INVARIANT SUBSPACES AND HYPER-REFLEXIVITY FOR FREE SEMIGROUP ALGEBRAS

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In this paper, we obtain a complete description of the invariant subspace structure of an interesting new class of algebras which we call free semigroup algebras. This enables us to prove that they are reflexive, and moreover to obtain a quantitative measure of the distance to these algebras in terms of the invariant subspaces. Such algebras are called hyper-reflexive. This property is very strong, but it has been established in only a very few cases. Moreover the prototypes of this class of algebras are the natural candidate for a non-commutative analytic Toeplitz algebra on n variables. The case we make for this analogy is very compelling. In particular, in this paper, the key to the invariant subspace analysis is a good analogue of the Beurling theorem for invariant subspaces of the unilateral shift. This leads to a notion of inner–outer factorization in these algebras. In a sequel to this paper [13], we add to this evidence by showing that there is a natural homomorphism of the automorphism group onto the group of conformal automorphisms of the ball in $\mathbb{C}^n$.

A *free semigroup algebra* is the weak operator topology closed algebra generated by a set $S_1, \ldots, S_n$ of isometries with pairwise orthogonal ranges. These conditions are described algebraically by

(F) $$ S_i^* S_j = \delta_{ij} I \quad \text{for} \quad 1 \le i, j \le n; $$

or equivalently by

(F′) $$ S_i^* S_i = I \quad \text{for} \quad 1 \le i \le n \qquad \text{and} \qquad \sum_{i=1}^{n} S_i S_i^* \le I. $$

Let $\mathcal{F}_n$ denote the unital free semigroup generated by $z_1, \ldots, z_n$. This semigroup consists of all non-commuting words w in the generators. We are considering those representations of $\mathcal{F}_n$ as isometries on Hilbert space determined by such an n-tuple of isometries; sending w to $w(S_1, \ldots, S_n)$. We allow n to be any positive integer or ∞, although for notational convenience we will act as if n is finite. When there is any distinction for the infinite case, it will be noted.

Ordered tuples of isometries satisfying (F) have been studied in various contexts. In C*-algebra theory, they arise as the generators of two important

algebras [**9**], the Cuntz algebras $\mathcal{O}_n$ when $\sum_i S_i S_i^* = I$ and the Cuntz–Toeplitz extension $\mathcal{E}_n$ of $\mathcal{O}_n$ by the compact operators when this sum is strictly less than the identity. These algebras have played an important role in the modern C* theory. The left regular representation of $\mathcal{F}_n$ acts on the full Fock space generated by $\mathbb{C}^n$. This representation occurs in certain C*-algebraic formulations of quantum mechanics (c.f. [**26**, X.7] and [**6**, 5.2]).

Recent work of Bratteli and Jorgensen [**5**] decompose certain permutation representations of the Cuntz algebra into irreducible representations in order to study wavelets. It turns out that they consider a class of representations which are a subset of those which we call *atomic*. We will obtain a complete classification of these representations. This will enable us to completely describe the invariant subspace structure of the atomic free semi-group algebras.

These algebras also arise in dilation theory. Frahzo [**14**, **15**] and Bunce [**7**] show that any n-tuple $T = (T_1, \ldots, T_n)$ of operators in $\mathcal{B}(\mathcal{H})$ satisfying $\sum_i T_i T_i^* \leq I$ may be dilated to an n-tuple of isometries satisfying (F'). This circle of ideas has been elaborated on in a series of papers by Popescu [**19**, **20**, **22**, **24**, **25**] generalizing many important results from the Sz. Nagy–Foiaş theory [**29**] to the multi-variable context.

Of particular interest is the left regular free semigroup algebra $\mathfrak{L}_n$, determined by the left regular representation of $\mathcal{F}_n$ which acts on $\ell_2(\mathcal{F}_n)$ by $\lambda(w)\xi_v = \xi_{wv}$ for v, w in $\mathcal{F}_n$. The algebra $\mathfrak{L}_n$ is generated by the isometries $L_i = \lambda(z_i)$ for $1 \leq i \leq n$. For $n = 1$, we obtain the algebra generated by the unilateral shift, the analytic Toeplitz algebra. For $n \geq 2$, we obtain a natural non-commutative analogue of the Toeplitz algebra. In particular, there is a reasonable analogue of Beurling's Theorem and inner–outer factorization. There is a plethora of point evaluations for the complex n-ball from eigenvalues of the adjoint. However, because of the non-commutative nature, these vectors do not span the whole space; and thus do not provide a complete picture. Nevertheless, the complex ball provides an important connection between this algebra and complex function theory.

Our study begins with a careful look at the left regular free semigroup algebra. We first show that $\mathfrak{L}_n$ is the commutant of $\mathfrak{R}_n$, the right regular representation algebra. This leads to information about the spectrum of operators of $\mathfrak{L}_n$. In particular, every non-scalar element has connected spectrum containing more than one point. These operators are always injective, and thus have no eigenvectors. This in turn leads to the fact that $\mathfrak{L}_n$ is inverse closed and semisimple.

We establish a detailed structure of the invariant subspace lattice of $\mathfrak{L}_n$ that parallels Beurling's Theorem [**4**] for the case $n = 1$. Every invariant subspace is determined by a wandering space, and can be written as a direct sum of cyclic invariant subspaces. Moreover, akin to the characterization of invariant subspaces of the shift by inner functions, every cyclic invariant subspace is the range of an isometry in the commutant $\mathfrak{R}_n$. This leads to the inner–outer factorization—every element A in $\mathfrak{L}_n$ factors as $A = LB$ in $\mathfrak{L}_n$,

where L is an isometry and B has dense range. Certain invariant subspaces correspond to point evaluations in the complex n-ball $\mathbb{B}_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$. Indeed, for each point λ in $\mathbb{B}_n$, there is a vector ν_λ which is an eigenvector for the adjoint which determines a WOT–continuous multiplicative linear function $\varphi_\lambda(A) = (A\nu_\lambda, \nu_\lambda)$. These functionals play a pervasive role in our analysis. However, unlike the $n = 1$ case, these vectors do not generate the whole lattice because they are symmetric (depending only on the evaluation of words on *commuting* variables).

An operator algebra $\mathfrak{A}$ is *reflexive* if the algebra can be recovered from its invariant subspace lattice $\mathcal{L} = \text{Lat}(\mathfrak{A})$ as the set $\text{Alg}(\mathcal{L})$ of all operators leaving each subspace invariant. The lattice $\mathcal{L}$ determines a seminorm on $\mathcal{B}(\mathcal{H})$ by

$$\beta_\mathcal{L}(T) := \sup_{L \in \mathcal{L}} \|L^\perp T L\|.$$

Clearly, $\beta_\mathcal{L}(T) = 0$ precisely when T belongs to $\text{Alg}(\mathcal{L})$. Moreover, it is elementary to show that

$$\beta_\mathcal{L}(T) \leq \text{dist}(T, \text{Alg}(\mathcal{L})) \quad \text{for all} \quad T \in \mathcal{B}(\mathcal{H}).$$

The algebra is said to be *hyper-reflexive* if these norms are comparable. In this case, the constant of hyper-reflexivity is the smallest number C such that

$$\text{dist}(T, \text{Alg}(\mathcal{L})) \leq C\beta_\mathcal{L}(T) \quad \text{for all} \quad T \in \mathcal{B}(\mathcal{H}).$$

The list of algebras known to be hyper-reflexive is rather short. It includes nest algebras [3] which have constant 1 and injective von Neumann algebras [8] which have constant at most 4 (von Neumann algebras with abelian commutant have constant at most 2, as do abelian von Neumann algebras [27]). And the case most closely related to our study is the analytic Toeplitz algebra $\mathcal{T}(H^\infty) = \mathfrak{L}_1$ which has distance constant at most 19 [11].

We will show that $\mathfrak{L}_n$ is hyper-reflexive with distance constant at most 51 for $n \geq 2$. We also show that this algebra has property $\mathbb{A}_1$, meaning that every weak-$*$ continuous linear functional φ on $\mathfrak{L}_n$ can be represented by a rank one functional $\varphi(A) = (A\xi, \zeta)$ such that $\|\xi\| \|\zeta\| \leq (1 + \varepsilon)\|\varphi\|$ for any positive ε. This leads to the conclusion that every unital WOT–closed subalgebra of $\mathfrak{L}_n$ is also hyper-reflexive. Since $\mathfrak{L}_n$ is its own double commutant, there is another estimate of the distance to $\mathfrak{L}_n$ obtained from the norm of the derivation δ_T restricted to the commutant $\mathfrak{L}_n' = \mathfrak{R}_n$. This seminorm is also shown to be equivalent to the norm distance.

In the last section, we turn our attention to more general free semigroup algebras. Say that a free semigroup algebra is *atomic* if there is an atomic masa containing the ranges of the isometries $w(S_1, \ldots, S_n)$. Equivalently, this means that there is an orthonormal basis $\{\xi_n\}$ for the Hilbert space $\mathcal{H}$, endomorphisms π_i of $\mathbb{N}$ and scalars $\lambda_{i,n}$ in $\mathbb{T}$ such that $S_i\xi_n = \lambda_{i,n}\xi_{\pi_i(n)}$. These yield an interesting and tractable class of representations of the Cuntz algebra $\mathcal{O}_n$ subsuming the class of permutation representations studied in [5]. As $\mathcal{O}_n$ is a simple C*-algebra which is not type I, its representations

cannot be classified up to unitary equivalence. So it is interesting that this class can be completely classified.

Surprisingly, some of the WOT-closed free semigroup algebras obtained from these representations contain proper projections, which is not the case for $\mathfrak{L}_n$. The invariant subspaces can again be classified; and these algebras are shown to be hyper-reflexive.

The key to the analysis is the fact that a basis vector ξ_n is either a wandering vector which sweeps out a copy of the left regular representation or it is fixed by some word $w(S_1, \ldots, S_n)$. This word determines an irreducible representation of a special form that has a *ring* of vectors permuted by those S_i occurring, in order, in the word w with $n-1$ copies of the left regular representation coming off each node of the ring. The projection onto this ring lies in the algebra, and every vector in its range is cyclic. The general atomic representation is shown to be a direct sum of representations of three special types—the left regular representation, some doubly infinite variants of the left regular representation, and these ring representations.

As we were in the process of preparing our results for publication, the paper of Arias and Popescu [1] appeared. Their results overlap somewhat with ours, most notably in section 2 below, however our point of view is somewhat different. They have also pointed out that several of the results we obtained were not new, and appear in previous papers of Popescu, particularly [21, 23]. We wish to thank Arias and Popescu for their comments and we will note the overlaps below.

1. The Left Regular Free Semigroup Algebra

Form a Hilbert space $\mathcal{H}_n = \ell_2(\mathcal{F}_n)$ with orthonormal basis vectors ξ_w for each word w in the unital semigroup $\mathcal{F}_n$. Define operators $L_i = \lambda(z_i)$ by $L_i\xi_w = \xi_{z_iw}$. It is immediately evident that each L_i is an isometry, and that the ranges are pairwise orthogonal. Indeed,

$$I - \sum_{i=1}^n L_iL_i^* = \xi_1\xi_1^*$$

is a rank one projection. Let $\mathfrak{L}_n$ denote the unital WOT–closed algebra generated by $L_1, \ldots, L_n$. It is then evident that $v(L_1, \ldots, L_n)\xi_w = \xi_{vw}$ for all words v and w in $\mathcal{F}_n$. For convenience of notation, we will write $L = (L_1, \ldots, L_n)$ and L_v or $v(L)$ will denote the corresponding word in the n-tuple.

Similarly, the right regular representation is defined by $\rho(w) = R_{\widetilde{w}}$ where

$$R_v\xi_w = \xi_{wv} \quad \text{for} \quad v, w \in \mathcal{F}_n$$

and $\widetilde{w}$ denotes the word w in reverse order. Let $R_i := R_{z_i} = \rho(z_i)$ denote the images of the generators for $1 \le i \le n$; and let $\mathfrak{R}_n$ denote the WOT–closed algebra generated by the R_i's. Denote by W the unitary which sends ξ_w to $\xi_{\widetilde{w}}$. It is easily seen that $WL_iW^* = R_i$. Thus $\mathfrak{R}_n$ is unitarily equivalent to $\mathfrak{L}_n$, $\rho = \mathrm{Ad}(W)\lambda$, and $WL_wW^* = R_{\widetilde{w}}$. Notice that $\mathfrak{L}_n$ and $\mathfrak{R}_n$ commute

with each other. It follows that the range projection $P_v = R_v R_v^*$ of each right shift is invariant for $\mathfrak{L}_n$.

The full Fock space of a Hilbert space $\mathcal{H}$ is the Hilbert space

$$F(\mathcal{H}) = \sum_{k \geq 0} \oplus \mathcal{H}^{\otimes k}$$

where $\mathcal{H}^{\otimes 0} = \mathbb{C}$ and $\mathcal{H}^{\otimes k}$ is the tensor product of k copies of $\mathcal{H}$. When $\mathcal{H} = \mathbb{C}^n$ with orthonormal basis ζ_i for $1 \leq i \leq n$, the Fock space has an orthonormal basis $\zeta_w = \zeta_{i_1} \otimes \cdots \otimes \zeta_{i_k}$ for all choices of $w = (i_1, \ldots, i_k)$ in $\{1, \ldots, n\}^k$ and $k \geq 0$ (with the convention that $\zeta_\varnothing$ spans $\mathcal{H}^{\otimes 0}$). For each vector ζ in $\mathcal{H}$, there is a left creation operator $\ell(\zeta)\xi = \zeta \otimes \xi$. Clearly, there is a natural isomorphism of Fock space onto $\mathcal{H}_n$, where $n = \dim \mathcal{H}$, given by sending ζ_w to ξ_w. This unitary equivalence sends $\ell(\zeta_i)$ to L_i.

