COMPLETE SPECTRAL SETS AND NUMERICAL RANGE

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Abstract. We define the complete numerical radius norm for homomorphisms from any operator algebra into $\mathcal{B}(\mathcal{H})$, and show that this norm can be computed explicitly in terms of the completely bounded norm. This is used to show that if K is a complete C-spectral set for an operator T, then it is a complete M-numerical radius set, where $M = \frac{1}{2}(C + C^{-1})$. In particular, in view of Crouzeix's theorem, there is a universal constant M (less than 5.6) so that if P is a matrix polynomial and $T \in \mathcal{B}(\mathcal{H})$, then $w(P(T)) \le M\|P\|_{W(T)}$. When $W(T) = \overline{\mathbb{D}}$, we have $M = \frac{5}{4}$.

In 2007, Michel Crouzeix [6] proved the remarkable fact that for any operator T on a Hilbert space $\mathcal{H}$, the numerical range is a complete C-spectral set for some constant with a universal bound of 11.08. Moreover in [5], he conjectures that the optimal constant is 2, which is the case for a disc. This inspired a result of Drury [7] who proved that if the numerical range of T is contained in the disc, then the numerical radius of any polynomial in T is bounded by $\frac{5}{4}$ times the supremum norm of the polynomial over the disc. Generally all that one can say about the relationship between the norm and numerical radius is that $w(X) \le \|X\|$, with equality for many operators, so the improvement from 2 to Drury's $\frac{5}{4}$ was unexpected.

In this note, we establish a precise relationship between the completely bounded norm of a homomorphism of an arbitrary operator algebra and what we call the *complete numerical radius norm* of the homomorphism. When applied to the case of the disc, our relationship yields Drury's result in the matrix polynomial case, and our result also applies to the general study of C-spectral sets.

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We now introduce the definitions and prior results that we shall need.

A set K is a *C-spectral set* for an operator T if for every rational function f with poles off of K, one has

$$\|f(T)\| \leq C\|f\|_K$$

where $\|f\|_K = \sup\{|f(z)| : z \in K\}$. It is a *complete C-spectral set* if this inequality holds for all matrices with rational coefficients. This inequality clearly extends to $R(K)$, the uniform closure of these rational functions in $\mathrm{C}(K)$. It is well-known that when the complement of K is connected (such as when K is convex), it suffices to verify this inequality for polynomials. When the constant $C = 1$, we call K a *(complete) spectral set*. The second author [11] showed in 1984 that K is a complete C-spectral set for T if and only if there is an invertible operator S so that K is a complete spectral set for STS^{-1} and $\|S\|\,\|S^{-1}\| \leq C$.

The numerical range of an operator T is the set

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}.$$

This set is always convex and contains the spectrum of T. The numerical radius of T is

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

It is well known that $w(T) \leq \|T\| \leq 2w(T)$, and that this inequality is sharp.

Early estimates for $w(p(T))$ for a polynomial p in the case of the disc were due to Berger and Stampfli [4]: if $w(T) \leq 1$ and $p(0) = 0$, then $w(p(T)) \leq \|p\|_{\mathbb{D}}$. A recent result of Drury [7] deals with the case of $p(0) \neq 0$:

$$w(p(T)) \leq \frac{5}{4}\|p\|_{\mathbb{D}}.$$

This inequality can be seen to be sharp for $T = \left[\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right]$ and the Möbius map that takes 0 to 1/2. Drury gives more precise information about the shape of $W(p(T))$ as a "teardrop". For another proof of this fact, see [9]. Since we are primarily interested in matrix polynomials, this geometric picture is no longer valid, but we prove the same $\frac{5}{4}$ inequality.

We will say that a compact set K is a *C'-numerical radius set* for T if

$$w(f(T)) \leq C'\|f\|_K \quad \text{for all} \quad f \in R(K),$$

and it is a *complete C'-numerical radius set* if the same inequality holds for matrices over $R(K)$. One of our key results is the following:

Theorem 3.1. *Let $C \geq 1$ and set $C' = \frac{1}{2}(C + C^{-1})$. A compact subset $K \subset \mathbb{C}$ is a complete C-spectral set for $T \in \mathcal{B}(\mathcal{H})$ if and only if it is a complete C'-numerical radius set for T.*

In fact, we will prove a general result about unital operator algebras. Recall that every unital operator algebra $\mathcal{A}$ has a family of norms on $\mathcal{M}_n(\mathcal{A})$, and that $\mathcal{A}$ may be represented completely isometrically as an algebra of operators on some Hilbert space. See [12] for details. If $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a bounded linear map, it induces coordinatewise maps $\Phi^{(n)} : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^{(n)})$; and one defines the *completely bounded norm* by

$$\|\Phi\|_{cb} = \sup_{n\geq 1} \|\Phi^{(n)}\|.$$

We will also define a *complete numerical radius norm* on such maps

$$\|\Phi\|_{wcb} := \sup_{n\geq 1} \sup_{A\in\mathcal{M}_n(\mathcal{A}),\ \|A\|\leq 1} w(\Phi^{(n)}(A)).$$

