ABSOLUTELY CONTINUOUS REPRESENTATIONS AND A KAPLANSKY DENSITY THEOREM FOR FREE SEMIGROUP ALGEBRAS

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Abstract. We introduce a notion of absolutely continuous functionals on the non-commutative disk algebra $\mathfrak{A}_n$. These functionals are used to help identify the type L part of the free semigroup algebra associated to a representation. A $*$-extendible representation of $\mathfrak{A}_n$ is *regular* if the absolutely continuous part coincides with the type L part. All known examples are regular. Absolutely continuous functionals are intimately related to maps which intertwine a given $*$-extendible representation with the left regular representation. A simple application of these ideas extends reflexivity and hyper-reflexivity results. Moreover the use of absolute continuity is a crucial device for establishing a density theorem that the unit ball of $\sigma(\mathfrak{A}_n)$ is weak-$*$ dense in the unit ball of the free semigroup algebra if and only if it is regular. We provide some explicit constructions related to the density theorem for specific representations. A notion of singular functionals is also defined, and every functional decomposes in a canonical way into the sum of its absolutely continuous and singular parts.

Free semigroup algebras were introduced in **[13]** as a method for analyzing the fine structure of n-tuples of isometries with commuting ranges. The C*-algebra generated by such an n-tuple is either the Cuntz algebra $\mathcal{O}_n$ or the Cuntz-Toeplitz algebra $\mathcal{E}_n$. As such, the free semigroup algebras can be used to reveal the fine spatial structure of representations of these algebras much in the same way as the von Neumann algebra generated by a unitary operator encodes the measure class and multiplicity which cannot be detected in the C*-algebra it generates.

This viewpoint yields critical information in the work of Bratteli and Jorgensen **[5, 6, 20, 21]** who use certain representations of $\mathcal{O}_n$ to construct and analyze wavelet bases.

From another point of view, free semigroup algebras can be used to study arbitrary (row contractive) n-tuples of operators. Frahzo **[17, 18]**, Bunce **[7]** and Popescu **[23]** show that every (row) contractive n-tuple of operators has a unique minimal dilation to an n-tuple of isometries which is a row contraction, meaning that the ranges are pairwise orthogonal. Thus every row contraction determines a free semigroup algebra. Popescu **[26]** establishes the n-variable von Neumann inequality which follows immediately from the dilation theorem. Popescu has pursued a program of establishing the analogues of the Sz. Nagy–Foiaş program in the n-variable setting **[24, 25, 27]**; the latter two papers deal with the free semigroup algebras from this point of view. Free semigroup algebras play the same role for non-commuting operator theory as the weakly closed unital algebra determined by the isometric dilation of a contraction plays for a single operator.

In **[12]**, the first author, Kribs and Shpigel use dilation theory to classify the free semigroup algebras which are obtained as the minimal isometric dilation of contractive n-tuples

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of operators on finite dimensional spaces. Such free semigroup algebras are called *finitely correlated*, because from the wavelet perspective, these algebras correspond to the finitely correlated representations of $\mathcal{E}_n$ or $\mathcal{O}_n$ introduced and studied by Bratteli and Jorgensen. It is interesting that this class of representations of $\mathcal{O}_n$ are understood in terms of an n-tuple of matrices, a reversal of the the single variable approach of analyzing arbitrary operators using the isometric dilation. Out of the analysis of finitely correlated free semigroup algebras emerged a structural result that appeared to rely on the special nature of the representation. However in **[11]**, two of the current authors and Katsoulis were able to expose a rather precise and beautiful structure for arbitrary free semigroup algebras. This structure plays a key role in this paper.