The following heuristic is useful when working with operators in $\mathfrak{L}_n$. If $A = \sum_w a_w L_w$ is a finite linear combination of the set $\{L_w : w \in \mathcal{F}_n\}$, then $A\xi_1 = \sum_w a_w \xi_w$; conversely, given a finite linear combination of basis vectors $\zeta = \sum_w a_w \xi_w$, the operator $A = \sum_w a_w L_w$ belongs to $\mathfrak{L}_n$ and satisfies $A\xi_1 = \zeta$. This correspondence of course cannot be extended to infinite combinations, however we note that for an arbitrary element A of $\mathfrak{L}_n$, A is completely determined by what it does on ξ_1: indeed, $A\xi_v = AR_v\xi_1 = R_v A\xi_1$. So if $A\xi_1 = \sum_w a_w \xi_w$, we have $A\xi_v = \sum a_w \ xi_{wv} = \sum_w a_w (L_w \xi_v)$. Thus it is sometimes useful to view A as the formal sum $\sum_w a_w L_w$ which serves as a Fourier expansion of A.

As in the case with classical harmonic analysis, we shall use Cesàro sums to enable us to make sense of such formal sums. Let Q_k denote the projection onto $\mathrm{span}\{\xi_w : |w| = k\}$. Let Φ_j denote the completely contractive projections on $\mathcal{B}(\mathcal{H})$ given by

$$\Phi_j(T) = \sum_{k \geq \max\{0,-j\}} Q_k T Q_{k+j}.$$

The proof of the following lemma follows from the standard estimates for Fejér's Theorem.

Lemma 1.1. *The Cesàro operators on $\mathcal{B}(\mathcal{H}_n)$ defined by*

$$\Sigma_k(T) = \sum_{|j|<k} \left(1 - \frac{|j|}{k}\right) \Phi_j(T) \quad for \quad k \geq 1$$

are completely contractive. Moreover, for each T in $\mathcal{B}(\mathcal{H})$, the sequence $\Sigma_k(T)$ converges to T in the strong operator topology.

Our first result is a direct analogue of the characterizations of the analytic Toeplitz algebra, the $n = 1$ case, where $\mathfrak{L}_1 = \mathfrak{R}_1 = \mathcal{T}(H^\infty)$.

We note that Theorem 1.2 and Corollary 1.3 also appear in [**23**], (see Theorem 1.2 and the material following it on page 35 of [**23**]). However, our proof is quite different.

Theorem 1.2. *For A in $\mathcal{B}(\mathcal{H}_n)$, the following are equivalent:*

> (i) A belongs to $\mathfrak{L}_n$
> (ii) A belongs to $\mathfrak{R}'_n$
> (iii) $A = R_v^* A R_v$ and P_v belongs to $\mathrm{Lat}(A)$ for all v in $\mathcal{F}_n$.

Proof. It is clear that (i) implies (ii), and (ii) implies (iii). Suppose that (iii) holds. Then

$$AR_v\xi_w = P_v AR_v\xi_w = R_v(R_v^* A R_v)\xi_w = R_v A\xi_w$$

for all v, w in $\mathcal{F}_n$. So (ii) holds.

To show that (ii) implies (i), fix A in $\mathfrak{R}'_n$ and let $A\xi_1 = \sum_w a_w\xi_w$. Recall that $|w|$ denotes the length of a word w. Consider the Cesàro sums

$$p_k(L) = \sum_{|w|<k} \left(1 - \tfrac{|w|}{k}\right) a_w L_w.$$

It is evident that $p_k(L)$ belong to $\mathfrak{L}_n$. We will show that $p_k(L)$ converges WOT to A.

Note that $\Phi_j(A)$ also lies in $\mathfrak{R}'_n$ because $R_i Q_k = Q_{k+1} R_i$; and thus

$$R_i \Phi_j(A) = \sum_k R_i Q_k A Q_{k+j} = \sum_k Q_{k+1} R_i A Q_{k+j}$$

$$= \sum_k Q_{k+1} A Q_{k+j+1} R_i = \Phi_j(A) R_i.$$

Hence $\Sigma_k(A)$ also lies in $\mathfrak{R}'_n$ and converges SOT to A. Notice that

$$\Sigma_k(A)\xi_1 = \sum_{|w|<k} \left(1 - \tfrac{|w|}{k}\right) a_w\xi_w = p_k(L)\xi_1.$$

By the remarks preceding Lemma 1.1, we conclude that $\Sigma_k(A) = p_k(L)$ belongs to $\mathfrak{L}_n$; and so A does also. ∎

By symmetry, it is also the case that $\mathfrak{R}_n = \mathfrak{L}'_n$. Thus we have:

Corollary 1.3. $\mathfrak{L}_n$ *is its own double commutant* $\mathfrak{L}_n = \mathfrak{L}''_n$.

Another automatic consequence of being a commutant is:

Corollary 1.4. $\mathfrak{L}_n$ *is inverse closed.*

We note that Corollary 1.4 also follows from Corollary 3.2 of [**23**].

Proof. This result holds for any algebra which is a commutant $\mathfrak{R}'$ of some algebra $\mathfrak{R}$. Suppose that A in $\mathfrak{R}'$ is invertible in $\mathcal{B}(\mathcal{H})$. Then for any R in $\mathfrak{R}$,

$$A^{-1}R = A^{-1}RAA^{-1} = A^{-1}ARA^{-1} = RA^{-1}.$$

Hence A^{-1} belongs to $\mathfrak{R}'$. ∎

Now we develop some properties of the individual elements of $\mathfrak{L}_n$. Some follow easily from the commutant theorem; while the detailed spectral picture requires additional work.

Corollary 1.5. *The only normal elements in* $\mathfrak{L}_n$ *are scalars.*

Proof. If N is a normal element of $\mathfrak{L}_n$, define $\alpha = (N\xi_1, \xi_1)$. As ξ_1 is an eigenvector for $\mathfrak{L}_n^*$, it follows that $N^*\xi_1 = \overline{\alpha}\xi_1$. By normality, $N\xi_1 = \alpha\xi_1$. Since N commutes with $\mathfrak{R}_n$, we obtain

$$N\xi_w = NR_w\xi_1 = R_wN\xi_1 = \alpha\xi_w.$$

Hence $N = \alpha I$. ∎

Corollary 1.6. *For all A in $\mathfrak{L}_n$, $\|A\| = \|A\|_e$. In particular, there are no non-zero compact operators in $\mathfrak{L}_n$.*

Proof. For any vector ξ and word w in $\mathcal{F}_n$,

$$\|AR_w\xi\| = \|R_wA\xi\| = \|A\xi\|.$$

Since $R_w\xi$ tends weakly to 0 as $|w|$ tends to infinity for every vector ξ in $\mathcal{H}$, it follows that $\|A\| = \|A\|_e$. ∎

Our analysis of the spectrum of elements of $\mathfrak{L}_n$ is quite direct. As with the proof of the commutant theorem, it relies on the fact that each element A of $\mathfrak{L}_n$ is uniquely determined by $A\xi_1$, which has a 'Fourier expansion' in the standard basis. It also makes use of certain evident invariant subspaces arising from this standard basis.

Theorem 1.7. *Every non-zero element of $\mathfrak{L}_n$ is injective and has non-zero spectrum. For $n \geq 2$ and all A in $\mathfrak{L}_n$, $\sigma(A) = \sigma_e(A)$.*

Proof. Let A in $\mathfrak{L}_n$ be a non-zero element; and let $A\xi_1 = \sum_w a_w\xi_w$. This is non-zero, for otherwise $A\xi_w = R_wA\xi_1 = 0$ for all w, whence $A = 0$. Choose a word v of minimal length such that $a_v \neq 0$. Then for any basis vector ξ_u,

$$A\xi_u = R_uA\xi_1 = \sum_w a_w\xi_{wu}.$$

Thus the non-zero terms correspond to words of length at least $|u| + |v|$. Therefore it follows recursively that

$$A^k\xi_1 = a_v^k\xi_{v^k} + \sum_{\substack{w \neq v^k \\ |w| \geq |v|^k}} a_{w,k}\xi_w.$$

Therefore

$$\|A^k\|^{1/k} \geq |(A^k\xi_1, \xi_{v^k})|^{1/k} = |a_v|.$$

So the spectral radius of A is at least $|a_v| > 0$.

Now suppose that $\zeta = \sum_w b_w\xi_w$ is a non-zero vector, and let u be a word of minimal length such that $b_u \neq 0$. Then computing as above, one sees that

$$(A\zeta, \xi_{vu}) = a_vb_u \neq 0.$$

Therefore A is injective.

Note that to show that $\sigma(A) \subseteq \sigma_e(A)$ for every A in $\mathfrak{L}_n$, it suffices to show that whenever A belongs to $\mathfrak{L}_n$ and 0 belongs to $\sigma(A)$, then 0 lies in $\sigma_e(A)$. So suppose 0 belongs to $\sigma(A)$. By Corollary 1.4 and the fact that non-zero

elements of $\mathfrak{L}_n$ are injective, we see that the only way this can happen is if A is not onto. If $\operatorname{Ran} A$ is not closed, then A is not Fredholm, so 0 is in $\sigma_e(A)$.

Suppose then that 0 belongs to $\sigma(A)$ because $\operatorname{Ran} A$ is closed and proper. The range of A is $\mathfrak{R}_n$ invariant. Thus it does not contain ξ_1, as this is cyclic for $\mathfrak{R}_n$. We claim that

$$(1) \qquad (A \operatorname{span}\{\xi_w : |w| \geq k\}) \cap \operatorname{span}\{\xi_w : |w| \leq k\} = \{0\}.$$

Indeed, $\operatorname{span}\{\xi_w : |w| \geq k\} = \sum_{|w|=k} \oplus R_w \mathcal{H}_n$ is invariant for $\mathfrak{L}_n$. Thus the only possible vectors in the intersection have the form

$$\xi = \sum_{|w|=k} a_w \xi_w = A \sum_{|w|=k} R_w \zeta_w$$

for some vectors ζ_w in $\mathcal{H}_n$. But if $|v| = k$, then

$$a_v \xi_1 = R_v^* \xi = \sum_{|w|=k} R_v^* R_w A \zeta_w = A \zeta_v.$$

Hence $a_v = 0$ for all v; whence $\xi = 0$. So (1) holds.

Therefore,

$$\dim\big(\operatorname{Ran}(A) \cap \operatorname{span}\{\xi_w : |w| \leq k\}\big) \leq \dim \operatorname{span}\{\xi_w : |w| < k\}$$
$$= \dim \operatorname{span}\{\xi_w : |w| \leq k\} - n^k.$$

Hence $\operatorname{Ran}(A)$ has codimension at least n^k for all $k \geq 0$.

Therefore, for $2 \leq n < \infty$, this shows that $\operatorname{Ran}(A)$ has infinite codimension. When $n = \infty$, the argument still shows that

$$\operatorname{Ran}(A) \cap \operatorname{span}\{\xi_w : |w| = 1\}$$

is at most 1–dimensional with the same conclusion. Thus 0 belongs to $\sigma_e(A)$. ∎

Corollary 1.8. *The algebra $\mathfrak{L}_n$ contains no non-trivial idempotents or non-zero quasinilpotent elements. Thus the spectrum of every non-scalar element is connected and contains more than one point.*

Proof. A non-trivial idempotent has kernel, and a non-zero quasinilpotent element has zero spectrum, both contradicting Theorem 1.7. The Riesz functional calculus provides a proper idempotent in the algebra generated by any operator with disconnected spectrum. Thus the spectrum of every element of $\mathfrak{L}_n$ is connected. If $\sigma(A) = \{\lambda\}$ is a singleton, then $A - \lambda I$ is quasinilpotent and therefore equals 0. So A is scalar. ∎

This yields the following important algebraic consequence.

Corollary 1.9. *$\mathfrak{L}_n$ is semisimple.*

Proof. The radical is contained in the set of quasinilpotent elements, and therefore is $\{0\}$. ∎

Remark 1.10. During recent correspondence with Arias, we learned that Katsoulis in 1994 also observed $\mathfrak{L}_n$ is semisimple, but we do not have a reference.

We include a computational lemma which will be useful later.

Lemma 1.11. *If A belongs to $\mathfrak{L}_n$, then $R_j^* A - AR_j^* = (R_j^* A\xi_1)\xi_1^*$ is a rank one operator. Therefore every element of the norm closed algebra generated by $\mathrm{C}^*(R_1, \ldots, R_n) + \mathfrak{R}_n$ commutes with $\mathfrak{L}_n$ modulo the compact operators.*

Proof. This is a straight-forward computation:

$$R_j^* A - AR_j^* = R_j^* A\Big(\xi_1\xi_1^* + \sum_{i=1}^{n} R_i R_i^*\Big) - AR_j^*$$

$$= (R_j^* A\xi_1)\xi_1^* + \sum_{i=1}^{n} R_j^* R_i AR_i^* - AR_j^* = (R_j^* A\xi_1)\xi_1^*.$$

Since each R_j and R_j^* commutes with $\mathfrak{L}_n$ modulo the compact operators, as does $\mathfrak{R}_n$, this fact extends to the norm-closed algebra that they generate. ∎

It would be interesting to know if the converse is true. This is the case for $n = 1$, where $\mathrm{C}^*(R_1) + \mathfrak{R}_1 = \mathcal{T}(H^\infty + C)$ is known to be the essential commutant of the Toeplitz algebra [**10**]. For $n \geq 2$, $\mathrm{C}^*(R_1, \ldots, R_n) + \mathfrak{R}_n$ is still closed but is no longer an algebra. The fact that it is closed can be established by mimicking the second proof given in [**28**]. One uses a Cesàro mean argument to establish that

$$\mathrm{dist}(X, \mathcal{A}_n) = \mathrm{dist}(X, \mathfrak{R}_n) \quad \text{for all} \quad X \in \mathrm{C}^*(R_1, \ldots, R_n)$$

where $\mathcal{A}_n$ is the norm closed algebra generated by R. Indeed, X is the *norm* limit $\lim_{k\to\infty} \Sigma_k(X)$. So if A in $\mathfrak{R}_n$ satisfies $\|X - A\| = \mathrm{dist}(X, \mathfrak{R}_n)$, then

$$\mathrm{dist}(X, \mathcal{A}_n) \leq \lim_{k\to\infty} \|X - \Sigma_k(A)\|$$

$$\leq \lim_{k\to\infty} \|X - \Sigma_k(X)\| + \|\Sigma_k(X - A)\| = \mathrm{dist}(X, \mathfrak{R}_n).$$

Hence the injection

$$\mathrm{C}^*(R_1, \ldots, R_n)/\mathcal{A}_n \hookrightarrow \mathcal{B}(\mathcal{H}_n)/\mathfrak{R}_n$$

is isometric, and thus has closed range. Pulling this back to $\mathcal{B}(\mathcal{H}_n)$ shows that $\mathrm{C}^*(R_1, \ldots, R_n) + \mathfrak{R}_n$ is closed. Using a result on the structure of finitely generated ideals in $\mathfrak{L}_n$ from [**13**], one can show that finite sums of the form $\sum A_w R_w^*$ with coefficients A_w in $\mathfrak{R}_n$ are dense in the algebra this generates.