Our main result is the following:

Theorem 2.3. *Let $\mathcal{A}$ be an operator algebra, and let Φ be a completely bounded homomorphism. Then*

$$\|\Phi\|_{wcb} = \frac{1}{2}\big(\|\Phi\|_{cb} + \|\Phi\|_{cb}^{-1}\big).$$

We complete this introduction with a bit more background on the numerical radius. Ando [1] provided some useful characterizations of $w(T)$. One we require is Ando's numerical radius formula:

$$w(T) = \min\Big\{ \tfrac{1}{2}\|A + B\| : \begin{bmatrix} A & T \\ T^* & B \end{bmatrix} \geq 0 \Big\}.$$

Numerical range is intimately connected to dilation theory. The first such result was due to Berger [3], who showed that if $w(T) \leq 1$, then there is a unitary operator U on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that

$$T^n = 2P_{\mathcal{H}} U^n|_{\mathcal{H}} \quad \text{for} \quad n \geq 1.$$

Easy examples show that the converse is not valid. Sz.Nagy and Foiaş [13] introduced the notion of a ρ-contraction (for $\rho \geq 1$) as an operator T for which there is a ρ-dilation, meaning a unitary operator U on $\mathcal{K} \supset \mathcal{H}$ such that

$$T^n = \rho P_{\mathcal{H}} U^n|_{\mathcal{H}} \quad \text{for} \quad n \geq 1.$$

Okubo and Ando [10] show that if T is a ρ-contraction, then there is an invertible S so that $\|STS^{-1}\| \leq 1$ and $\|S\|\,\|S^{-1}\| \leq \rho$. By the remarks above, this shows that $\overline{\mathbb{D}}$ is a complete ρ-spectral set for T. In particular, if $w(T) \leq 1$, then T is a 2-contraction by Berger's dilation, and thus $\overline{\mathbb{D}}$ is a complete 2-spectral set for T.

In view of the work of Crouzeix [6], there has been a lot of renewed interest in numerical range. See the monograph [8] and the recent survey [2] for many relevant references.

2. The main theorem

We begin with a key observation which yields one direction of our theorem.

Lemma 2.1. *If $\|T\| \leq 1$ and $\|S\|\,\|S^{-1}\| \leq C$, then*

$$w(S^{-1}TS) \leq \frac{1}{2}(C + C^{-1}).$$

Proof. Using polar decomposition, $S = U|S|$, we may replace T by the unitarily equivalent U^*TU and suppose that $S > 0$. After scaling, we may suppose that $C^{-1/2}I \leq S \leq C^{1/2}I$. Since $\|T\| \leq 1$ we have that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} S^{-1} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & T \\ T^* & I \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & S \end{bmatrix} = \begin{bmatrix} S^{-2} & S^{-1}TS \\ ST^*S^{-1} & S^2 \end{bmatrix}$$

By Ando's numerical radius formula, we obtain that

$$\begin{aligned} w(S^{-1}TS) &\leq \frac{1}{2}\|S^{-2} + S^2\| \\ &\leq \sup\{\frac{1}{2}(t + t^{-1}) : C^{-1} \leq t \leq C\} \\ &= \frac{1}{2}(C + C^{-1}). \end{aligned}$$ ∎

To establish the converse, we first need a simple computational lemma.

Lemma 2.2. *Let $B \in \mathcal{B}(\mathcal{H})$ and let $T = \begin{bmatrix} \alpha I & B \\ 0 & \alpha I \end{bmatrix}$. Then*

$$\|T\| = \frac{\|B\| + \sqrt{\|B\|^2 + 4|\alpha|^2}}{2} \quad \text{and} \quad w(T) = |\alpha| + \tfrac{1}{2}\|B\|.$$

In particular, $\|T\| = 1$ if and only if $|\alpha|^2 + \|B\| = 1$.

