Corollary 5.11, we also see that there is an n_0 so that $\sigma_{n_0} q_{n_0}(a_1)$ is invertible. Therefore $\rho(a_1)$ is invertible. This is a faithful representation, so a_1 is invertible. The result follows from the Key Lemma 3.4. \blacksquare

As one last result, we make an improvement to Proposition 5.6. Let ρ be a faithful representation of R_∞ on \mathcal{H} . This induces a representation of A , which we also call ρ , which factors through

$$A \rightarrow A/\text{ann}(R_\infty) \rightarrow M(R_\infty),$$

where $M(R_\infty)$ is the multiplier algebra of R_∞ . Since ρ extends to a faithful representation of $M(R_\infty)$, we may consider $\rho(A)$ as a subalgebra of $M(R_\infty)$.

Proposition 5.15. *Suppose that φ is an isometric isomorphism of the semicrossed product $A \times_\alpha \mathbb{Z}_+$ onto $B \times_\beta \mathbb{Z}_+$. If $\rho(\alpha(A)'),$ the image of the commutant of $\alpha(A)$ in $M(R_\infty)$, is a finite C^* -algebra, then (A, α) and (B, β) are outer conjugate.*

Proof. The argument again is to show that a_1 is invertible. A faithful representation of A is obtained from $\pi q_\infty \oplus \rho$, where π is a faithful representation of A/R_∞ . By Proposition 5.10, $\pi q_\infty(a_1)$ is invertible. Now $\rho(a_1 \gamma^{-1}(b_1))$ belongs to $\rho(\alpha(A)'),$ by Lemma 3.1(iii). This element is right invertible by Lemma 3.2. By hypothesis, $\rho(\alpha(A)'),$ is finite. Therefore $\rho(a_1 \gamma^{-1}(b_1))$ is invertible. Hence $\rho(\gamma^{-1}(b_1))$ is invertible, and

so $\rho(a_1)$ is also invertible. Combining these two facts, we obtain that a_1 is invertible. Thus the Key Lemma 3.4 applies to show that (A, α) and (B, β) are outer conjugate. ■

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