2. Invariant Subspaces for $\mathfrak{L}_n$

Following Popescu's version of the Wold decomposition [**19**], say that a subspace $\mathcal{W}$ is *wandering* for an n-tuple of isometries $S = (S_1, \ldots, S_n)$ satisfying (F') if $w(S)\mathcal{W}$ are pairwise orthogonal for distinct words w in $\mathcal{F}_n$. Clearly, every wandering subspace generates the invariant subspace $\mathfrak{A}[\mathcal{W}] = \sum_w \oplus w(S)\mathcal{W}$ for $\mathfrak{A} = \mathrm{Alg}\{S_1, \ldots, S_n\}$. When $\mathcal{W}$ is spanned by

a single unit vector ζ, we shall also use the notation $\mathfrak{A}[\zeta]$ for the cyclic subspace generated by ζ. The subspaces $\mathfrak{A}[\mathcal{W}]$ are easily seen to be a direct sum of cyclic subspaces. Indeed, let $\{\zeta_j\}$ denote an orthonormal basis for $\mathcal{W}$. Then $\mathfrak{A}[\zeta_j]$ are pairwise orthogonal $\mathfrak{A}$-invariant subspaces which sum to $\mathfrak{A}[\mathcal{W}]$. Popescu shows that every $\mathfrak{A}$-invariant subspace decomposes as the direct sum of an $\mathfrak{A}[\mathcal{W}]$ space and an invariant subspace $\mathcal{M}$ such that $T_i = S_i|_{\mathcal{M}}$ satisfy

$$\sum_{i=1}^{n} T_i T_i^* = I.$$

Unlike the $n = 1$ case, this latter condition does not yield much information. When we study other free semigroup algebras, a more delicate analysis will be required.

In the case of the left regular representation, all invariant subspaces are generated by a wandering subspace. This result should be seen as the analogue of Beurling's Theorem for the analytic Toeplitz algebra. In the $n = 1$ case, every invariant subspace is cyclic of the form ωH^2 where ω is an inner function. These subspaces are therefore the range of the isometries T_ω, which are all the isometries in $\mathcal{T}(H^\infty) = \mathfrak{L}_1 = \mathfrak{R}_1$. The following result has a significant overlap with Theorem 2.3 of [1].

Theorem 2.1. *Every invariant subspace of $\mathfrak{L}_n$ is generated by a wandering subspace. Thus it is the direct sum of cyclic subspaces. The cyclic invariant subspaces of $\mathfrak{L}_n$ are precisely the ranges of isometries in $\mathfrak{R}_n$; and the choice of isometry is unique up to a scalar.*

Proof. Given an invariant subspace $\mathcal{M}$, form the subspace

$$\mathcal{W} = \mathcal{M} \ominus \Big(\sum_{i=1}^{n} \oplus L_i \mathcal{M} \Big).$$

It is clear that $\mathcal{W}$ is orthogonal to $L_w \mathcal{W}$ for all $w \neq 1$; and hence it is easy to see that the $L_w \mathcal{W}$ are pairwise orthogonal. So $\mathcal{W}$ is wandering. It remains to show that $\mathcal{M} = \mathfrak{L}_n[\mathcal{W}]$. By the Wold decomposition, the complement $\mathcal{N}$ of $\mathfrak{L}_n[\mathcal{W}]$ in $\mathcal{M}$ is invariant for $\mathfrak{L}_n$ and $\mathcal{N} = \sum_{i=1}^{n} \oplus L_i \mathcal{N}$. If $\mathcal{N}$ is non-zero, choose the smallest integer k_0 such that $Q_{k_0} \mathcal{N} \neq 0$. (Recall that Q_{k_0} is the projection onto span$\{\xi_w : |w| = k_0\}$.) But then

$$Q_{k_0} \mathcal{N} \subseteq \sum_{i=1}^{n} Q_{k_0} L_i \mathcal{N} = \sum_{i=1}^{n} L_i Q_{k_0-1} \mathcal{N} = 0.$$

This contradiction shows that $\mathcal{M} = \mathfrak{L}_n[\mathcal{W}]$ as desired.

Now $\mathcal{M}$ is cyclic precisely when $\mathcal{W}$ is one dimensional. Consider a cyclic invariant subspace $\mathfrak{L}_n[\zeta]$ where ζ is a unit wandering vector. Define an isometry R_ζ by the rule

$$R_\zeta \xi_w = L_w \zeta \quad \text{for} \quad w \in \mathcal{F}_n.$$

It is evident by construction that R_ζ commutes with the operators L_w. So R_ζ belongs to $\mathfrak{R}_n$ by Theorem 1.2. Moreover, the range of R_ζ is equal to $\mathfrak{L}_n[\zeta]$ by design.

Conversely, if R is an isometry in $\mathfrak{R}_n$, the range of R is invariant for $\mathfrak{L}_n$. Moreover, $\zeta = R\xi_1$ is a wandering vector for this range and $R = R_\zeta$. So it is cyclic. Finally, suppose that R' is another isometry in $\mathfrak{R}_n$ with the same range. Then both ζ and $\zeta' = R'\xi_1$ are wandering vectors for $R\mathcal{H}_n$. Therefore they both lie in the one dimensional wandering space $\mathbb{C}\zeta$. Hence $\zeta' = \lambda\zeta$ and $R' = \lambda R$ for some scalar λ in $\mathbb{T}$. ∎

Another classical result of function theory is that every function f in H^1 has an inner–outer factorization $f = \omega h$. This factorization is useful both in H^∞ and in H^2. We have already noted that the inner functions in H^∞ correspond to those analytic Toeplitz operators which are isometries. Thus we define the *inner* elements of $\mathfrak{L}_n$ to be the isometries. It is the case that h in H^∞ is outer if and only if T_h has dense range. So we define *outer* elements of $\mathfrak{L}_n$ to be those with dense range. This allows us to obtain a natural inner–outer factorization in $\mathfrak{L}_n$. Our definition of outer elements differs from the definition which appears in [21]. However, it is not difficult to show that the definitions coincide for operators in $\mathfrak{L}_n$. Thus the following result is an immediate consequence of Theorem 4.2 of [21]. We include it for completeness of exposition and to provide a somewhat different proof.

Corollary 2.2. *Every A in $\mathfrak{L}_n$ factors as $A = L_\zeta B$ where L_ζ is an isometry in $\mathfrak{L}_n$ and B belongs to $\mathfrak{L}_n$ and has dense range. This factorization is unique up to a scalar. The operator B is invertible if and only if A has closed range.*

Proof. The closed range $\mathcal{M} = \overline{\mathrm{Ran}(A)} = \overline{A\mathfrak{R}_n\xi_1} = \mathfrak{R}_n[A\xi_1]$ is a cyclic invariant subspace for $\mathfrak{R}_n$. Therefore, it has a unit wandering vector ζ, which is unique up to a scalar, such that $\mathcal{M} = L_\zeta\mathcal{H}_n$. Let $B = L_\zeta^*A$. Evidently, $A = L_\zeta B$, and B has dense range. To verify that B belongs to $\mathfrak{L}_n$, we show that it commutes with each R_j. Indeed, by Lemma 1.11,

$$R_j B - B R_j = (R_j L_\zeta^* - L_\zeta^* R_j)A = -\xi_1(R_j^* L_\zeta \xi_1)^*A = -\xi_1(A^* R_j^* \zeta)^*.$$

Recall that ζ lies in $\mathcal{M} \ominus \sum_{i=1}^n R_i\mathcal{M}$. Hence $R_j^*\zeta$ is orthogonal to the subspace $\mathcal{M} = \overline{\mathrm{Ran}\,A}$, so $R_j^*\zeta$ belongs to $\ker A^*$, and therefore $A^* R_j^*\zeta = 0$. Thus $R_j B - B R_j = 0$ as claimed.

Uniqueness follows from the uniqueness of ζ up to a scalar. If A has closed range, then B is surjective. By Theorem 1.7, B is injective and thus is invertible in $\mathfrak{L}_n$ by Corollary 1.4. The converse is clear. ∎

Another of Beurling's results is that the cyclic vectors of the shift are precisely the outer functions in H^2. So if we take this as our notion of an outer function in $\mathcal{H}_n$, we obtain an analogue of inner–outer factorization in Fock space. However, we are missing some sort of intrinsic characterization of cyclic vectors and wandering vectors for $\mathfrak{L}_n$. This version of inner–outer

factorization in $\mathcal{H}_n$ is not as satisfying as the algebra version. Compare the following result with Theorem 2.1 of [**1**].

Corollary 2.3. *Every vector η in $\mathcal{H}_n$ factors as $\eta = R_\zeta \xi$ where ζ is a wandering vector and ξ is a cyclic vector for $\mathfrak{L}_n$. This factorization is unique up to a scalar of modulus* 1.

Proof. The cyclic subspace $\mathfrak{L}_n[\eta]$ has a wandering vector ζ. Since R_ζ is an isometry in $\mathfrak{L}'_n$ onto $\mathfrak{L}_n[\eta]$, it follows that $\xi = R_\zeta^* \eta$ is cyclic and $\eta = R_\zeta \xi$. The uniqueness up to a scalar follows from the uniqueness of ζ up to a scalar. ∎

To complete the picture of inner and outer functions, we provide a more intrinsic characterization of these elements. The characterization for inner operators is analogous to the characterization in H^∞ that h is inner if and only if $\|h\|_\infty = \|h\|_2 = 1$.

Proposition 2.4. *An element A in $\mathfrak{L}_n$ is inner (i.e. an isometry) if and only if $\|A\| = \|A\xi_1\| = 1$.*

Proof. If A is an isometry, it is clear that $\|A\| = \|A\xi_1\| = 1$. Conversely, if $\|A\xi_1\| = 1$, then $A\xi_w = R_w A\xi_1$ are unit vectors for all w in $\mathcal{F}_n$. Since $\|A\| = 1$, it follows that these vectors are pairwise orthogonal. Hence A is an isometry. ∎

The characterization of outer functions is parallel to the fact that among all H^∞ functions with a given absolute value on the boundary, outer functions have the greatest absolute value at the origin.

Proposition 2.5. *An element A in $\mathfrak{L}_n$ is outer if and only if $|\varphi_0(A)| = |(A\xi_1, \xi_1)|$ is maximal among*

$$\{|\varphi_0(B)| : B \in \mathfrak{L}_n, \ B^*B = A^*A\}.$$

Two outer functions with the same absolute value differ by a multiple of modulus one.

Before beginning the proof, we note that this result is part of Theorem 2.5 of [**23**].

Proof. Suppose that A is outer and that B in $\mathfrak{L}_n$ satisfies $B^*B = A^*A$. Then since A has dense range, there is an isometry L in $\mathcal{B}(\mathcal{H}_n)$ such that $B = LA$. Consequently, for $1 \le i \le n$,

$$0 = R_i B - B R_i = (R_i L - L R_i)A + L(R_i A - A R_i) = (R_i L - L R_i)A.$$

Since A has dense range, it follows that L commutes with each R_i; and therefore belongs to $\mathfrak{L}_n$ by Theorem 1.2. Since φ_0 is multiplicative and contractive on $\mathfrak{L}_n$,

$$|\varphi_0(B)| = |\varphi_0(L)\varphi_0(A)| \le |\varphi_0(A)|.$$

Conversely, this inequality is strict unless $L\xi_1 = \lambda \xi_1$ for some scalar λ of modulus one. But this clearly implies that $L = \lambda I$ and $B = \lambda A$. So if A is not outer, it does not achieve this maximum. ∎

For the analytic Toeplitz algebra, the eigenvalues of the adjoint algebra correspond to point evaluations in the unit disk. For each λ in $\mathbb{D}$, there is a unit vector

$$k_\lambda = (1 - |\lambda|^2)^{1/2}(1 - \overline{\lambda}z)^{-1} = (1 - |\lambda|^2)^{1/2}\sum_{k \geq 0}(\overline{\lambda}z)^k$$

such that $T_z^* k_\lambda = \overline{\lambda}k_\lambda$. Moreover

$$(T_h k_\lambda, k_\lambda) = h(\lambda) \quad \text{for all} \quad h \in H^\infty.$$

Their orthogonal complements are given by $\{k_\lambda\}^\perp = b_{\overline{\lambda}}H^2$ where

$$b_\lambda(z) = \frac{z - \lambda}{1 - \overline{\lambda}z}$$

is a Mobius map of the unit disk. The lattice generated by these subspaces contains bH^2 for every Blaschke product b without multiple roots. These subspaces are SOT–dense in the lattice of all invariant subspaces because the Blaschke products are SOT–dense (and even norm dense) in the set of inner functions.

For $n \geq 2$, the eigenvectors of $\mathfrak{L}_n^*$ turn out to be symmetric, meaning that the coefficients for ξ_w depend only on the evaluation of w on *commuting* variables. So they cannot be sufficient to determine the whole invariant subspace lattice. Indeed, they are also eigenvectors for $\mathfrak{R}_n^*$. Nevertheless, they will yield an important class of invariant subspaces. Let $\mathbb{B}_n$ denote the unit ball in n-dimensional Hilbert space. This makes sense even for $n = \infty$.

We note that eigenvectors of $\mathfrak{L}_n^*$ are also discussed in example 8 of [**1**].