Proof. It is straightforward to show that $\|T\| = \left\| \begin{bmatrix} \|B\| & |\alpha| \\ |\alpha| & 0 \end{bmatrix} \right\|$, and computation of the eigenvalues of this self-adjoint matrix yields the desired formula. Routine manipulation now shows that $\|T\| = 1$ if and only if $|\alpha|^2 + \|B\| = 1$. It is also easy to see that $W\left(\left[\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right]\right) = W\left(\left[\begin{smallmatrix} 0 & \|B\| \\ 0 & 0 \end{smallmatrix}\right]\right)$ is a disc centred at 0 of radius $\|B\|/2$. Hence $W(T) = \alpha + \frac{\|B\|}{2}\mathbb{D}$, and therefore $w(T) = |\alpha| + \frac{1}{2}\|B\|$. ∎

Theorem 2.3. *Let $\mathcal{A}$ be a unital operator algebra, and let Φ be a unital completely bounded homomorphism. Then*

$$\|\Phi\|_{wcb} = \frac{1}{2}\big(\|\Phi\|_{cb} + \|\Phi\|_{cb}^{-1}\big).$$

Proof. Let $C = \|\Phi\|_{cb}$. By Paulsen's similarity theorem [11], there is an invertible operator S so that $\operatorname{Ad} S \circ \Phi$ is completely contractive and $\|S\|\,\|S^{-1}\| = C$. (Here $\operatorname{Ad} S(T) = STS^{-1}$.) Let $A \in \mathcal{M}_n(\mathcal{A})$ with $\|A\| = 1$. Then $T := (\operatorname{Ad} S \circ \Phi)^{(n)}(A)$ satisfies $\|T\| \leq 1$ and $\Phi(A) = \operatorname{Ad} S^{-1(n)}(T)$. Hence by Lemma 2.1, $w(\Phi^{(n)}(A)) \leq \frac{1}{2}(C + C^{-1})$. Thus

$$\|\Phi\|_{wcb} \leq \frac{1}{2}(C + \frac{1}{C}).$$

Conversely, suppose that $A \in \mathcal{M}_n(\mathcal{A})$ with $\|A\| = 1$ such that $\|\Phi^{(n)}(A)\| > C - \varepsilon$ for some $\varepsilon > 0$. Define $B \in \mathcal{M}_{2n}(\mathcal{A})$ by

$$B = \begin{bmatrix} C^{-1} I_n & (1 - C^{-2})A \\ 0 & C^{-1} I_n \end{bmatrix}.$$

Then by Lemma 2.2, $\|B\| = 1$. Moreover by the second part of that lemma,

$$\begin{aligned} \|\Phi\|_{wcb} &\geq w(\Phi^{(2n)}(B)) \\ &= w\Big(\begin{bmatrix} C^{-1} I_n & (1 - C^{-2})\Phi^{(n)}(A) \\ 0 & C^{-1} I_n \end{bmatrix} \Big) \\ &> C^{-1} + \frac{1}{2}(1 - C^{-2})(C - \varepsilon) \\ &> \frac{1}{2}(C + \frac{1}{C}) - \frac{\varepsilon}{2}. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we obtain

$$\|\Phi\|_{wcb} = \frac{1}{2}(C + \frac{1}{C}) = \frac{1}{2}\big(\|\Phi\|_{cb} + \|\Phi\|_{cb}^{-1}\big).$$ ∎

Remark 2.4. Inverting the above function shows that for a unital homomorphism Φ,

$$\|\Phi\|_{cb} = \|\Phi\|_{wcb} + \sqrt{\|\Phi\|_{wcb}^2 - 1}.$$

3. Consequences

As an immediate application, we obtain the second theorem stated in the introduction. Note that convexity of K is not required.

Theorem 3.1. *Let $C \geq 1$ and set $C' = \frac{1}{2}(C + C^{-1})$. A compact subset $K \subset \mathbb{C}$ is a complete C-spectral set for $T \in \mathcal{B}(\mathcal{H})$ if and only if it is a complete C'-numerical radius set for T.*

Proof. If K is a complete C-spectral set for T, then the map $\Phi_T(f) = f(T)$ for $f \in R(K)$ has $\|\Phi_T\|_{cb} \leq C$. Hence by Theorem 2.3,

$$\|\Phi_T\|_{wcb} = \frac{1}{2}\big(\|\Phi_T\|_{cb} + \|\Phi_T\|_{cb}^{-1}\big) \leq \frac{1}{2}(C + C^{-1}) = C'.$$

Thus K is a complete C'-numerical radius set for T.

Conversely, since $\|A\| \leq 2w(A)$, if K is a complete C'-spectral set for T, it follows that Φ is completely bounded. Then

$$\frac{1}{2}(C + C^{-1}) = C' \geq \|\Phi\|_{wcb} = \frac{1}{2}\big(\|\Phi\|_{cb} + \|\Phi\|_{cb}^{-1}\big)$$

implies that $\|\Phi_T\|_{cb} \leq C$. So K is a complete C-spectral set for T. ∎

We apply this to the family of C_ρ-contractions. For these operators, the set K is the unit disc.