The prototype for free semigroup algebras is the algebra $\mathfrak{L}_n$ determined by the left regular representation of the free semigroup $\mathbb{F}_n^+$ on Fock space. This representation arises naturally in the formulation of quantum mechanics. We have named it the *non-commutative analytic Toeplitz algebra* because of the striking analytic properties that it has **[1, 13, 14, 15, 2]**. In particular, the vacuum vector (and many other vectors in this representation) has the property that its image under all words in the n isometries forms an orthonormal set. We call such vectors *wandering vectors.* Such vectors play a crucial role in these representations, and a deeper understanding of when they occur is one of the main open questions in this theory.

The norm-closed algebra $\mathfrak{A}_n$ generated by n isometries with orthogonal ranges is even more rigid than the C*-algebra. Indeed, it sits inside the C*-algebra $\mathcal{E}_n$, but the quotient onto $\mathcal{O}_n$ is completely isometric on this subalgebra. As $\mathcal{O}_n$ is simple, it is evidently the C*-envelope of $\mathfrak{A}_n$. This algebra has been dubbed the *non-commutative disk algebra* by Popescu. It plays the same role in this theory as the disk algebra plays in the study of a single isometry.

In this paper, we explore in greater depth the existence of wandering vectors. The major new device is the notion of an *absolutely continuous* linear functional on $\mathfrak{A}_n$. In the one variable case, a functional on $\mathrm{A}(\mathbb{D})$ is given by integration against a representing measure supported on the Shilov boundary $\mathbb{T}$. Absolute continuity is described in terms of Lebesgue measure. In our setting, we do not have a boundary, and we have instead defined absolute continuity in terms of its relationship to the left regular representation.

A related notion that plays a key role are intertwining maps from the left regular representation to an arbitrary free semigroup algebra. The key observation is that the range of such maps span the vectors which determine absolutely continuous functionals, and they serve to identify the type L part of the representation (see below). These results will be used to clarify precisely when a free semigroup is reflexive. For type L representations, we establish hyper-reflexivity whenever there are wandering vectors—the reflexive case. Basically the only obstruction to hyper-reflexivity is the possibility that there may be a free semigroup algebra which is type L (isomorphic to $\mathfrak{L}_n$) but has no wandering vectors, and hence will be reductive (all invariant subspaces have invariant ortho-complements).

The ultimate goal of this paper is to obtain an analogue of the Kaplansky density theorem. This basic and well-known result states that given any C*-algebra and any $*$-representation, the image of the unit ball is WOT-dense in the unit ball of the WOT-closure. In the non-selfadjoint setting, such a result is not generally true. However, in the context of completely isometric representations of $\mathfrak{A}_n$, we have a rather rigid structure, and we shall show that in fact such a Kaplansky type theorem does hold. Let σ be a $*$-extendible representation of $\mathfrak{A}_n$, that is, σ is the restriction of a $*$-representation of $\mathcal{O}_n$ or $\mathcal{E}_n$ to $\mathfrak{A}$. We call it regular if the type L part coincides with the absolutely continuous part. It is precisely this case in which a density theorem holds, and the unit ball of $\sigma(\mathfrak{A}_n)$ is weak-$*$ dense in the unit ball of the free semigroup algebra. In particular, we shall see that this holds in the presence of a wandering vector. In fact, the only possible obstruction to a Kaplansky density result for all representations of $\mathfrak{A}_n$ is the existence of a representation where the free semigroup algebra is a von Neumann algebra and is also absolutely continuous. No such representation is known to exist.

1. Preliminaries

In this section, we will remind the reader of some of the more technical aspects which we need, and will establish some notation for what follows.