Theorem 2.6. *The eigenvectors for $\mathfrak{L}_n^*$ are the vectors*

$$\nu_\lambda = (1 - \|\lambda\|^2)^{1/2}\sum_{w \in \mathcal{F}_n}\overline{w(\lambda)}\xi_w = (1 - \|\lambda\|^2)^{1/2}(I - \sum_{i=1}^n \overline{\lambda_i}L_i)^{-1}\xi_1$$

for λ in the unit ball $\mathbb{B}_n$. They satisfy

$$L_i^*\nu_\lambda = \overline{\lambda_i}\nu_\lambda$$

and $(p(L)\nu_\lambda, \nu_\lambda) = p(\lambda)$ for every polynomial $p = \sum_w a_w w$ in the semigroup algebra $\mathbb{C}\mathcal{F}_n$. This extends to the map $\varphi_\lambda(A) = (A\nu_\lambda, \nu_\lambda)$, which is a WOT–continuous multiplicative linear functional on $\mathfrak{L}_n$. The vector ν_λ is cyclic for $\mathfrak{L}_n$. The subspace $\mathcal{M}_\lambda = \{\nu_\lambda\}^\perp$ is $\mathfrak{L}_n$ invariant, and its wandering subspace $\mathcal{W}_\lambda$ is n-dimensional, spanned by

$$\zeta_{\lambda,i} = \lambda_i\xi_1 - (1 - \|\lambda\|^2)^{1/2}L_i\nu_\lambda \quad \text{for} \quad 1 \leq i \leq n.$$

Proof. First note that if λ is in $\mathbb{B}_n$, then

$$\sum_w |w(\lambda)|^2 = \sum_{k \geq 0} \sum_{|w|=k} |w(\lambda)|^2$$

$$= \sum_{k \geq 0} \sum_{s_i \geq 0, \sum s_i = k} \frac{k!}{s_1! \ldots s_n!} |\lambda_1|^{2s_1} \ldots |\lambda_n|^{2s_n}$$

$$= \sum_{k \geq 0} \left(\sum_{i=1}^n |\lambda_i|^2 \right)^k = (1 - \|\lambda\|^2)^{-1} < \infty.$$

So ν_λ is defined for λ in $\mathbb{B}_n$. Also note that

$$\| \sum_{i=1}^n \overline{\lambda_i} L_i \|^2 = \sum_{i=1}^n |\lambda_i|^2 = \|\lambda\|^2 < 1.$$

So $I - \sum_{i=1}^n \overline{\lambda_i} L_i$ is invertible, and its inverse is given by the power series

$$(I - \sum_{i=1}^n \overline{\lambda_i} L_i)^{-1} = \sum_{k \geq 0} \left(\sum_{i=1}^n \overline{\lambda_i} L_i \right)^k = \sum_{w \in \mathcal{F}_n} \overline{w(\lambda)} L_w.$$

Thus the second identity for ν_λ is evident. Since

$$(I - \sum_{i=1}^n \overline{\lambda_i} L_i) \nu_\lambda = (1 - \|\lambda\|^2)^{1/2} \xi_1,$$

it is evident that ν_λ is cyclic.

A straightforward computation shows that

$$L_i^* \nu_\lambda = (1 - \|\lambda\|^2)^{1/2} L_i^* \sum_w \overline{(z_i w)(\lambda)} \xi_{z_i w}$$

$$= (1 - \|\lambda\|^2)^{1/2} \overline{\lambda_i} \sum_w \overline{w(\lambda)} \xi_w = \overline{\lambda_i} \nu_\lambda.$$

Thus it follows that

$$(L_w \nu_\lambda, \nu_\lambda) = w(\lambda) \|\nu_\lambda\|^2 = w(\lambda).$$

This extends to polynomials by linearity. Evidently φ_λ is WOT–continuous. If A, B are in $\mathfrak{L}_n$, then the scalars $\alpha = \varphi_\lambda(A)$ and $\beta = \varphi_\lambda(B)$ are determined by the identities $A^* \nu_\lambda = \overline{\alpha} \nu_\lambda$ and $B^* \nu_\lambda = \overline{\beta} \nu_\lambda$. Hence

$$\varphi_\lambda(AB) = (AB\nu_\lambda, \nu_\lambda) = (\nu_\lambda, B^* A^* \nu_\lambda) = \alpha\beta = \varphi_\lambda(A)\varphi_\lambda(B).$$

Therefore φ_λ is multiplicative.

Conversely, if $\nu = \sum_w a_w \xi_w$ satisfies $L_i^* \nu = \overline{\lambda_i} \nu$, then reversing the calculation in the first paragraph shows that $a_w = \overline{w(\lambda)} a_1$. Since ν is a vector of finite norm, this forces the condition $\|\lambda\| < 1$, and $\nu = a_1 \nu_\lambda$.

Since $\mathbb{C}\nu_\lambda$ is invariant for $\mathfrak{L}_n^*$, it follows that $\mathcal{M}_\lambda = \{\nu_\lambda\}^\perp$ is invariant for $\mathfrak{L}_n$. Let $\mathcal{W}_\lambda = \mathcal{M}_\lambda \ominus \sum_{i=1}^n L_i \mathcal{M}_\lambda$ denote the corresponding wandering space. Notice that

$$L_i \mathcal{M}_\lambda = L_i \mathcal{H}_n \ominus \mathbb{C} L_i \nu_\lambda.$$

If ζ belongs to $\mathcal{W}_\lambda$, we may write $\zeta = \alpha \xi_1 + \sum_{i=1}^n \zeta_i$ for ζ_i in $L_i \mathcal{H}_n$. Since ζ is orthogonal to $L_i \mathcal{M}_\lambda$, it follows that ζ_i is a multiple of $L_i \nu_\lambda$, say $\zeta_i = \alpha_i L_i \nu_\lambda$. And since ζ is orthogonal to

$$\nu_\lambda = (1 - \|\lambda\|)^{1/2} \xi_1 + \sum_{i=1}^n \overline{\lambda_i} L_i \nu_\lambda,$$

we deduce that

$$0 = (\zeta, \nu_\lambda) = (1 - \|\lambda\|)^{1/2} \alpha + \sum_{i=1}^n \lambda_i \alpha_i.$$

This equation has an n-dimensional solution space spanned by the n solutions given by $\alpha = \lambda_i$, $\alpha_i = -(1 - \|\lambda\|)^{1/2}$ and $\alpha_j = 0$ otherwise. These yield the vectors $\zeta_{\lambda,i}$. ∎

We note that the ν_λ's all lie in, and indeed span, the symmetric Fock space. So while the $\mathcal{M}_\lambda$'s are invariant for $\mathfrak{L}_n$ and $\mathfrak{R}_n$, there are many other operators leaving them invariant as well. In particular, they do not come close to determining the invariant subspace lattice of $\mathfrak{L}_n$.

Now we turn to the problem of showing that $\mathfrak{L}_n$ is hyper-reflexive. We begin with a folklore result that algebras of infinite multiplicity are hyper-reflexive. A WOT–closed algebra $\mathfrak{A}$ is said to have infinite multiplicity if it is unitarily equivalent to an algebra of the form $\mathfrak{B} \otimes I$ where I is the identity operator on an infinite dimensional space.

Theorem 2.7. *Every* WOT*–closed algebra of infinite multiplicity is hyper-reflexive with distance constant at most* 9.

Proof. The algebra $\mathfrak{A} = \mathfrak{B} \otimes I$ is contained in $\mathcal{B}(\mathcal{H}) \otimes I$ which is an AF von Neumann algebra and thus is hyper-reflexive [8] with distance constant at most 4. It is a result of Arveson [2] that for any weak-∗ continuous linear functional f on $\mathcal{B}(\mathcal{H})$, there are vectors ξ and ζ in $\mathcal{H} \otimes \mathcal{H}'$ (where $\mathcal{H}'$ is any separable Hilbert space) such that for all T in $\mathcal{B}(\mathcal{H})$,

$$f(T) = ((T \otimes I)\xi, \zeta) \qquad \text{and} \qquad \|\xi\| \, \|\zeta\| = \|f\|.$$

Therefore $\mathcal{B}(\mathcal{H}) \otimes I$ has property $\mathbb{A}_1$. So the result now follows from [11, 18]. ∎

Remark 2.8. Probably the number 9 is quite crude. However, this constant can be greater than 1 even for von Neumann algebras of infinite multiplicity. The usual example of a von Neumann algebra with constant greater than 1 is the 3×3 diagonal subalgebra $\mathcal{D}_3$ of $\mathcal{M}_3$ with respect to a basis e_1, e_2, e_3, which has distance constant $\sqrt{3/2}$ [12]. In fact, $\mathcal{D} = \mathcal{D}_3 \otimes I$ acting on

$\mathbb{C}^3 \otimes \mathcal{H}$ also has distance constant at least $\sqrt{3/2}$. To see this, let T in $\mathcal{M}_3$ be the matrix

$$T = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

which has the property that $\|T\| = \mathrm{dist}(T, \mathcal{D}_3) = 1$ but $\beta_{\mathcal{L}_3}(T) = \sqrt{2/3}$, where $\mathcal{L}_3$ is the lattice of projections in $\mathcal{D}_3$. Let E denote a rank one projection $E = \xi_0 \xi_0^*$ in $\mathcal{B}(\mathcal{H})$. Consider $X = T \otimes E$. First notice that for any D in $\mathcal{D}_3$,

$$\begin{aligned} 1 = \|T\| &\le \|T - D\| \\ &= \|(I \otimes E)(X - D \otimes I)(I \otimes E)\| \le \|X - (D \otimes I)\|. \end{aligned}$$

Thus $\mathrm{dist}(X, \mathcal{D}) = \|X\| = 1$.

To compute $\beta(X) = \sup_{P \in \mathcal{L}} \|P^\perp X P\|$ as P runs over $\mathcal{L} = \mathrm{Lat}(\mathcal{D})$, it suffices to consider $\|P^\perp X P x\|$ where x is a unit vector in the range of P. So one may then replace P by the projection P_x onto the cyclic invariant subspace for $\mathcal{D}$ generated by x. Let

$$x = \sum_{i=1}^{3} x_i (e_i \otimes \zeta_i),$$

where ζ_i are unit vectors in $\mathcal{H}$. We may assume that $(\zeta_i, \xi_0) \ge 0$, and so can write

$$\zeta_i = \cos \theta_i \, \xi_0 + \sin \theta_i \, \xi_i$$

where ξ_i is a unit vector orthogonal to ξ_0. Then

$$\mathcal{D}x = \sum_{i=1}^{3} \oplus \mathbb{C}(e_i \otimes \zeta_i).$$

Let $f = \sum_{i=1}^{3} x_i \cos \theta_i e_i$. Then

$$Xx = (Tf) \otimes \xi_0 \qquad \text{and} \qquad \|Xx\|^2 = \|Tf\|^2 = \sum_{i=1}^{3} |(Tf, e_i)|^2.$$

Let us assume the convention that addition of indices is calculated modulo 3. Then

$$Tf = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} (x_{i+1} \cos \theta_{i+1} - x_{i-1} \cos \theta_{i-1}) \, e_i.$$

Now compute

$$\|P_x X x\|^2 = \sum_{i=1}^{3} |(Tf \otimes \xi_0, e_i \otimes \zeta_i)|^2 = \sum_{i=1}^{3} |(Tf, e_i)|^2 \cos^2 \theta_i$$

To simplify notation, let us write $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$. Hence

$$\|P_x^\perp Xx\|^2 = \|Xx\|^2 - \|P_x Xx\|^2 = \sum_{i=1}^{3} |(Tf, e_i)|^2 s_i^2$$

$$= \frac{1}{3}\sum_{i=1}^{3} |x_{i+1}c_{i+1} - x_{i-1}c_{i-1}|^2 s_i^2$$

$$= \frac{1}{3}\Big(\sum_{i=1}^{3} (s_{i+1}^2 + s_{i-1}^2)c_i^2|x_i|^2 - 2\operatorname{Re}\sum_{i=1}^{3} s_i^2 c_{i-1}c_{i+1}x_{i-1}\overline{x_{i+1}}\Big)$$

$$= (Ax, x),$$

where we now think of x as $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and

$$A = \frac{1}{3}\begin{bmatrix} (s_2^2 + s_3^2)c_1^2 & -c_1 c_2 s_3^2 & -c_1 c_3 s_2^2 \\ -c_1 c_2 s_3^2 & (s_1^2 + s_3^2)c_2^2 & -c_2 c_3 s_1^2 \\ -c_1 c_3 s_2^2 & -c_2 c_3 s_1^2 & (s_1^2 + s_2^2)c_3^2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}\begin{bmatrix} s_2^2 + s_3^2 & -s_3^2 & -s_2^2 \\ -s_3^2 & s_1^2 + s_3^2 & -s_1^2 \\ -s_2^2 & -s_1^2 & s_1^2 + s_2^2 \end{bmatrix}\begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

$$= \frac{1}{3}\operatorname{diag}\{c_1, c_2, c_3\}\Big(\sum_{i=1}^{3} s_i^2 g_i g_i^*\Big)\operatorname{diag}\{c_1, c_2, c_3\} = \frac{1}{3}\sum_{i=1}^{3} s_i^2 h_i h_i^*$$

where $g_i = e_{i+1} - e_{i-1}$ and $h_i = c_{i+1}e_{i+1} - c_{i-1}e_{i-1}$. Thus A is positive, and hence

$$\|A\| \le \operatorname{Tr}(A) = \frac{1}{3}\sum_{i=1}^{3} c_i^2(s_{i+1}^2 + s_{i-1}^2).$$

A calculus computation shows that this achieves its maximum when one of the c_i's equals 0 and another equals 1, which yields the value $2/3$. And, when $c_1 = 0$ and $c_2 = c_3 = 1$, one achieves $\|A\| = 2/3$. Therefore $\beta(X) = \sqrt{2/3}$. So the distance constant for $\mathcal{D}$ is at least $\sqrt{3/2}$ just as in the case for $\mathcal{D}_3$.

We can now prove the main result of this section. To prove hyper-reflexivity, we make use of the fact that there are natural invariant subspaces with infinite dimensional wandering space that almost fill up the whole space. Then we reach the vector ξ_1 by making use of the invariant subspaces arising from the eigenvectors of the adjoint. It seems to us that a proof just of reflexivity would require at least this much. Surprisingly, because of the availability of these subspaces of infinite multiplicity, the hyper-reflexivity result is much more accessible than in the case $n = 1$. Because of the commutant theorem and the structure of invariant subspaces, we also obtain a norm estimate from the corresponding derivation on $\mathfrak{R}_n$.

We note here that Arias and Popescu prove the reflexivity of $\mathfrak{L}_n$. Since hyper-reflexive algebras are reflexive, the following result subsumes Theorem 4.1 of [1].

