$\mathcal{B}(\mathcal{H})$ IS A FREE SEMIGROUP ALGEBRA

KENNETH R. DAVIDSON

Abstract. We provide a simplified version of a construction of Charles Read. For any $n \geq 2$, there are n isometries with orthogonal ranges with the property that the nonself-adjoint weak-$*$-closed algebra that they generate is all of $\mathcal{B}(\mathcal{H})$.

A free semigroup algebra is the weak operator topology closed algebra generated by an n-tuple of isometries with pairwise orthogonal ranges. The C*-algebra generated by these isometries is called the Cuntz algebra $\mathcal{O}_n$, introduced in [**4**]. Cuntz algebras arise frequently in the theory of C*-algebras as fundamental building blocks. They also play a crucial role in the analysis of endomorphisms of $\mathcal{B}(\mathcal{H})$ beginning with the seminal work of Powers [**20**]. Free semigroup algebras have been studied extensively in recent years [**1, 5, 6, 7, 8, 9, 10, 11, 14, 16, 17, 18, 19**]. In part, this is because of their intrinsically interesting properties as nonself-adjoint algebras. However it is also the case that these algebras reveal invariants of the corresponding C*-algebra representations. As such, they have allowed the classification of certain families of representations of the Cuntz algebra [**9, 7**] which are relevant to recent work of Bratteli and Jorgensen, who use the Cuntz algebras to generate wavelets [**2, 3, 12, 13**].

In [**9**], we first raised the question of whether a free semigroup algebra could be a von Neumann algebra. In [**6, 8**], the special structure theory of these algebras has been revealed to a significant degree, and this question was seen as a central unresolved issue. Recently, Charles Read [**21**] constructed a representation of $\mathcal{O}_2$ for which the free semigroup is all of $\mathcal{B}(\mathcal{H})$. The purpose of this note is to provide a more transparent view of his construction.

1. The Construction

For convenience, we will identify points in the unit interval I by their diadic expansion $x = 0.x_1x_2x_3\dots$ where $x_i \in \{0,1\}$. Let $\mathbb{F}_2^+$ denote the free semigroup on two letters, which we will think of as

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the set of all words in symbols $\{0,1\}$, including the empty word $\emptyset$, with multiplication being concatenation of words. Let $|w|$ denote the length of the word w. For each $w = i_1 \dots i_k$ in $\mathbb{F}_2^+$, define a map τ_w of I into itself by $\tau_w(x) = 0.i_1 \dots i_k x_1 x_2 x_3 \dots$ which is a homeomorphism of $[0,1]$ onto the diadic interval I_w determined by w of length $2^{-|w|}$. Also define the function σ of I into itself by $\sigma(x) = 0.x_2 x_3 x_4 \dots$. Observe that if $k = |w|$, then $\sigma^k \tau_w$ is the identity map, and $\tau_w \sigma^k$ is the identity restricted to I_w.

Let $\{h_i : i \geq 1\}$ denote the Walsh basis for $L^2(0,1)$, which is obtained as products of the Rademacher functions. See Paley [**15**] or Zygmund [**22**, vol.1, p.34, Ex.6]. This is an orthonormal basis with the properties that each h_i takes only the values ± 1, $h_1 = 1$, and $\operatorname{span}\{h_i : i \leq 2^k\}$ is the set of functions which are constant on each diadic interval of length 2^{-k}. Thus for each $i \leq 2^k$, there are constants $\mu_{i,w} \in \{\pm 1\}$ for words w with $|w| = k$ so that $h_i = \sum_{|w|=k} \mu_{i,w} \chi_{I_w}$.

For $i \geq 1$, let $b_i = 2^i$ and let $w_i = 011\dots110$ denote the word of length b_i beginning and ending with a 0 and with $b_i - 2$ ones in between. Observe that the intervals I_{w_i} are pairwise disjoint. Define a function on I by

$$g(x) = \begin{cases} h_i(\sigma^{b_i} x) & \text{if } x \in I_{w_i} \\ 1 & \text{otherwise} \end{cases}$$

Note that $|g| = 1$ on I; and $g = 1$ on I_1. In particular, multiplication by $g\tau_0$ is unitary on $L^2(I)$; and therefore we can define a new orthonormal basis for $L^2(I)$ by

$$g_i(x) := g(\tau_0 x) h_i(x) \quad \text{for} \quad i \geq 1.$$

Define two isometries S_0 and S_1 by

$$S_i f(x) = \begin{cases} \sqrt{2} g(x) f(\sigma x) & \text{if } x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

Observe that the range of S_i is $L^2(I_i)$ for $i = 0, 1$, and that these are complementary orthogonal subspaces. Since $g = 1$ on I_1,

$$S_1 f(x) = \sqrt{2}\ \chi_{I_1}(x) f(\sigma x).$$

For each $w = i_1 \dots i_k$ in $\mathbb{F}_2^+$, write S_w for the product $S_{i_1} \dots S_{i_k}$. Observe that the isometry S_w carries $L^2(I)$ onto $L^2(I_w)$. In fact, this is the 'standard' isometry carrying $L^2(I)$ onto $L^2(I_w)$ multiplied by a certain function of modulus one.