Corollary 3.2. *Suppose that T is a C_ρ-contraction for $\rho \geq 1$. If $F : \mathbb{D} \to M_n$ is a matrix polynomial (or has coefficients in $A(\mathbb{D})$), then*

$$w(F(T)) \leq \frac{1}{2}(\rho + \rho^{-1})\|F\|_\infty.$$

Proof. By [10, Theorem 2], there is an invertible operator S such that $\|S^{-1}TS\| \leq 1$ and $\|S\|\,\|S^{-1}\| \leq \rho$. After scaling, we may suppose that $\|F\|_\infty = 1$. Then by the generalized von Neumann inequality, we have

$$1 \geq \|F(S^{-1}TS)\| = \|(S^{-1} \otimes I_n)F(T)(S \otimes I_n)\|.$$

Now an application of Lemma 2.1 yields the conclusion. ∎

The case $\rho = 2$ includes all operators T with $w(T) \leq 1$. This provides a matrix polynomial version of Drury's scalar inequality [7].

Corollary 3.3. *Suppose that T has $w(T) \leq 1$. If $F : \mathbb{D} \to M_n$ is a matrix polynomial (or has coefficients in $A(\mathbb{D})$), then*

$$w(F(C)) \leq \frac{5}{4}\|F\|_\infty.$$

Remark 3.4. Note that the class of operators which have the disc an a complete 2-spectral set contains many operators which do not have numerical radius 1. For example, let

$$T = \begin{bmatrix} 1/2 & 3/2 \\ 0 & 1/2 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\|S^{-1}TS\| = 1$ and $\|S\|\,\|S^{-1}\| = 2$ but $w(T) = 5/4$.

As we mentioned in the introduction, Crouzeix showed [6] that the numerical range $W(T)$ is a complete C-spectral set for T for a universal constant $C < 11.08$. Crouzeix conjectures [5] that the optimal constant is 2, which is the case for a disc by [10]. The following are immediate from Theorem 3.1.

Corollary 3.5. *Let T be a bounded operator on $\mathcal{H}$. Suppose that $W(T)$ has a complete Crouzeix constant of C, and let $C' = \frac{1}{2}(C + C^{-1})$. If $F : W(T) \to M_n$ is a matrix polynomial (or has coefficients in $A(W(T))$), then*

$$w(F(T)) \leq C'\|F\|_{W(T)}.$$

In particular, the constant $C' = 5.6$ is valid.

Corollary 3.6. *Let T be a bounded operator on $\mathcal{H}$. Then $W(T)$ is a complete 2-spectral set for T if and only if*

$$w(F(T)) \leq \frac{5}{4}\|F\|_{W(T)}$$

for every matrix polynomial F.

Thus Crouzeix's conjecture is true for the norm case if and only if the above 5/4's inequality holds for every operator T. Also, we know that 2 and $\frac{5}{4}$ are the best possible constants in each case.

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References

[1] T. Ando, *Structure of operators with numerical radius one*, Acta Sci. Math. (Szeged) **34** (1973), 11–15.
[2] C. Badea and B. Beckermann, *Spectral sets*, preprint, arxiv:1303:0546v1.
[3] C.A. Berger, *A strange dilation theorem*, Notices Amer. Math. Soc. **12** (1965), p. 590
[4] C.A. Berger and J.G. Stampfli, *Mapping theorems for the numerical range*, Amer. J. Math. **89** (1967), 1047–1055.

[5] M. Crouzeix, *Bounds for analytical functions of matrices*, Integral Equations and Operator Theory **48** (2004), 461–477, .

[6] M. Crouzeix, *Numerical range and functional calculus in Hilbert space*, J. Funct. Anal. **244** (2007), 668–690.

[7] S.W. Drury, *Symbolic calculus of operators with unit numerical radius*, Linear Algebra Appl. **428** (2008), 2061–2069.

[8] K.E. Gustafson and D.K.M. Rao, *Numerical Range: The Field of Values of Linear Operators and Matrices*, Universitext, Springer-Verlag, New York, 1997.

[9] H. Klaja, J. Mashreghi and T. Ransford, *On mapping theorems for numerical range*, Proc. Amer. Math. Soc. **144** (2016), 3009–3018.

[10] K. Okubo and T. Ando, *Constants related to operators of class C_ρ*, Manuscripta Math. **16** (1975), 385–394.

[11] V.I. Paulsen, *Completely bounded homomorphisms of operator algebras*, Proc. Amer. Math. Soc. **92** (1984), 225–228.

[12] V.I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics **78**, Cambridge University Press, Cambridge, 2002.

[13] B. Sz.-Nagy, C. Foiaş, H. Bercovici and L. Kerchy, *Harmonic analysis of operators on Hilbert space. Second edition*, Universitext Springer, New York, 2010.

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