A typical n-tuple of isometries acting on a Hilbert space $\mathcal{H}$ and having pairwise orthogonal ranges will be denoted by $S_1, \dots, S_n$. This may be recognized algebraically by the relations $S_j^* S_j = I \geq \sum_{i=1}^n S_i S_i^*$, $1 \leq j \leq n$. The C*-algebra that they generate is the Cuntz algebra $\mathcal{O}_n$ when $\sum_{i=1}^n S_i S_i^* = I$ and the Cuntz-Toeplitz algebra $\mathcal{E}_n$ when $\sum_{i=1}^n S_i S_i^* < I$. The norm-closed unital subalgebra generated by $S_1, \dots, S_n$ (but not their adjoints) is completely isometrically isomorphic to Popescu's non-commutative disk algebra $\mathfrak{A}_n$. The ideal of $\mathcal{E}_n$ generated by $I - \sum_{i=1}^n S_i S_i^*$ is isomorphic to the compact operators $\mathfrak{K}$, and the quotient by this ideal is $\mathcal{O}_n$. Let the canonical generators of $\mathcal{E}_n$ be denoted by $\mathfrak{s}_1, \dots, \mathfrak{s}_n$. Then every such n-tuple of isometries arises from a $*$-representation σ of $\mathcal{E}_n$ (write $\sigma \in \mathrm{Rep}(\mathcal{E}_n)$) as $S_i = \sigma(\mathfrak{s}_i)$.

We shall call a representation σ of $\mathfrak{A}_n$ *$*$-extendible* if σ is the restriction to $\mathfrak{A}_n$ of a $*$-representation of $\mathcal{E}_n$ or $\mathcal{O}_n$ to the canonical copy of $\mathfrak{A}_n$. It is easy to see that σ is $*$-extendible if and only if $\sigma(\mathfrak{S}_i)$ are isometries with orthogonal ranges; or equivalently, σ is contractive and $\sigma(\mathfrak{s}_i)$ are isometries.

Let $\mathbb{F}_n^+$ denote the unital free semigroup on n letters. (Probably we should use the algebraist's term 'monoid' here, but our habit of using the term semigroup is well entrenched.) This semigroup consists of all words w in $1, 2, \dots, n$ including the empty word $\varnothing$. The Fock space $\ell^2(\mathbb{F}_n^+)$ has an orthonormal basis $\{\xi_w : w \in \mathbb{F}_n^+\}$, and is the natural Hilbert space for the left regular representation λ. This representation has generators, denoted by $L_i := \lambda(\mathfrak{s}_i)$, which act by $L_i \xi_w = \xi_{iw}$. The WOT-closed algebra that they generate is denoted by $\mathfrak{L}_n$.

In general, each n-tuple $S_1, \dots, S_n$ will generate a unital algebra, and the WOT-closure will be denoted by $\mathfrak{S}$. When a representation σ of $\mathcal{E}_n$ is given and $S_i = \sigma(\mathfrak{s}_i)$, we may write $\mathfrak{S}_\sigma$ for clarity. For each word $w = i_1 \dots i_k$ in $\mathbb{F}_n^+$, we will use the notation S_w to denote the corresponding operator $S_{i_1} \cdots S_{i_k}$. In particular, $L_w \xi_v = \xi_{wv}$ for $w, v \in \mathbb{F}_n^+$.

It is a basic fact of C*-algebra theory that every representation of $\mathcal{E}_n$ splits as a direct sum of the representation induced from its restriction to $\mathfrak{K}$ and a representation that factors through the quotient by $\mathfrak{K}$. However, $\mathfrak{K}$ has a unique irreducible representation, and it induces the left regular representation λ of $\mathbb{F}_n^+$, described above. So $\sigma \simeq \lambda^{(\alpha)} \oplus \tau$ where α is some cardinal and τ is a representation of $\mathcal{O}_n$. This is equivalent to the spatial result known as the Wold decomposition. The Wold decomposition is the observation that the range $\mathcal{M}$ of the projection $I - \sum_{i=1}^n S_i S_i^*$ is a wandering subspace, meaning that the subspaces $\{S_w \mathcal{M} : w \in \mathbb{F}_n^+\}$ are pairwise orthogonal, and together span the subspace $\mathfrak{S}[\mathcal{M}]$. Any orthonormal basis for $\mathcal{M}$ will consist of wandering vectors which generate orthogonal copies of the left regular representation; moreover, the restriction of the S_i to $\mathfrak{S}[\mathcal{M}]^\perp$ will be a representation which factors through $\mathcal{O}_n$. We call the representation τ the *Cuntz part* of σ, and when $\alpha = 0$, i.e. when $\sum_{i=1}^n \sigma(\mathfrak{s}_i\mathfrak{s}_i^*) = I$, we say simply that the representation σ is of *Cuntz type*.