Theorem 2.9. *The algebras $\mathfrak{L}_n$ are hyper-reflexive. Moreover, for all T in $\mathcal{B}(\mathcal{H}_n)$,*

$$\frac{1}{51}\operatorname{dist}(T,\mathfrak{L}_n) \leq \sup_{L\in\operatorname{Lat}(\mathfrak{L}_n)} \|L^\perp T L\| \leq \|\delta_T|_{\mathfrak{R}_n}\| \leq 2\operatorname{dist}(T,\mathfrak{L}_n).$$

Proof. The case $n = 1$ is hyper-reflexive by [11] with constant less than 19. So we assume that $n \geq 2$.

Fix T in $\mathcal{B}(\mathcal{H}_n)$ and let $\beta = \beta_{\operatorname{Lat}\mathfrak{L}_n}(T)$. Notice that $\mathfrak{L}_n$ is unitarily equivalent to the restriction of $\mathfrak{L}_n$ to any cyclic subspace. So the restriction of $\mathfrak{L}_n$ to an invariant subspace $\mathcal{M}$ with an infinite dimensional wandering space has infinite multiplicity, and therefore is hyper-reflexive with distance constant 9 by Theorem 2.7. Moreover, since the invariant subspaces of the restriction algebra on $\mathcal{M}$ are merely the invariant subspaces of $\mathfrak{L}_n$ contained in $\mathcal{M}$, it is clear that the distance estimate,

$$\sup\{\|P^\perp(T|_{\mathcal{M}})P\| : P \in \operatorname{Lat}(\mathfrak{L}_n|_{\mathcal{M}})\},$$

for the restriction of T to $\mathcal{M}$ is at most β.

Next we will show that there is an A in $\mathfrak{L}_n$ such that

$$\|(T - A)|_{\{\xi_1\|^\perp}\| \leq 18\sqrt{2}\beta.$$

It will also follow that for any λ in $\mathbb{B}_n$ and any wandering vector ζ for $\{\nu_\lambda\}^\perp$, there is a B in $\mathfrak{L}_n$ such that

$$(2) \qquad\qquad \|(T - B)|_{\mathfrak{L}_n[\zeta]}\| \leq 9\beta.$$

Choose two subspaces $\mathcal{M}_1$, $\mathcal{M}_2$ in $\operatorname{Lat}(\mathfrak{L}_n)$ with infinite dimensional wandering spaces such that $\mathcal{M}_1 + \mathcal{M}_2 = \{\xi_1\}^\perp$ and the projections onto $\mathcal{M}_i$ commute. For example, let

$$\mathcal{M}_i = \sum_{j\neq i}\sum_{k\geq 0}\oplus\mathfrak{L}_n[\xi_{z_j z_i^k}].$$

Also notice that with a minor modification, it can be arranged that any wandering vector for $\{\xi_1\}^\perp$ may be included in $\mathcal{M}_1$. By the preceding remarks, there are elements A_i in $\mathfrak{L}_n$ such that

$$\|(T - A_i)|_{\mathcal{M}_i}\| \leq 9\beta \quad \text{for} \quad i = 1, 2.$$

Notice that Theorem 2.1 implies that for any L in $\operatorname{Lat}(\mathfrak{L}_n)$, the restriction mapping $X \mapsto X|_L$ is isometric on $\mathfrak{L}_n$. Hence, as $\mathcal{M}_1$ and $\mathcal{M}_2$ have non-trivial intersection, we deduce that $\|A_1 - A_2\| \leq 18\beta$. Set $A = (A_1 + A_2)/2$, and note that $\|A - A_i\| \leq 9\beta$. Since the projections onto $\mathcal{M}_i$ commute,

given any vector η in $\{\xi\}^\perp$, we may find orthogonal vectors η_i in $\mathcal{M}_i$ such that $\eta = \eta_1 + \eta_2$. Then by the Cauchy–Schwarz inequality,

$$\|(T - A)|_{\{\xi_1\}^\perp}\| \leq \sup_{\eta \perp \xi_1,\, \|\eta\|=1} \|(T-A)\eta_1\| + \|(T-A)\eta_2\|$$
$$\leq \left(\|(T-A)|_{\mathcal{M}_1}\|^2 + \|(T-A)|_{\mathcal{M}_2}\|^2\right)^{1/2} \leq 18\sqrt{2}\beta.$$

Now if λ is in $\mathbb{B}_n$, Theorem 2.6 shows that $\{\nu_\lambda\}^\perp$ has an n-dimensional wandering space. Thus the restriction of $\mathfrak{L}_n$ to $\{\nu_\lambda\}^\perp$ is unitarily equivalent to its restriction to $\{\xi_1\}^\perp$. Therefore the above analysis applies equally well with ν_λ in place of ξ_1. In particular, given any wandering vector ζ for $\{\nu_\lambda\}^\perp$, there is a subspace $\mathcal{M}$ with infinite dimensional wandering space containing ζ. Thus, as above, there is a B in $\mathfrak{L}_n$ such that

$$\|(T - B)|_{\mathfrak{L}_n[\zeta]}\| \leq \|(T - B)|_{\mathcal{M}}\| \leq 9\beta.$$

Take $\lambda_r = (r, 0, \ldots, 0)$ for $0 \leq r < 1$, set $\zeta_r = \zeta_{\lambda_r,1}/\|\zeta_{\lambda_r,1}\|$ and choose an element B_r in $\mathfrak{L}_n$ such that $\|(T - B_r)|_{\mathfrak{L}_n[\zeta_r]}\| \leq 9\beta$. Then since $\mathfrak{L}_n[\zeta_r]$ intersects $\{\xi_1\}^\perp$ in an infinite dimensional subspace, we deduce again that $\|B_r - A\| < (9 + 18\sqrt{2})\beta$. Hence

$$\|(T - A)\zeta_r\| \leq 18(1 + \sqrt{2})\beta.$$

But $\lim_{r \to 1} \zeta_r = \xi_1$, whence

$$\|(T - A)\xi_1\| \leq 18(1 + \sqrt{2})\beta.$$

Thus, using the Cauchy–Schwarz inequality again,

$$\|T - A\| \leq \left(\|(T - A)|_{\{\xi_1\}^\perp}\|^2 + \|(T - A)\xi_1\|^2\right)^{1/2}$$
$$< 18\sqrt{5 + 2\sqrt{2}}\,\beta < 51\beta.$$

Hence $\mathfrak{L}_n$ is hyper-reflexive.

Choose $\mathcal{M}$ in $\mathrm{Lat}(\mathfrak{L}_n)$ and a unit vector ξ in $\mathcal{M}$ such that β is approximated by $\|P_{\mathcal{M}}^\perp T\xi\|$. Without loss of generality, it may be supposed that $\mathcal{M}$ is the cyclic subspace generated by ξ. Hence there is a wandering vector ζ so that $\mathcal{M} = \mathfrak{L}_n[\zeta]$. Therefore $P_{\mathcal{M}} = R_\zeta R_\zeta^*$ and

$$\|\delta_T|_{\mathfrak{R}_n}\| \geq \|TR_\zeta - R_\zeta T\| \geq \|P_{\mathcal{M}}^\perp (TR_\zeta - R_\zeta T)R_\zeta^*\|$$
$$= \|P_{\mathcal{M}}^\perp T P_{\mathcal{M}}\| \geq \|P_{\mathcal{M}}^\perp T\xi\|.$$

It follows that $\|\delta_T|_{\mathfrak{R}_n}\| \geq \beta$, which establishes the second inequality.

The third inequality is standard. If A is in $\mathfrak{L}_n$ and R belongs to $\mathfrak{R}_n$,

$$\|\delta_T(R)\| \leq \|(T - A)R - R(T - A)\| \leq 2\|R\|\,\|T - A\|.$$

Minimizing over A in $\mathfrak{L}_n$ shows that $\|\delta_T|_{\mathfrak{R}_n}\| \leq 2\,\mathrm{dist}(T, \mathfrak{L}_n)$. ∎

In the case $n = \infty$, the restriction to $\{\xi_1\}^\perp$ is already infinite multiplicity and thus has distance constant at most 9. So a review of the argument shows that the algebra $\mathfrak{L}_\infty$ has a distance constant of at most $18\sqrt{2}$.

We also show that $\mathfrak{L}_n$ has the property $\mathbb{A}_1$, namely that every weak-$*$ continuous linear functional f on $\mathfrak{L}_n$ is given by a rank one operator acting as a functional on $\mathcal{B}(\mathcal{H})$. This means, in particular, that every weak-$*$ continuous linear functional is WOT–continuous. Hence these two topologies coincide on $\mathfrak{L}_n$.

Theorem 2.10. *Suppose that f is a weak-$*$ continuous linear functional on $\mathfrak{L}_n$ for $n \geq 2$ with $\|f\| < 1$. Then there are vectors ξ and ζ with $\|\xi\| \, \|\zeta\| < 1$ such that $f(A) = (A\xi, \zeta)$ for all A in $\mathfrak{L}_n$.*

Proof. This follows immediately from the fact that there are countably many pairwise orthogonal cyclic subspaces $\mathfrak{L}_n[\nu_k]$ such that the restriction of $\mathfrak{L}_n$ to each $\mathfrak{L}_n[\nu_k]$ is canonically unitarily equivalent to $\mathfrak{L}_n$. Given f with $\|f\| < 1$, there is (by the Hahn–Banach Theorem) a trace class operator K with $\|K\|_1 < 1$ so that $f(A) = \mathrm{Tr}(AK)$. The singular decomposition of K yields

$$K = \sum_{k \geq 1} s_k \xi_k \zeta_k^*$$

where ξ_k and ζ_k are unit vectors and $s_k \geq 0$ such that $\sum_k s_k < 1$. Let

$$\xi := \sum_k s_k^{1/2} R_{\nu_k} \xi_k \qquad \text{and} \qquad \zeta := \sum_k s_k^{1/2} R_{\nu_k} \zeta_k.$$

Then $\|\xi\| = \|\zeta\| < 1$ and by part (iii) of Theorem 1.2,

$$(A\xi, \zeta) = \sum_k s_k (A R_{\nu_k} \xi_k, R_{\nu_k} \zeta_k) = \sum_k s_k (A\xi_k, \zeta_k) = f(A). \qquad \blacksquare$$

An immediate consequence of [11] or [18] for hyper-reflexivity is:

Corollary 2.11. *Every WOT–closed unital subalgebra of $\mathfrak{L}_n$ is hyper-reflexive with constant at most* 103.

Another immediate consequence that will prove to be important in [13] is:

Corollary 2.12. *The weak-$*$ and WOT topologies on $\mathfrak{L}_n$ coincide.*

3. Atomic Free Semigroup Algebras

In this section, we will examine a more general class of isometric representations of the free semigroup. Say that an n-tuple of isometries $S = (S_1, \ldots, S_n)$ is *free atomic* if $\sum_{i=1}^n S_i S_i^* \leq I$ and there is an orthonormal basis $\{\xi_k\}$ for $\mathcal{H}$ for which there are endomorphisms $\pi_i : \mathbb{N} \to \mathbb{N}$ (where $1 \leq i \leq n$) and scalars $\lambda_{i,k} \in \mathbb{T}$ satisfying $S_i \xi_k = \lambda_{i,k} \xi_{\pi_i(k)}$. We call the corresponding representation of $\mathcal{F}_n$ *atomic* as well. Equivalently, this says that there is an atomic masa containing all the range projections

$P_w = w(S)w(S)^*$. A *free atomic semigroup algebra* will be the WOT–closed algebra $\mathfrak{A} = \mathrm{Alg}(S_1, \ldots, S_n)$ generated by a set of free atomic isometries.

There is a connection between these representations and the permutation representations of $\mathcal{O}_n$ recently introduced and studied by Bratteli and Jorgensen in [5]. Permutation representations are a subclass of atomic representations: they are the same as above except that all scalars $\lambda_{i,k} = 1$ and $\sum_{i} = 1^n S_i S_i^* = I$. Bratteli and Jorgensen were are interested in decomposing permutation representations into irreducible representations. However, the condition that $\lambda_{i,k} = 1$ forced them to make certain restrictive assumptions. In general, to obtain a decomposition into irreducible representations, arbitrary scalars are needed as we shall see.

It turns out that atomic free semigroup algebras have a very nice structure theory. We show that the irreducible atomic representations of $\mathcal{F}_n$ can be completely classified up to unitary equivalence. This will then allow us to determine the invariant subspace structure. Every free atomic semigroup algebra will be shown to be hyper-reflexive.

Classifying all free semigroup algebras up to unitary equivalence is essentially equivalent to classifying all representations of the C*-algebra $\mathcal{O}_n$ up to unitary equivalence. However, this C*-algebra is NGCR, meaning that it has no type I quotients. (Indeed, it is simple.) Such C*-algebras do not have a nice representation space in the sense [16] that there is no countable collection of Borel functions that distinguish the unitary invariants. Hence it is perhaps surprising that this class of atomic representations has such a nice classification.

There are three classes of atomic isometric representations of $\mathcal{F}_n$ which we will now describe.

(i) The left regular representation λ.

(ii) Let $x = z_{i_1} z_{i_2} \ldots z_{i_m} \ldots$ be an infinite word in the generators of $\mathcal{F}_n$; and define the sequence

$$x_m = z_{i_1} z_{i_2} \ldots z_{i_m} \quad \text{for} \quad m \geq 0.$$

Let $\mathcal{F}_n x^{-1}$ denote the collection of words in the free group on n generators of the form $v = u x_m^{-1}$ for u in $\mathcal{F}_n$ and some $m \geq 0$. Identify words which are the same after cancellation, namely $u x_m^{-1} = (u x_{i_{m+1}}) x_{m+1}^{-1}$. Let $\mathcal{H}_x$ be the Hilbert space with orthonormal basis $\{\xi_v : v \in \mathcal{F}_n x^{-1}\}$. Define a representation π_x of the free semigroup $\mathcal{F}_n$ on $\mathcal{H}_x$ by

$$\pi_x(z_i)\xi_v = \xi_{z_i v} \quad \text{for} \quad v \in \mathcal{F}_n x^{-1}.$$

Two words x and $x' = z_{j_1} z_{j_2} \ldots z_{j_m} \ldots$ are said to be *tail equivalent* if there are integers k and ℓ so that $i_{m+k} = j_{m+\ell}$ for all $m \geq 0$. Let $[x]$ denote the tail equivalence class of x. When x and x' are tail equivalent, there is a unitary operator $U : \mathcal{H}_x \to \mathcal{H}_{x'}$ given by $U\xi_{vx_{m+k}^{-1}} = \xi'_{vx'^{-1}_{m+\ell}}$ for v in $\mathcal{F}_n$ and $m \geq 0$. It is easy to check that U is a well defined bijection between the bases which intertwines the two representations. Thus U extends to a unitary

operator which implements the unitary equivalence of the representations π_x and $\pi_{x'}$.