We record a couple of useful identities as a lemma:

Lemma 1.1. *For each word* $w \in \mathbb{F}_2^+$,

$$S_w f(x) = 2^{|w|/2} G_{|w|}(x) \chi_{I_w}(x) f(\sigma^{|w|}x) \tag{1.2}$$

where $G_k(x) = \prod_{0 \le j < k} g(\sigma^j x)$ *is a function of modulus one. And*

$$S_{w_j} g_i(x) = 2^{b_j/2} \chi_{I_{w_j}}(x)(h_j h_i)(\sigma^{b_j} x). \tag{1.3}$$

In particular, $S_{w_j} g_j = 2^{b_j/2} \chi_{I_{w_j}}$*; and* $S_{w_j} g_i$ *is a function supported on* I_{w_j} *which has constant modulus and mean* 0 *when* $i \ne j$.

Proof. The computation of $S_w f$ is routine. Note that

$$S_0 g_i(x) = \sqrt{2}\, \chi_{I_0}(x) g(\tau_0 x)^2 h_i(\sigma x) = \sqrt{2}\, \chi_{I_0}(x) h_i(\sigma x).$$

Hence if we let v_j be the word w_j with the initial 0 truncated,

$$S_1^{b_j-2} S_0 g_i = 2^{(b_j-1)/2} \chi_{I_{v_j}} h_i(\sigma^{b_j-1} x).$$

Therefore

$$\begin{aligned} S_{w_j} g_i(x) &= 2^{(b_j-1)/2} S_0 \chi_{I_{v_j}} h_i(\sigma^{b_j-1} x) \\ &= 2^{b_j/2} \chi_{I_{w_j}}(x)(h_j h_i)(\sigma^{b_j} x). \end{aligned}$$

The properties of $S_{w_j} g_i$ are immediate. ∎

Definition 1.4. Let $\mathcal{G}_{i,k} = \{w : |w| = b_k - b_i$ and $G_{b_k-b_i} \tau_{ww_i}$ is constant$\}$. In this case, let $\lambda_{i,k,w}$ denote the value of $G_{b_k-b_i}$ on the interval I_{ww_i}.

Lemma 1.5. *If* $k > i$, *then*

$$1 - 2^{-6} < 2^{b_i - b_k} |\mathcal{G}_{i,k}| \le 1.$$

Moreover, membership in $\mathcal{G}_{i,k}$ *is not dependent on the first* $b_k/4 = 2^{k-2}$ *coefficients of* w.

Proof. $G_{b_k-b_i} \tau_{ww_i}(x) = \prod_{0 \le j < b_k - b_i} g(\sigma^j \tau_{ww_i} x)$ is a product of ± 1's. The only time there is the possibility that $g(\sigma^j \tau_{ww_i} x)$ is not constant is if $\sigma^j \tau_{ww_i} x$ begins with a w_s. Since w_i begins with a 0 followed by $b_i - 2$ ones, such a word is contained entirely within w or ends with the initial 0 of w_i. In particular $s < k$; and $j \le b_k - b_i - b_s + 1$. Moreover, the value of g will then be determined by the next $\lceil \log_2 s \rceil$ coefficients. This extends beyond the end of ww_i only if $j + b_s + \lceil \log_2 s \rceil > b_k$; whence

$$b_k - b_s - \lceil \log_2 s \rceil < j \le b_k - b_i - b_s + 1.$$

This allows $\lceil \log_2 s \rceil + 1 - b_i$ possibilities for j when this number is positive—namely for $2^{b_i - 1} < s < k$.

The probability that $\sigma^j \tau_{ww_i} x$ begins with w_s is 2^{-b_s} except in the case $j = b_k - b_i - b_s + 1$ when it is 2^{1-b_s} because the initial 0 of w_i is fixed. Thus the proportion of omitted words is at most

$$\begin{aligned}\sum_{2^{b_i - 1} < s \le k-1} \big(\lceil \log_2 s \rceil + 2 - b_i\big) 2^{-b_s} &\le \sum_{s \ge 3} \lceil \log_2 s \rceil 2^{-b_s} \\ &= 2(2^{-8} + 2^{-16}) + 3(2^{-32} + \dots + 2^{-256}) + 4(2^{-512} + \dots + 2^{-65536}) + \dots \\ &< 2^{-6}\end{aligned}$$

The smallest value of j which can achieve one of these inequalities is

$$b_k - b_{k-1} - \lceil \log_2 (k-1) \rceil + 1 > 2^{k-2} = b_k/4.$$

So the first 2^{k-2} letters of w are irrelevant to determining membership in $\mathcal{G}_{i,k}$. ∎

In view of this lemma, we may define $\mathcal{G}'_{i,k} = \sigma^{b_k/4} \mathcal{G}_{i,k}$ to be the set of tails of $\mathcal{G}_{i,k}$. Note that $|\mathcal{G}'_{i,k}| = 2^{-b_k/4} |\mathcal{G}_{i,k}|$ and

$$\mathcal{G}_{i,k} = \{ v w' : |v| = b_k/4 \text{ and } w' \in \mathcal{G}'_{i,k} \}.$$