Recall **[13]** that every $A \in \mathfrak{L}_n$ has a Fourier series $A \sim \sum_{w\in\mathbb{F}_n^+} a_w L_w$ determined by $A\xi_\varnothing = \sum_{w\in\mathbb{F}_n^+} a_w \xi_w$. The representation λ is a canonical completely isometric map from $\mathfrak{A}_n$ into $\mathfrak{L}_n$ which sends $\mathfrak{s}_i$ to L_i. Hence elements of $\mathfrak{A}_n$ inherit corresponding Fourier series, and we will write $A \sim \sum_{w\in\mathbb{F}_n^+} a_w \mathfrak{s}_w$. The functional φ_0 reads off the coefficient $a_\varnothing$. The kernel of φ_0 in $\mathfrak{A}_n$ and $\mathfrak{L}_n$ are denoted by $\mathfrak{A}_{n,0}$ and $\mathfrak{L}_{n,0}$ respectively. These are the norm and WOT-closed ideals, respectively, generated by the generators $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ and $L_1, \dots, L_n$. Even when φ_0 is not defined on a free semigroup algebra $\mathfrak{S}$, we still denote by $\mathfrak{S}_0$ the WOT-closed ideal generated by $S_1, \dots, S_n$. This will either be codimension one or equal to the entire algebra.

The ideals $\mathfrak{A}_{n,0}^k$ and $\mathfrak{L}_{n,0}^k$ consist of those elements with zero Fourier coefficients for all words w with $|w| < k$; and are generated as a right ideal by $\{\mathfrak{s}_w : |w| = k\}$. Moreover **[14]**, each element in $\mathfrak{L}_{n,0}^k$ may be uniquely represented as $A = \sum_{|w|=k} L_w A_w$ and $\|A\|$ is equal to the norm of the column operator with entries A_w.

One can recover an element of $\mathfrak{A}_n$ or $\mathfrak{L}_n$ from its Fourier series in the classical way using a summability kernel. For $t \in \mathbb{T}$, let α_t be the gauge automorphism of $\mathcal{O}_n$ determined by the mapping $\mathfrak{s}_i \mapsto t\mathfrak{s}_i$. Let $V_n(t) = \sum_{k=-2n-1}^{2n+1} c_k t^k$ be the de la Vallée Poussin summability kernel on $\mathbb{T}$ from harmonic analysis. Recall that V_n is a trigonometric polynomial of degree $2n+1$ with Fourier transform $\hat{V}_n(k) = 1$ for $|k| \le n+1$. Let m be normalized Lebesgue measure on $\mathbb{T}$. Define linear maps Σ_k on $\mathcal{O}_n$ by

$$\Sigma_k(X) = \int_{\mathbb{T}} V_k(t)\alpha_t^{-1}(X)\, dm(t).$$

Then Σ_k is a unital completely positive map on $\mathcal{O}_n$ which leaves $\mathfrak{A}_n$ invariant and moreover, for every $X \in \mathcal{O}_n$, $\Sigma_k(X)$ converges in norm to X. It has the additional property that the Fourier coefficients of $\Sigma_k(X)$ agree with those of X up to the k-th level. Indeed, if $A \sim \sum_{w\in\mathbb{F}_n^+} a_w L_w$ lies in $\mathfrak{A}_n$, then $\Sigma_k(A) = \sum_{|w|\le 2k+1} c_{|w|} a_w \mathfrak{s}_w$. Notice that for $A \in \mathfrak{L}_n$, $\Sigma_k(A)$ converges to A in the strong operator topology.