These representations are irreducible except when $[w]$ is periodic. The structure for the periodic case will be described later. Two representations π_x and $\pi_{x'}$ are unitarily equivalent if and only if x and x' are tail equivalent.

(iii) Let $u = z_{i_1} \ldots z_{i_k}$ be a non-trivial word in $\mathcal{F}_n$ and let λ be a scalar of modulus one. The idea behind the following construction is to create a representation of $\mathcal{F}_n$ under which the word u maps to an isometry which has λ as an eigenvalue.

Let $\mathcal{K}_u$ be the Hilbert space with orthonormal basis,

$$\{\xi_{s,w} : 1 \le s \le k \text{ and } w \in \mathcal{F}_n \setminus \mathcal{F}_n z_{i_s}\}.$$

Define a representation $\sigma_{u,\lambda}$ of $\mathcal{F}_n$ by

$$\sigma_{u,\lambda}(z_i)\xi_{s,1} = \xi_{s-1,1} \quad \text{if} \quad i = i_s,\ s > 1$$
$$\sigma_{u,\lambda}(z_i)\xi_{1,1} = \lambda\xi_{k,1} \quad \text{if} \quad i = i_1$$
$$\sigma_{u,\lambda}(z_i)\xi_{s,1} = \xi_{s,z_i} \quad \text{if} \quad i \ne i_s$$
$$\sigma_{u,\lambda}(z_i)\xi_{s,w} = \xi_{s,z_i w} \quad \text{if} \quad w \ne 1$$

Notice that $\sigma_{u,\lambda}(u)\xi_{k,1} = \lambda\xi_{k,1}$. The word u will be called the *central generator* for this representation; and the sequence $\xi_{s,1}$ for $1 \le s \le k$ of basis vectors which are cyclically permuted will be called the *central cycle*. Each basis vector $\xi_{s,1}$ is mapped by $\sigma_{u,\lambda}(z_{i_s})$ to the next vector in the cycle, $\xi_{s-1,1}$; but when $j \ne i_s$, $\sigma_{u,\lambda}(z_j)\xi_{s,1} = \xi_{s,z_j}$. This is a wandering vector which sweeps out a copy of the left regular representation. So this representation consists of a central ring of k nodes, and from each node, there are $n - 1$ 'spokes' equivalent to the left regular representation.

It is evident that if the terms of u are cyclically rotated, one obtains an equivalent representation. Let $\langle u \rangle$ denote the equivalence class of u up to cyclic permutation.

The word u is said to be *primitive* if it is not the power of a smaller word. It turns out that $\sigma_{u,\lambda}$ is irreducible precisely when u is primitive.

Example 3.1. Let $\mathcal{H} = \ell^2(\mathbb{N})$. Define $S_1\xi_n = \xi_{2n-1}$ and $S_2\xi_n = \xi_{2n}$. Then $S_1\xi_1 = \xi_1$. Clearly, ξ_1 is a cyclic vector for (S_1, S_2). This is a representation of $\mathfrak{L}_2$ of type $\sigma_{z_1,1}$.

Example 3.2. Let $\mathcal{H} = \ell^2(\mathbb{Z} \times \mathbb{N}_0)$. Define two isometries in $\mathcal{B}(\mathcal{H})$ by $S_1\xi_{k,n} = \xi_{k,2n+1}$, $S_2\xi_{k,0} = \xi_{k+1,0}$ and $S_2\xi_{k,n} = \xi_{k,2n}$ for $K \in \mathbb{Z}$ and $n > 0$. In this case, every basis vector is a wandering vector. This example is a representation of $\mathfrak{L}_2$ equivalent to $\pi_{z_2^\infty}$.

Example 3.3. Consider the $n = 1$ case. The left regular representation is just the unilateral shift since $\mathbb{N}$ acts on $\ell_2(\mathbb{N})$ by translation. The only infinite word is z_1^∞. This yields the shift on $\ell_2(\mathbb{Z})$, namely the bilateral shift. The only primitive word is z_1. The representations $\sigma_{z_1,\lambda}$ are the one–dimensional representations as scalars. From the spectral representation of

the bilateral shift as multiplication by z on $L^2(\mathbb{T})$, it follows that $\pi_{z_1^\infty}$ is the direct integral of the $\sigma_{z_1,\lambda}$'s with respect to Lebesgue measure. Finally, the general word is z_1^k. The representation $\sigma_{z_1^k,\lambda}$ is a cyclic weighted shift on k–dimensional space with all weights equal to 1 except one which is λ. This is unitary, and by the spectral theorem, it is diagonalizable with eigenvalues equal to the k-th roots of λ, say $\mu_1, \ldots, \mu_k$. Thus $\sigma_{z_1^k,\lambda} \simeq \sum_{i=1}^k \oplus \sigma_{z_1,\mu_i}$.

The main result of this section can now be stated:

Theorem 3.4. *Every representation of the free semigroup as a free atomic semigroup algebra is unitarily equivalent to a direct integral of representations which are each unitarily equivalent to one of*

- (i) *the left regular representation λ*
- (ii) *π_x corresponding to an aperiodic infinite word x which is unique up to tail equivalence, or*
- (iii) *$\sigma_{u,\lambda}$ for a primitive central generator u, which is unique up to a cyclical permutation, and a constant λ in $\mathbb{T}$.*

This decomposition is canonical.

We begin with an elementary lemma. For the rest of this section, let $\mathfrak{A}$ be an atomic free semigroup algebra $\mathrm{Alg}(S_1, \ldots, S_n)$, and let $\{\xi_n\}$ be a standard basis on which it acts.

Lemma 3.5. *If ξ_m and ξ_n are basis vectors, then either $\overline{\mathfrak{A}\xi_m}$ and $\overline{\mathfrak{A}\xi_n}$ are orthogonal or one contains the other, in which case there is a word w and scalar λ in $\mathbb{T}$ so that $w(S)\xi_m = \lambda\xi_n$ or vice versa.*

Proof. If there is an intersection, it is spanned by standard basis vectors; and thus there are words v and w so that

$$v(S)\xi_m = \lambda w(S)\xi_n.$$

If both v and w are non-trivial, they must have the same initial term because the ranges of distinct S_i are orthogonal. Cancelling off terms from the left, we obtain a minimal pair (v', w') such that $v'(S)\xi_m = \lambda w'(S)\xi_n$. It follows that one of v' or w' is equal to 1. ∎

Corollary 3.6. *A standard basis vector ξ is either a wandering vector for $\mathfrak{A}$ (so that $\{w(S)\xi : w \in \mathcal{F}_n\}$ is an orthonormal set), or there is a word $u \neq 1$ and scalar λ in $\mathbb{T}$ such that $u(S)\xi = \lambda\xi$. The minimal choice of u is unique, say $u = z_{i_1} \ldots z_{i_k}$. For any $j \neq i_k$, $S_j\xi$ is a wandering vector.*

Proof. If ξ is not a wandering vector, then the definition of atomic free semigroup algebra shows that there are distinct words v and w and a scalar λ in $\mathbb{T}$ such that $v(S)\xi = \lambda w(S)\xi$. Apply the lemma and use cancellation to obtain a word $u = z_{i_1} \ldots z_{i_k}$ so that $u(S)\xi = \lambda\xi$. Moreover, if u and u' are two words with this property, then one divides the other. Hence the minimal choice of the word u is unique; and any other choice is a power of this minimal word.

Fix j with $1 \leq j \leq k$ and set $\zeta = S_j \xi$. If ζ is not a wandering vector, then there is a scalar μ in $\mathbb{T}$ and a word w so that $w(S)\zeta = \mu\zeta$. But then $(wz_j)(S)\xi = \mu S_j \xi$. Therefore, since the ranges of the S_i's are orthogonal, we find that $w = z_j w'$ and so $(w'z_j)(S)\xi = \mu\xi$. Hence $w'z_j$ is a power of u. But this means that $j = i_k$. ∎

Let $W = \{\xi_n : \xi_n \text{ is wandering}\}$. Then $\mathcal{W} = \operatorname{span}(W)$ is invariant for $\mathfrak{A}$. This is because Corollary 3.6 shows that when ξ_n belongs to W, then $w(S)\xi_n$ is also in W for every word w in $\mathcal{F}_n$.

Lemma 3.7. *For each primitive word u and scalar λ in $\mathbb{T}$, the projection $P_{u,\lambda}$ onto the subspace $\{\xi : u(S)\xi = \lambda\xi\}$ belongs to $\mathfrak{A}$.*

Proof. We will show that

$$P_{u,\lambda} = \operatorname{SOT\!-\!lim}_{m \to \infty} p_m(\overline{\lambda}u(S))$$

where $p_m(x) = \frac{1}{m!}\sum_{j=1}^{m!} x^{m!+j}$. To verify convergence, it is enough to check each standard basis vector because the sequence is bounded. If ξ is a standard basis vector such that $u^k(S)\xi$ is never a multiple of ξ, then by Corollary 3.6, $u^k(S)\xi$ is eventually a wandering vector. Hence large powers $u^k(S)\xi$ are pairwise orthogonal, and consequently their average $p_m(\overline{\lambda}u(S))\xi$ converges to 0 in norm.

Next suppose ξ is a unit vector such that $u(S)^q\xi = \mu\xi$ for some positive integer q and scalar μ in $\mathbb{T}$. Assume q is the least such integer, and let

$$M = \operatorname{span}\{u(S)^j\xi : 1 \leq j \leq q\}.$$

Clearly M reduces $u(S)$. Let α_j, $1 \leq j \leq q$ be the qth roots of μ. The vectors

$$y_j = \frac{1}{\sqrt{q}}\sum_{k=1}^{q}\overline{\alpha}_j^k u(S)^k\xi$$

satisfy $u(S)y_j = \alpha_j y_j$ and are also an orthonormal basis for M. Hence $u(S)|_M$ is diagonal with respect to this basis. Also $\sum_{j=1}^{q} y_j = \sqrt{q}\xi$.

Evidently, $p_m(1) = 1$ for every $m \geq 1$, and

$$p_m(x) = \frac{x}{1-x}\left[\frac{x^{m!} - (x^{m!})^2}{m!}\right], \quad \text{for} \quad x \neq 1.$$

So $\lim p_m(\tau) = 0$ for all τ in $\mathbb{T}$, $\tau \neq 1$. Therefore,

$$(7) \qquad \lim_{m\to\infty} p_m(\overline{\lambda}u(S))\xi = \begin{cases} 0 & \text{if } \lambda^q \neq \mu \\ (\xi, y_j)y_j = \frac{1}{\sqrt{q}}y_j & \text{if } \lambda = \alpha_j,\ 1 \leq j \leq q. \end{cases}$$

Hence $P_{u,\lambda}$ is defined in $\mathfrak{A}$ as a SOT limit.

Next observe that when $u(S)\xi = \lambda\xi$, then $p_m(\overline{\lambda}u(S))\xi = \xi$ for all m and thus $P_{u,\lambda}\xi = \xi$. Conversely, if $P_{u,\lambda}\zeta = \xi$, then

$$
\begin{aligned}
u(S)\xi &= \lim_{m\to\infty} u(S)\frac{1}{m!}\sum_{t=1}^{m!}\left(\overline{\lambda}u(S)\right)^{m!+t}\zeta \\
&= \lim_{m\to\infty} \lambda\frac{1}{m!}\sum_{t=1}^{m!}\left(\overline{\lambda}u(S)\right)^{m!+t+1}\zeta \\
&= \lambda\xi + \lim_{m\to\infty}\frac{1}{m!}\left((\overline{\lambda}u(S))^{2n!} - I\right)u(S)\zeta = \lambda\xi.
\end{aligned}
$$

Thus $P_{u,\lambda}$ is a projection with the desired range. ∎

Corollary 3.8. *If u is a primitive word and q is the least positive integer such that $u(S)^q\xi$ is a multiple $\mu\xi$ of ξ, then $\xi = \sum_{i=1}^{q} P_{u,\lambda_i}\xi$ where λ_i are the q-th roots of μ for $1 \le i \le q$.*

Proof. Equation (7) in the proof of Lemma 3.7 yields $P_{u,\lambda_j}\xi = \frac{1}{\sqrt{q}}\,y_j$. Since $\xi = \frac{1}{\sqrt{q}}\sum_{i=1}^{q} y_j$, the result follows. ∎

Corollary 3.9. *The projection P onto $\mathcal{W}^\perp$ lies in $\mathfrak{A}$.*

Proof. We will show that P equals $Q := \sum P_{u,\lambda}$ as this sum runs over all primitive words u and all scalars of modulus 1. This sum is SOT–convergent because the projections $P_{u,\lambda}$ are pair-wise orthogonal projections; and hence Q belongs to $\mathfrak{A}$. A basis vector ξ is either wandering, or satisfies $v(S)\xi = \lambda\xi$ for some non-trivial word v and scalar λ in $\mathbb{T}$ by Corollary 3.6. Clearly, $Q\xi = 0$ for every wandering vector ξ. In the latter case, Corollary 3.8 shows that ξ lies in the range of Q. Hence $Q = P$ as claimed. ∎

Now we can decide which of the atomic representations of types (i)–(iii) are irreducible, by which we mean that the image does not commute with any proper projection.

Proposition 3.10. *Consider the representations of type (i)–(iii).*

  (i) *The left regular representation λ is irreducible.*
  (ii) *The representation π_x is irreducible except when x is tail equivalent to a periodic word. In this case, there is a primitive word u so that x is equivalent to the infinite repetition of u. Then π_x is unitarily equivalent to the direct integral $\int_{\mathbb{T}} \oplus\sigma_{u,\lambda}\,d\lambda$ with respect to normalized Lebesgue measure.*
  (iii) *The representation $\sigma_{u,\lambda}$ is irreducible when u is primitive.*
       When $u = v^r$ and v is primitive, then $\sigma_u \simeq \sum_{j=1}^{r} \oplus\sigma_{v,\mu_j}$ where μ_j are the r-th roots of λ.