Lemma 1.6. *Let $i, j \in \mathbb{N}$. For each $k > \max\{i, j\}$, define*

$$T_{i,j,k} = |\mathcal{G}_{i,k}|^{-1/2} \sum_{w \in \mathcal{G}_{i,k}} \overline{\lambda}_{i,k,w} \mu_{j,w} S_{ww_i}.$$

Then $T_{i,j,k}$ is an isometry with the property that

$$\langle T_{i,j,k}\, g_{i'}, h_{j'} \rangle = \delta_{ii'} \delta_{jj'} 2^{-b_i/2} \big(2^{b_i - b_k} |\mathcal{G}_{i,k}| \big)^{1/2}$$

for all $i', j' \in \mathbb{N}$ with $j' \le 2^{k-2}$.

Proof. By equation (1.3), we see that

$$S_{w_i} g_{i'} = 2^{b_i/2} \chi_{I_{w_i}}(x) (h_i h_{i'})(\sigma^{b_i} x).$$

Hence by equation (1.2) for $w \in \mathcal{G}_{i,k}$,

$$\overline{\lambda}_{i,k,w} \mu_{j,w} S_{ww_i} g_{i'}(x) = 2^{b_k/2} \mu_{j,w} \chi_{I_{ww_j}}(x) (h_i h_{i'})(\sigma^{b_k} x),$$

which is supported on $I_{ww_j} \subset I_w$. If $i' \ne i$, then since this function has mean zero for each $w \in \mathcal{G}_{i,k}$ and $h_{j'}$ is constant on the interval I_w, the inner product is 0.

So suppose that $i' = i$. Observe that $\mu_{j',vw'} = \mu_{j',v}$ depends on at most the first $2^{k-2} = b_k/4$ terms v of $w = vw'$. From the orthogonality of h_j and $h_{j'}$, we obtain $\sum_{|v|=b_k/4} \mu_{j,v}\mu_{j',v} = \delta_{jj'} 2^{b_k/4}$. Hence

$$\begin{aligned}\langle T_{i,j,k}\ g_i, h_{j'}\rangle &= |\mathcal{G}_{i,k}|^{-1/2} \sum_{w\in\mathcal{G}_{i,k}} 2^{b_k/2}\mu_{j,w}\mu_{j',w}\langle \chi_{I_{ww_j}}, \chi_{I_w}\rangle \\ &= |\mathcal{G}_{i,k}|^{-1/2} \sum_{|v|=b_k/4} \mu_{j,v}\mu_{j',v} \sum_{w'\in\mathcal{G}'_{i,k}} 2^{b_k/2}\langle \chi_{I_{vw'w_j}}, \chi_{I_{vw'}}\rangle \\ &= |\mathcal{G}_{i,k}|^{-1/2}\delta_{jj'}2^{b_k/4}|\mathcal{G}'_{i,k}|2^{-b_k} \\ &= \delta_{jj'}2^{-b_i/2}\big(2^{b_i-b_k}|\mathcal{G}_{i,k}|\big)^{1/2}\end{aligned}$$ ∎

Theorem 1.7. *For each $s \geq 1$, the weak-$*$ closed algebra generated by $\{S_w : |w| = 2^s\}$ is all of $\mathcal{B}(\mathcal{H})$.*

Proof. The previous lemma exhibits a bounded sequence of operators $T_{i,j,k}$ built from words S_{ww_i} of length 2^k which converge in the weak operator topology to a positive multiple of the rank one operator $h_j g_i^*$. Hence the limit also exists in the weak-$*$ topology. For $k \geq s$, these operators lie in the desired algebra. These limit operators span the set of compact operators, which is weak-$*$ dense in $\mathcal{B}(\mathcal{H})$. ∎

Corollary 1.8. *For every $n \geq 2$, $\mathcal{B}(\mathcal{H})$ is a free semigroup algebra for a representation of $\mathcal{O}_n$.*

Proof. Between the algebras $\mathcal{O}_{2^k}$ and $\mathcal{O}_{2^{k+1}}$ generated by words S_w where w has length 2^k and 2^{k+1} respectively, there is a copy of $\mathcal{O}_n$ for $2^k < n < 2^{k+1}$. For example, $\mathcal{O}_3$ is generated by S_{00}, S_{01} and S_1. Use the representation constructed above. ∎

Remark 1.9. Since Lemma 1.6 is valid for all i', it follows that

$$T^*_{i,j,k}h_{j'} = \delta_{jj'}\lambda_{i.k}g_i$$

where $\lambda_{i,k} = 2^{-b_i/2}\big(2^{b_i-b_k}|\mathcal{G}_{i,k}|\big)^{1/2}$. So these adjoints actually converge in the strong operator topology.

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Pure Math. Dept., U. Waterloo, Waterloo, ON N2L–3G1, CANADA
E-mail address: krdavids@uwaterloo.ca