Let σ be a $*$-extendible representation and let $\mathfrak{S} = \mathfrak{S}_\sigma$. We now recall some facts from **[11]** regarding the ideals $\mathfrak{S}_0^k$. The intersection $\mathfrak{J}$ of these ideals is a left ideal of the von Neumann algebra $\mathfrak{W}$ generated by the S_i; therefore, $\mathfrak{J}$ has the form $\mathfrak{W}P_\sigma$ for some projection $P_\sigma \in \mathfrak{S}$. (When the context is clear, we will write P instead of P_σ.) The Structure Theorem for free semigroup algebras **[11]** shows that P is characterized as the largest projection in $\mathfrak{S}$ such that $P\mathfrak{S}P$ is self-adjoint. Moreover $P^\perp\mathcal{H}$ is invariant for $\mathfrak{S}$ and when $P \ne I$, the restriction of $\mathfrak{S}$ to the range of $P^\perp$ is canonically isomorphic to $\mathfrak{L}_n$. Indeed, the map taking $S_i|_{P^\perp\mathcal{H}}$ to L_i extends to a completely isometric isomorphism which is also a weak-$*$ homeomorphism. Algebras which are isomorphic to $\mathfrak{L}_n$ are called *type L*. When $P \ne I$, the restriction of σ to the range of $P^\perp$ again determines a $*$-extendible representation of $\mathfrak{A}_n$, and we call this restriction the *type L part* of σ.

2. Absolute Continuity

In the study of the disk algebra, those functionals which are absolutely continuous to Lebesgue measure play a special role. Of course, the Shilov boundary of the disk algebra is the unit circle, and the Lebesgue probability measure m is Haar measure on it. Moreover, every representing measure for evaluation at points interior to the disk is absolutely continuous. We have been seeking an appropriate analogue of this for free semigroup algebras for some time. That is, which functionals on the non-commutative disk algebra $\mathfrak{A}_n$ should be deemed to be absolutely continuous? Unfortunately, there is no clear notion of boundary or representing measure. However there is a natural analogy, and we propose it here.

Our starting point is the left regular representation of the semigroup $(\mathbb{N},+)$. Under this representation, the generator of $(\mathbb{N},+)$ is mapped to the unilateral shift S and elements of $\mathbb{A}(\mathbb{D})$ are analytic functions $h(S)$ of the shift, which may be regarded as multipliers of $H^2(\mathbb{T})$. With this perspective, every vector functional $h \mapsto \langle h(S)f_1, f_2\rangle = \int_{\mathbb{T}} hf_1\overline{f_2}\,dm$ corresponds to a measure which is absolutely continuous with respect to Lebesgue measure.

On the other hand, suppose φ is a functional on $\mathbb{A}(\mathbb{D})$ given by integration over $\mathbb{T}$ by an absolutely continuous measure, so that $\varphi(h) = \int_{\mathbb{T}} hf\,dm$ for some $f \in L^1(\mathbb{T})$. It is not difficult to show that such functionals on $\mathbb{A}(\mathbb{D})$ can be approximated by vector functionals from the Hilbert space of the left regular representation. Moreover, if one allows infinite multiplicity, one can represent φ as a vector state, that is, there are vectors x_1 and x_2 in $H^{2\,(\infty)}$ such that

$$\varphi(h) = \langle h(S^{(\infty)})x_1, x_2\rangle \quad\text{and}\quad \|\varphi\| = \|x_1\|\,\|x_2\|.$$

Another view is that the absolutely continuous functionals on $\mathrm{A}(\mathbb{D})$ are the functionals in the predual of $H^\infty(\mathbb{T})$. Our analogue of this algebra is $\mathfrak{L}_n$.

So we are motivated to make the following definition:

Definition 2.1. For $n \geq 2$, a functional on the non-commutative disk algebra $\mathfrak{A}_n$ is *absolutely continuous* if it is given by a vector state on $\mathfrak{L}_n$; i.e. if there are vectors $\zeta,\, \eta \in \ell^2(\mathbb{F}_n^+)$ so that $\varphi(A) = \langle \lambda(A)\zeta, \eta\rangle$. Let $\mathfrak{A}_n^a$ denote the set of all absolutely continuous functionals on $\mathfrak{A}_n$.