Proof. First consider the left regular representation λ. The rank one projection $\xi_1\xi_1^* = I - \sum_i S_i S_i^*$ lies in $\mathrm{C}^*(\mathfrak{L}_n)$. Thus if Q is a reducing projection for $\mathfrak{L}_n$, it must commute with $\xi_1\xi_1^*$. By replacing Q by $Q^\perp$ if necessary,

we may assume that ξ_1 lies in the range of Q. However, ξ_1 is cyclic and therefore $Q = I$. Hence λ is irreducible.

Next consider the case (iii) representations $\sigma_{u,\lambda}$ when u is primitive. The projection $P = P_{u,\lambda}$ is the rank one projection onto the span of $\xi_{k,1}$, which is a cyclic vector. So as in case (i), this representation is irreducible.

When $u = v^r$ and $|v| = k$, decompose the space $\mathcal{K}_u$ as follows. Let $\mathcal{K}_v^{(r)}$ be the direct sum of r copies of $\mathcal{K}_v$ with basis $\xi_{s,w}^j$ for $0 \le j < r$, $1 \le s \le k$ and w in $\mathcal{F}_n \setminus \mathcal{F}_n z_{i_s}$. There is a unitary operator W from $\mathcal{K}_v^{(r)}$ onto $\mathcal{K}_u$ given by $W\xi_{s,w}^j = \xi_{s+kj,w}$. Consider $\sigma = W^* \sigma_{u,\lambda} W$. Then a calculation shows that

$$\sigma(z_i)\xi_{1,1}^0 = \lambda \xi_{k,1}^{r-1} \quad \text{if} \quad i = i_1$$
$$\sigma(z_i)\xi_{1,1}^j = \xi_{k,1}^{j-1} \quad \text{if} \quad i = i_1 \text{ and } j > 0$$
$$\sigma(z_i)\xi_{s,1}^j = \xi_{s-1,1}^j \quad \text{if} \quad i = i_s, \text{ and } s > 1$$
$$\sigma(z_i)\xi_{s,1}^j = \xi_{s,z_i}^j \quad \text{if} \quad i \ne i_s$$
$$\sigma(z_i)\xi_{s,w}^j = \xi_{s,z_i w}^j \quad \text{if} \quad w \ne 1$$

Let $\mathcal{E}_{s,w} = \text{span}\{\xi_{s,w}^j : 0 \le j < r\}$ for $1 \le s \le k$ and $w \in \mathcal{F}_n \setminus \mathcal{F}_n z_{i_s}$. Thus we see that $\sigma(v)$ maps the subspace $\mathcal{E}_{1,1}$ onto itself via the unitary U which is the weighted shift given by $U\xi_{1,1}^j = \xi_{1,1}^{j-1}$ for $2 \le j \le k$ and $U\xi_{1,1}^1 = \lambda \xi_{1,1}^{r-1}$. The spectrum of U is the set of r-th roots of λ, given by μ_j for $0 \le j < r$.

Diagonalize U with respect to a new basis $\zeta_{1,1}^j$ for $0 \le j < r$ so that $U\zeta_{1,1}^j = \mu_j \zeta_{1,1}^j$. Then, using the fact that $\sigma(z_{i_s})\mathcal{E}_{s,1} = \mathcal{E}_{s-1,1}$ for $1 < s \le k$, define a basis $\zeta_{s,w}^j$ for $\mathcal{K}_v^{(r)}$ by setting $\zeta_{s-1,1}^j = \sigma(z_{i_s})\zeta_{s,1}^j$ for $0 \le j < r$ and $1 < s \le k$; and let $\zeta_{s,w}^j = \sigma(w)\zeta_{s,1}^j$ for $0 \le j < r$, $1 \le s \le k$ and w in $\mathcal{F}_n \setminus \mathcal{F}_n z_{i_s}$. Then it follows that

$$\sigma(z_{i_1})\zeta_{1,1}^j = \mu_j \zeta_{k,1}^j.$$

Thus this exhibits a unitary equivalence between σ and $\sum_{j=0}^{r-1} \oplus \sigma_{v,\mu_j}$.

Now turn to case (ii) when $[x]$ is aperiodic. Consider the standard basis vector ξ_1. We claim that the rank one projection $\xi_1 \xi_1^*$ belongs to the von Neumann algebra $W^*(\pi_x(z_1), \dots, \pi_x(z_n))$. Indeed, $P_m = x_m(S)x_m(S)^*$ is the range projection onto the span of those basis vectors ξ_w for w in $x_m \mathcal{F}_n x^{-1}$. This is a decreasing sequence, and thus the SOT–limit of the P_m's is a projection P. For each word w and positive integer k, there is a unique word w_k of length k so that ξ_w lies in the range of $w_k(S)$. This determines an infinite word which is easily seen to be wx, and w_k consists of the first k terms of this (reduced) word. Since x is aperiodic, wx and x are different except for $w = 1$. Hence for some sufficiently large k, $w_k \ne x_k$ and so $P_k \xi_w = 0$. Therefore $P = \xi_1 \xi_1^*$. This vector is clearly cyclic for $W^*(\pi_x(z_1), \dots, \pi_x(z_n))$, and hence π_x is irreducible.

Finally, suppose that x is periodic, being the infinite product of a primitive word $u = z_{i_1} \dots z_{i_k}$. Denote a basis for $\mathcal{K}_u \otimes \ell_2(\mathbb{Z})$ by $\xi_{s,w}^j$ for j in $\mathbb{Z}$, $1 \le s \le k$

and w in $\mathcal{F}_n \setminus \mathcal{F}_n z_{i_s}$. Define a unitary W from this space onto $\mathcal{H}_x$ by

$$W\xi_{s,w}^j = \xi_{wx_s^{-1}u^j} \quad \text{for} \quad j \in \mathbb{Z},\ 1 \le s \le k,\ w \in \mathcal{F}_n \setminus \mathcal{F}_n z_{i_{k-s}}$$

where we adopt the convention that $x_0 = 1$ and $x_{-t}^{-1} = z_{i_t} \dots z_{i_1}$ for $t \ge 1$.
Then a calculation shows that $\sigma = W^*\pi_x W$ is given by

$$\sigma(z_i)\xi_{s,1}^j = \xi_{s-1,1}^j \quad \text{if} \quad i = i_s,\ s > 1$$

$$\sigma(z_i)\xi_{1,1}^j = \xi_{k,1}^{j-1} \quad \text{if} \quad i = i_1$$

$$\sigma(z_i)\xi_{s,1}^j = \xi_{s,z_i}^j \quad \text{if} \quad i \ne i_s$$

$$\sigma(z_i)\xi_{s,w}^j = \xi_{s,z_i w}^j \quad \text{if} \quad w \ne 1.$$

The compression of $\sigma(u)$ to $\mathcal{E}_{1,1} = \mathrm{span}\{\xi_{1,1}^j : j \in \mathbb{Z}\}$ is the unitary bilateral
shift $U\xi_{1,1}^j = \xi_{1,1}^{j-1}$. This operator is unitarily equivalent to multiplication
by z on $L^2(\mathbb{T})$. So as in the case (iii) above, we see that σ decomposes as a
direct integral of $\sigma_{u,\lambda}$'s over $\mathbb{T}$ with respect to Lebesgue measure. ∎

Proof of Theorem 3.4. Decompose the space into minimal subspaces
spanned by standard basis vectors which are reducing for $\mathfrak{A}$. The subspace
containing a basis vector ξ will contain its image under all words in the S_i's
and S_j^*'s. However, because of the relation (F), one need only consider words
of the form $u(S)v(S)^*$. (Note that $v(S)^*\xi$ is either a non-zero multiple of
another standard basis vector or it is 0.) For the rest of the argument, we
may assume that the whole space is a minimal diagonally reducing subspace
in this sense.

Let $W = \{\xi_n : \xi_n \text{ is wandering}\}$. Then $\mathcal{W} = \mathrm{span}(W)$ is invariant for
$\mathfrak{A}$. Suppose that $\mathcal{W} = \mathcal{H}$. Applying Lemma 3.5, we see that either $\mathfrak{A}$ has a
cyclic wandering vector ξ, and so is unitarily equivalent to the left regular
representation; or every ξ in W is a modulus-one scalar multiple of the image
of another member of W under some S_i. Starting with any ξ in W, there is
a unique i_1 so that ξ belongs to $S_{i_1}\mathcal{H}$. Hence there is a unique sequence

$$x_m = z_{i_1}z_{i_2}\dots z_{i_m}$$

for which there is an ζ_m in W and scalar λ_m so that $\xi = \lambda_m x_m(S)\zeta_m$. Let x
be the infinite word $x = z_{i_1}z_{i_2}\dots z_{i_m}\dots$. Given another element ξ' of W, let
$x' = z_{j_1}z_{j_2}\dots z_{j_m}\dots$ be the infinite word determined by ξ'. The diagonal
irreducibility of W means that there is a word uv^* so that $\xi = u(S)v(S)^*\xi'$.
It is then evident that $u = x_k$ and $v = x'_\ell$ for certain integers k and ℓ. Hence
$\zeta = u(S)^*\xi = v(S)^*\xi'$ is an element of W; and the two sequences x and x'
have the same tails $i_{m+k} = j_{m+\ell}$ for all $m \ge 0$. This equivalence class $[x]$ is
therefore independent of the choice of basis vector.

For each word w in $\mathcal{F}_n x^{-1}$, say $w = vx_m^{-1}$, define $\xi_w = v(S)x_m(S)^*\xi$. This
is always a modulus one multiple of a standard basis vector. The diagonal
irreducibility shows that every basis vector is achieved in this way. And
because each basis vector is wandering, this map is also one-to-one. So we
have a new basis for our space. With respect to this basis, it is evident that

$u(S)\xi_w = \xi_{uw}$. So this shows that the representation is unitarily equivalent to π_x.

In fact, the class $[x]$ is a unitary invariant of the representation. To see this, consider a wandering vector $\zeta = \sum_w a_w \xi_w$ in $\mathcal{H}_x$ with the property that there is an infinite word x' so that for each $m \geq 1$, there is a vector ζ_m in $\mathcal{H}_x$ so that $x'_m(S)\zeta_m = \zeta$. Then this infinite word coincides with the infinite word associated to any basis vector ξ_w for which $a_w \neq 0$. Hence the class $[x]$ is determined intrinsically from the representation independent of the choice of basis; and thus is a unitary invariant.

Now suppose that $\mathcal{W} \neq \mathcal{H}$, and that $\mathcal{H}$ is diagonally irreducible. Then by Corollary 3.6, there is a minimal word $u = z_{i_1} \ldots z_{i_k} \neq 1$, a scalar λ in $\mathbb{T}$ and a standard basis vector ξ such that $u(S)\xi = \lambda\xi$. We will show that π is unitarily equivalent to $\sigma_{u,\lambda}$. Let $\xi_{k,1} = \xi$ and $\xi_{s-1,1} = S_{i_s}\xi_{s,1}$ for $2 \leq s \leq k$. Note that $S_{i_1}\xi_{1,1} = \lambda\xi_{k,1}$. Hence if $u_s = z_{i_{s+1}} \ldots z_{i_k} z_{i_1} \ldots z_{i_s}$ is the word obtained from u by cyclical permutation of the letters, it follows that

(8) $$u_s(S)\xi_{s,1} = \lambda\xi_{s,1} \quad \text{for} \quad 1 \leq s \leq k.$$

Now define $\xi_{s,w} = w(S)\xi_{s,1}$ for all w in $\mathcal{F}_n \setminus \mathcal{F}_n z_{i_s}$. Since $\xi_{s,w} = v(S)\xi$ for some word v in $\mathcal{F}_n$, we see that $\xi_{s,w}$ is just a modulus one multiple of a standard basis vector. Also, note that by Corollary 3.6 and equation (8), each ξ_{s,z_i} for $i \neq i_s$ is a wandering vector. Moreover, the set

$$\{\xi_{s,w} : 1 \leq s \leq k, \ w \in \mathcal{F}_n \setminus \mathcal{F}_n z_{i_s}\}$$

spans a diagonally irreducible subspace because it is mapped *onto* itself by the n isometries S_i. Therefore it is a basis for $\mathcal{H}$. It is evident that this is unitarily equivalent to $\sigma_{\lambda,u}$.

It is clear that a cyclical permutation of u does not change the unitary equivalence class of the representation. Suppose that $u = v^s$ where v is primitive. Let $k = |v|$, and let v_i denote the k cyclic permutations of v; and let μ_j for $1 \leq j \leq s$ be the s-th roots of λ. To see that u up to permutation and λ are unitary invariants, consider the space $\mathcal{W}^\perp$. This is the complement of the span of all wandering vectors in $\mathcal{H}$. By Lemma 3.7 and Corollary 3.8, this subspace is the range of $P_{\langle u \rangle, \lambda} := \sum_{i=1}^k \sum_{j=1}^s P_{v_i, \mu_j}$. Moreover, for any other primitive word v', the projection $P_{v', \mu} = 0$; as is $P_{v_i, \mu} = 0$ when $\mu^s \neq \lambda$. Hence the v_i's and μ_j's are uniquely determined by the representation. Since $u_i = v_i^s$ are the cyclic permutations of u and $\lambda = \mu_i^k$ where $k = |v_i|$ and s is the number of roots, it follows that $\langle u \rangle$ and λ can be recovered from the representation.

Proposition 3.10 shows that when u is a higher power of a primitive word, then $\sigma_{u,\lambda}$ decomposes as the direct sum $\sum_{j=1}^s \oplus \sigma_{v,\mu_j}$ which are irreducible.

Finally, we show that this decomposition is canonical. The projection P onto $\mathcal{W}^\perp$ can be used as above to determine the summands. For each primitive word v and scalar λ in $\mathbb{T}$, the multiplicity of the representation $\sigma_{v,\lambda}$ in an atomic representation σ is given by the rank of $P_{v,\lambda}$. Of course, only one of the cyclic permutations of a primitive word should be used.

The multiplicity of the left regular representation is given by the rank of $I - \sum_{i=1}^{n} S_i S_i^*$. If x is an aperiodic infinite word, then the rank of the projection $P_x = \text{SOT–}\lim_{n\to\infty} x_n(S) x_n(S)^*$ determines the multiplicity of π_x (compare with the proof of Proposition 3.10). Finally, for each primitive word u, consider the periodic word $x = u^\infty$. The range projections $P_n = P^\perp u^n(S) u^n(S)^*$ decrease to the projection P_u onto those vectors in $\mathcal{W}$ which are in the range of $u^n(S)$ for all $n \geq 1$. The multiplicity of π_x equals the multiplicity of the bilateral shift $P_u u(S)$. By Proposition 3.10, π_x decomposes uniquely as a direct integral of the irreducible representations $\sigma_{u,\lambda}$'s.   ■

We collect the more precise information that came out in the proof as a corollary.