For $n \geq 2$, $\mathfrak{L}_n$ has enough "infinite multiplicity" that it is unnecessary to take the closure of vector functionals; in fact we shall see shortly that $\mathfrak{A}_n^a$ is already norm closed.

The following result shows that the notion of being representable as a vector state and being in the predual of $\mathfrak{L}_n$ are equivalent to each other and to a natural norm condition on the functional.

Proposition 2.2. *For $\varphi \in \mathfrak{A}_n^*$, the following are equivalent:*

(1) *φ is absolutely continuous.*
(2) *$\varphi \circ \lambda^{-1}$ extends to a weak-$*$ continuous functional on $\mathfrak{L}_n$.*
(3) $\lim\limits_{k\to\infty} \|\varphi|_{\mathfrak{A}_{n,0}^k}\| = 0$.

Moreover, given $\varepsilon > 0$, the vectors ζ and η may be chosen so that $\|\zeta\|\,\|\eta\| < \|\varphi\| + \varepsilon$.

Proof. (1) implies (2) by definition. The converse follows from **[13]** where it is shown that every weak-$*$ continuous functional on $\mathfrak{L}_n$ is given by a vector state. The norm condition on the vectors ζ and η is also obtained there.

Next, suppose (1) holds. The map λ carries $\mathfrak{A}_{n,0}^k$ into $\mathfrak{L}_{n,0}^k$. Let Q_k denote the projection of $\ell^2(\mathbb{F}_n^+)$ onto $\overline{\operatorname{span}}\{\xi_w : w \in \mathbb{F}_n^+, |w| \geq k\}$. Elements of $\mathfrak{L}_{n,0}^k$ are characterized by $A = Q_kA$. Thus for $A \in \mathfrak{A}_{n,0}^k$,

$$|\varphi(A)| = |\langle \lambda(A)\zeta, \eta\rangle| = |\langle \lambda(A)\zeta, Q_k\eta\rangle| \leq \|A\|\,\|\zeta\|\,\|Q_k\eta\|.$$

Hence it follows that

$$\lim_{k\to\infty} \|\varphi|_{\mathfrak{A}^k_{n,0}}\| \le \lim_{k\to\infty} \|\zeta\|\,\|Q_k\eta\| = 0.$$

Conversely suppose that (3) holds. Then given $A \in \mathfrak{A}_n$, we use the fact that $\Sigma_k(A)$ converges to A in norm. Note that when $m \ge k$, $\Sigma_k(X) - \Sigma_m(X)$ belongs to $\mathfrak{A}^k_{n,0}$ and has norm at most $2\|A\|$. It follows therefore that the adjoint maps satisfy,

$$\|\Sigma^*_k(\varphi) - \Sigma^*_m(\varphi)\| \le 2\|\varphi|_{\mathfrak{A}^k_{n,0}}\|.$$

So $\Sigma^*_k(\varphi)$ is a Cauchy sequence in $\mathfrak{A}^*$.

We claim that $\Sigma^*_k(\varphi)$ is absolutely continuous. Indeed, consider the Fourier series $A \sim \sum_{w\in\mathbb{F}^+_n} a_w \mathfrak{s}_w$. Then

$$\begin{aligned}\Sigma^*_k(\varphi)(A) &= \sum_{|w|\le 2k+1} c_{|w|} a_w \varphi(\mathfrak{s}_w)\\ &= \sum_{|w|\le 2k+1} c_{|w|} \varphi(\mathfrak{s}_w)\langle \lambda(A)\xi_\varnothing, \xi_w\rangle\\ &= \langle \lambda(A)\xi_\varnothing, \sum_{|w|\le 2k+1} c_{|w|}\varphi(\mathfrak{s}_w)\xi_w\rangle.\end{aligned}$$

From (1) implies (2), we found that the set of absolutely continuous functionals is norm closed. Hence the limit φ is also absolutely continuous. ∎

The following is immediate.