Corollary 3.11. *Every atomic representation ρ of the free semigroup $\mathcal{F}_n$ may be decomposed uniquely as*

$$\rho \simeq \lambda^{(\alpha)} \oplus \sum \oplus \pi_{[x]}^{(\beta_{[x]})} \oplus \sum \oplus \sigma_{\langle u\rangle,\lambda}^{(\gamma_{\langle u\rangle,\lambda})}$$

where $\alpha = \text{rank}(I - \sum_{i=1}^{n} \rho(z_i)\rho(z_i)^*)$; for each infinite aperiodic word x with tail equivalence class $[x]$,

$$\beta_{[x]} = \text{rank}(P_x) \quad \text{where} \quad P_x = \text{SOT–}\lim_{m\to\infty} \rho(x_m)\rho(x_m)^*;$$

when x is (equivalent to) a periodic word u^∞, $\beta_{[x]}$ is the multiplicity of the bilateral shift $P_x\rho(u)$; and for each primitive word u with cyclic permutation class $\langle u\rangle$, $\gamma_{\langle u\rangle,\lambda} = \text{rank}(P_{u,\lambda})$ where

$$P_{u,\lambda} = \text{SOT–}\lim_{m\to\infty} p_m(\overline{\lambda}\rho(u)) \quad \text{and} \quad p_m(x) = \frac{1}{m!}\sum_{j=1}^{m!} x^{m!+j}.$$

In order to understand the invariant subspace structure of free atomic algebras, we now study the algebra $P\mathfrak{A}P$ in detail. As above, P is the projection onto the orthogonal complement of

$$\mathcal{W} = \text{span}\{\xi_n : \xi_n \text{ is a wandering standard basis vector}\}.$$

Lemma 3.12. *Suppose that u in $\mathcal{F}_n$ is a primitive word of length k, λ is in $\mathbb{T}$ and let $\mathfrak{A}_{u,\lambda}$ be the WOT–closed algebra generated by the representation $\sigma_{u,\lambda}$ of $\mathcal{F}_n$. Then*

$$P\mathfrak{A}_{u,\lambda}P \simeq \mathcal{M}_k \qquad \text{and} \qquad \mathfrak{A}_{u,\lambda}P = \mathcal{B}(\mathcal{K}_u)P.$$

In particular, a vector ξ in $\mathcal{K}_u$ is cyclic for $\mathfrak{A}_{u,\lambda}$ if and only if $P\xi \neq 0$.

Proof. The projection $P_{u,\lambda}$ is the rank one projection onto the span of the basis vector $\xi_{k,1}$. The operator $U = \sum_{i=1}^{n} PS_iP$ is a unitary on $P\mathcal{K}_u$ which cyclically shifts the basis vectors $\xi_{s,1}$ for $1 \leq s \leq k$. Clearly these two operators generate $\mathcal{B}(P\mathcal{K}_u) \simeq \mathcal{M}_k$.

The vector $\xi_{1,1}$ is a cyclic vector for $\mathfrak{A}_{u,\lambda}$. Thus every vector ξ such that $P\xi \neq 0$ is cyclic since the previous paragraph shows that $\xi_{1,1}(P\xi)^*$ belongs

to $\mathfrak{A}_{u,\lambda}$. Finally, notice that if ζ is in $\mathcal{K}_u$ and A_n is a sequence in $\mathfrak{A}_{u,\lambda}$ such that $\lim_{n\to\infty} A_n\xi_{1,1} = \zeta$, then

$$\lim_{n\to\infty} A_n\big(\xi_{1,1}\xi_{s,1}^*\big) = \zeta\xi_{s,1}^*$$

belongs to $\mathfrak{A}_{u,\lambda}$. Hence $\mathfrak{A}_{u,\lambda}P = \mathcal{B}(\mathcal{K}_u)P$.   ■

Lemma 3.13. *$P\mathfrak{A}P$ is a finite type I von Neumann algebra.*

Proof. Clearly $P\mathfrak{A}P$ is WOT–closed and is generated by PS_iP. To see that it is self-adjoint, it suffices to show that their adjoints also lie in $P\mathfrak{A}P$. Consider a primitive word $u = z_{i_1}\dots z_{i_k}$ and scalar λ in $\mathbb{T}$. Given a non-zero vector η in $P_{u,\lambda}\mathcal{H}$, an argument similar to the proof of Corollary 3.6 shows that if $S_i\eta$ is not a wandering vector, then $i = i_k$. Since $P = \sum P_{v,\mu}$, (where this SOT–sum is taken over all primitive words v in $\mathcal{F}_n$ and all μ in $\mathbb{T}$) we see that wandering vectors for $\mathfrak{A}$ belong to $\mathcal{W}$. Hence $S_i\eta$ belongs to $\mathcal{W}$ unless $i = i_k$. Since $\mathcal{W}$ is invariant for $\mathfrak{A}$ and $S_{i_1}\dots S_{i_{k-1}}(S_{i_k}\eta) = \lambda\eta$, it follows that $PS_{i_k}\eta = S_{i_k}\eta$. Therefore,

$$PS_iP_{u,\lambda} = \begin{cases} 0 & \text{if } i \neq i_k \\ S_iP_{u,\lambda} & \text{if } i = i_k, \end{cases}$$

and hence $PS_iP_{u,\lambda}$ is a partial isometry. Then letting $u' = z_{i_1}\dots z_{i_{k-1}}$, one sees that

$$(\overline{\lambda}P_{u,\lambda}u'(S)P)(PS_iP_{u,\lambda}) = P_{u,\lambda}.$$

Hence $(PS_iP_{u,\lambda})^* = \overline{\lambda}P_{u,\lambda}u'(S)P$ belongs to $\mathfrak{A}$. Since P is the WOT–convergent sum of the $P_{u,\lambda}$'s, it follows that PS_i^*P belongs to $P\mathfrak{A}P$.

For each primitive word u, let u_s for $1 \leq s \leq k$ be the cyclic permutations of u. The projection $P_{\langle u\rangle,\lambda} = \sum_{s=1}^{k} P_{u_s,\lambda}$ is invariant for $P\mathfrak{A}P$ by Theorem 3.4 because this is the projection onto the summand corresponding to multiples of $\sigma_{u,\lambda}$. Therefore it lies in the centre of $P\mathfrak{A}P$. By the previous lemma, $P_{\langle u\rangle,\lambda}\mathfrak{A}P_{\langle u\rangle,\lambda} \simeq \mathcal{M}_k$. Since P is the WOT–convergent sum of all the $P_{\langle u\rangle,\lambda}$'s, it follows that $P\mathfrak{A}P$ is the direct product of full matrix algebras. ■

We now are prepared to establish the second important result of this section.

Theorem 3.14. *Every atomic free semigroup algebra is hyper-reflexive with constant less than 51.*

Proof. The left regular representation yields the algebra $\mathfrak{L}_n$ which is hyper-reflexive with constant less than 51 by Theorem 2.9.

The algebra $\mathfrak{A}_{[x]}$ determined by an infinite word x acts on the space $\mathcal{H}_x$. This space is the increasing union of the cyclic subspaces $\mathfrak{A}[\xi_{x_m^{-1}}]$. As $\xi_{x_m^{-1}}$ is a wandering vector, the restriction to this subspace is unitarily equivalent to the left regular representation. On any subspace which is the direct sum of countably many pairwise orthogonal cyclic subspaces, the distance constant for the restriction is at most 9 by Theorem 2.7. In particular, inside $\mathfrak{A}[x_{m+1}^{-1}]$ there are countably many orthogonal cyclic subspaces including

$\mathfrak{A}[x_m^{-1}]$ which are obtained exactly as in section 2. Fix an operator T in $\mathcal{B}(\mathcal{H}_x)$. For each $m \geq 1$, one finds an operator A_m in $\mathfrak{A}$ so that

$$\|(T - A_m)|_{\mathfrak{A}[x_m^{-1}]}\| \leq 9\beta(T).$$

Thus an easy estimate shows that any WOT–limit A of a subsequence of $\{A_m\}$ satisfies $\|T - A\| \leq 9\beta(T)$.

Now consider $\mathfrak{A}_{u,\lambda}$. By Lemma 3.12,

$$\operatorname{dist}(T, \mathfrak{A}_{u,\lambda}) = \operatorname{dist}(TP^\perp, \mathfrak{A}_{u,\lambda}P^\perp).$$

The algebra $\mathfrak{A}_{u,\lambda}P^\perp$ is the direct sum of $k(n-1)$ copies of the left regular representation. Hence this algebra is hyper-reflexive with constant at most $18\sqrt{2}$ as in the proof of Theorem 2.9.

Now look at the general case of an algebra $\mathfrak{A}$, which by Corollary 3.11 corresponds to a representation

$$\rho \simeq \lambda^{(\alpha)} \oplus \sum \oplus \pi_{[x]}^{(\beta_{[x]})} \oplus \sum \oplus \sigma_{\langle u \rangle, \lambda}^{(\gamma_{\langle u \rangle, \lambda})}$$

with two or more summands. Then $\mathfrak{A}P^\perp\mathcal{H}$ is spanned by wandering vectors. When there is no $\pi_{[x]}$, this space is determined by a wandering space of dimension at least 2 and has a distance constant at most $18\sqrt{2}$. When there is at least one $\pi_{[x]}$, the space $P^\perp\mathcal{H}$ is the increasing union of such spaces with infinite dimensional wandering space. So it has distance constant at most 9. The constant 9 is also valid if there are countably many summands in ρ.

So it remains to analyze $\mathfrak{A}P$. By Theorem 3.4 and Lemma 3.12, this space has the form

$$\mathfrak{A}P \simeq \sum_{\langle u \rangle, \lambda} \oplus \big(\mathcal{B}(\mathcal{K}_u)P_{\langle u \rangle, \lambda}\big)^{(\gamma_{\langle u \rangle, \lambda})}.$$

This space therefore equals $\mathfrak{W}P$ where $\mathfrak{W}$ is the type I von Neumann algebra

$$\mathfrak{W} = \sum_{\langle u \rangle, \lambda} \oplus \mathcal{B}(\mathcal{K}_u)^{(\gamma_{\langle u \rangle, \lambda})}.$$

As P belongs to $\mathfrak{W}$ and $\mathfrak{W}$ has a distance constant at most 4 by Christensen's result[8], it follows that $\mathfrak{A}P$ also has distance constant at most 4.

Combining these two results, we see that

$$\operatorname{dist}(T, \mathfrak{A}) \leq \big(\operatorname{dist}(TP, \mathfrak{A}P)^2 + \operatorname{dist}(TP^\perp, \mathfrak{A}P^\perp)^2\big)^{1/2}$$
$$\leq \sqrt{664}\beta(T) < 26\beta(T).$$

This constant can be improved to $\sqrt{97} < 10$ when the constant 9 can be used instead of $18\sqrt{2}$. ∎

Example 3.15. This example is to show that a certain natural representation of $\mathcal{F}_2$ on $L^2(0,1)$ can be analyzed using the structure of free atomic

algebras. The natural analogue for all finite $n \geq 2$ is valid. Let $\mathcal{H} = L^2([0,1])$ equipped with Lebesgue measure and define isometries S_i by

$$(S_1 f)(t) = \sqrt{2} f(2t) \chi_{[0,\frac{1}{2}]}(t) \quad \text{and} \quad (S_2 f)(t) = \sqrt{2} f(2t-1) \chi_{[\frac{1}{2},1]}(t).$$

Then $S_1 S_1^* + S_2 S_2^* = I$. Let $\mathfrak{A}$ be the unital WOT–closed algebra generated by S_1 and S_2.

Let $\eta_0 = \chi_{[0,1]}$, $\eta_1 = \chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}$, and for every word $w \neq 1$ in $\mathcal{F}_n$, let $\eta_w = w(S)\eta_1$; note that these vectors are pairwise orthogonal since η_1 is a wandering vector for $\mathfrak{A}$.

Let $\mathcal{E}_n$ denote the span of $\{\eta_0, \eta_w : |w| < n\}$. Then it is easy to see that $\mathcal{E}_n$ equals the span of the characteristic functions of diadic intervals of length 2^{-n} and that this set is an orthonormal basis of $\mathcal{E}_n$. As the union of the $\mathcal{E}_n$ is dense in $L^2(0,1)$, altogether this yields an orthonormal basis of $L^2(0,1)$. In fact, this is the well–known Haar basis for $L^2(0,1)$. Therefore we see that η_1 is a wandering vector for $\mathfrak{A}[\eta_1] = \{\eta_0\}^\perp$. Since $(S_1 - S_2)\eta_0 = \sqrt{2}\eta_1$, it follows that η_0 is a cyclic vector for $\mathfrak{A}$.

Let $T_1 = (S_1 + S_2)/\sqrt{2}$ and $T_2 = (S_1 - S_2)/\sqrt{2}$. Then it is easy to verify that T_1 and T_2 are isometries with orthogonal ranges that generate the same algebra $\mathfrak{A}$. Moreover, η_1 is a wandering vector for this pair, and determines a basis $\zeta_w = w(T_1, T_2)\eta_1$, $w \in \mathcal{F}_2$, for $\{\eta_0\}^\perp$. This basis consists of real functions of modulus 1; and for $|w| = n$, these functions are constant on diadic intervals of length 2^{-n-1}. Moreover, $T_1\eta_0 = \eta_0$ and $T_2\eta_0 = \eta_1 = \zeta_1$. So this pair determines a free atomic representation of $\mathcal{F}_2$. As it has a cyclic vector η_0 such that $T_1\eta_0 = \eta_0$, this is evidently the representation σ_{z_1} as in Example 3.1.

Let U be the isometry of $\mathcal{H}_2$ onto $\{\eta_0\}^\perp$ given by $U\xi_w = \eta_w$. Then it follows that $\mathfrak{A}$ consists of all operators of the form $\eta\eta_0^* + UAU^*$ for η in $L^2(0,1)$ and A in $\mathfrak{L}_2$.

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