Corollary 2.3. *The set $\mathfrak{A}^a_n$ is the closed subspace of the dual of $\mathfrak{A}_n$ which forms the predual of $\mathfrak{L}_n$.*

Definition 2.4. Let σ be a $*$-extendible representation of $\mathfrak{A}_n$ on the Hilbert space $\mathcal{H}_\sigma$. A vector $x \in \mathcal{H}_\sigma$ is called an *absolutely continuous* vector if the corresponding vector state taking $A \in \mathfrak{A}_n$ to $\langle \sigma(A)x, x\rangle$ is absolutely continuous.

Another straightforward but useful consequence is:

Corollary 2.5. *If σ is a $*$-extendible representation of $\mathfrak{A}_n$ and x, y are vectors lying in the type L part of $\mathfrak{S} = \overline{\sigma(\mathfrak{A}_n)}^{\text{WOT}}$ (or even in the type L part of $\mathfrak{T} = \overline{(\sigma\oplus\lambda)(\mathfrak{A}_n)}^{\text{WOT}}$), then $\varphi(A) = \langle \sigma(A)x, y\rangle$ is absolutely continuous. In particular, every vector lying in the type L part of $\mathcal{H}_\sigma$ is absolutely continuous.*

Proof. Relative to $\mathcal{H} \oplus \ell^2(\mathbb{F}^+_n)$, the structure projection $P_{\sigma\oplus\lambda}$ for $\mathfrak{T}$ decomposes as $P_1 \oplus 0$, with $P_1 \le P_\sigma$. By considering vectors of the form $x \oplus 0$, where x is in the range of P_σ, we may regard the type L part of $\mathfrak{S}$ as contained in the type L part of $\mathfrak{T}$. Thus, we may assume to be working with the representation $\sigma \oplus \lambda$ from the start. By [**11**, Theorem 1.6], the type L part of $\sigma \oplus \lambda$ is spanned by wandering vectors . For any wandering vector w, the functional $\varphi_w(A) = \langle (\sigma\oplus\lambda)(A)w, y\oplus 0\rangle$ is absolutely continuous because the cyclic subspace $(\sigma\oplus\lambda)(\mathfrak{A}_n)[w]$ is unitarily equivalent to $\ell^2(\mathbb{F}^+_n)$ and $y \oplus 0$ may be replaced with its projection into this subspace. By the previous corollary, the set of absolutely continuous functionals is a closed subspace. Taking linear combinations and limits shows that φ is in this closure, and hence also absolutely continuous. ∎

Now we wish to develop a connection between absolute continuity and certain intertwining operators.

Definition 2.6. Let σ be a $*$-extendible representation of $\mathfrak{A}_n$ on a Hilbert space $\mathcal{H}$ with generators $S_i = \sigma(\mathfrak{s}_i)$. Say that an operator $X \in \mathcal{B}(\ell^2(\mathbb{F}_n^+), \mathcal{H})$ *intertwines* S and L if $S_i X = X L_i$ for $1 \le i \le n$. Let $\mathcal{X}(\sigma)$ denote the set of all such intertwiners. We denote the range of the subspace $\mathcal{X}(\sigma)$ by $\mathcal{V}_{ac}(\sigma)$, that is, $\mathcal{V}_{ac}(\sigma) = \mathcal{X}(\sigma)\ell^2(\mathbb{F}_n^+)$.

Notice that $\mathcal{V}_{ac}(\sigma)$ is an invariant linear manifold for $\sigma(\mathfrak{A}_n)$. The next result shows that $\mathcal{V}_{ac}(\sigma)$ is also closed and equals the set of absolutely continuous vectors for σ.

Theorem 2.7. *For a representation σ of $\mathcal{E}_n$ on $\mathcal{H}$, let Q be the structure projection of $\mathfrak{T} = \overline{(\sigma \oplus \lambda)(\mathfrak{A}_n)}^{\text{WOT}}$. The following statements hold.*

i) *For $x, y \in \mathcal{V}_{ac}(\sigma)$, the functional $\psi(A) = \langle \sigma(A)x, y\rangle$ is absolutely continuous on $\mathfrak{A}_n$.*
ii) *If $x \in \mathcal{H}$ and $\psi(A) = \langle \sigma(A)x, x\rangle$ is absolutely continuous, then $x \in \mathcal{V}_{ac}(\sigma)$.*
iii) *The subspace $\mathcal{V}_{ac}(\sigma)$ is closed and is $\sigma(\mathfrak{A}_n)$-invariant. Moreover, it is the subspace of $\mathcal{H}$ corresponding to the projection onto $\mathcal{H}$ of the type L part of $(\sigma \oplus \lambda)(\mathfrak{A}_n)$, that is, $\mathcal{V}_{ac}(\sigma) = \operatorname{Ran}(P_\mathcal{H} Q^\perp)$.*

Proof. Let $x, y \in \mathcal{V}_{ac}(\sigma)$, and choose vectors ζ, η in $\ell^2(\mathbb{F}_n)$ and X, $Y \in \mathcal{X}(\sigma)$ with $X\zeta = x$ and $Y\eta = y$. Then

$$\varphi(A) = \langle \sigma(A)x, y\rangle = \langle \sigma(A)X\zeta, Y\eta\rangle = \langle X\lambda(A)\zeta, Y\eta\rangle = \langle \lambda(A)\zeta, X^*Y\eta\rangle,$$

so φ is absolutely continuous.

Suppose that $\psi(A) = \langle \sigma(A)x, x\rangle$ is absolutely continuous, say $\psi(A) = \langle \lambda(A)\zeta, \eta\rangle$. Theorem 1.6 of **[11]** shows that $x \oplus \zeta$ is a cyclic vector for an invariant subspace $\mathcal{M}$ of $(\sigma\oplus\lambda)(\mathfrak{A}_n)$ on which the restriction is unitarily equivalent to λ. Indeed, while the hypothesis of that theorem requires that σ be type L, this condition is used only to establish that ψ is absolutely continuous (in our new terminology). It is evident that a subspace of this type is the range of an intertwining isometry $V \in \mathcal{X}(\sigma \oplus \lambda)$. Let $X = P_\mathcal{H} V$. Then X intertwines S and L. Moreover, since $x \oplus \zeta$ is in the range of V, it follows that x is in the range of X, so (ii) holds.

We now push this argument a little further. Observe that as in the proof of [**11**, Theorem 1.6], given $t > 0$, ζ may be replaced by $t\zeta$. Therefore, if $x \in \mathcal{V}_{ac}(\sigma)$, the argument of the previous paragraph also shows that x belongs to the closed span of the wandering vectors for $(\sigma\oplus\lambda)(\mathfrak{A}_n)$. Thus x belongs to the type L part of $(\sigma\oplus\lambda)(\mathfrak{A}_n)$, whence $\mathcal{V}_{ac}(\sigma) \subseteq \operatorname{Ran}(P_\mathcal{H} Q^\perp)$. Conversely, since $P_\mathcal{H}$ and Q commute, any vector $x \in \operatorname{Ran}(P_\mathcal{H} Q^\perp)$ lies in the type L part of $\mathfrak{T}$, and thus $\psi(A) = \langle Ax, x\rangle$ is absolutely continuous by Corollary 2.5. But then $x \in \mathcal{V}_{ac}(\sigma)$ by part (ii). So $\operatorname{Ran}(P_\mathcal{H} Q^\perp) = \mathcal{V}_{ac}(\sigma)$. That $\mathcal{V}_{ac}(\sigma)$ is closed is now obvious. ∎

We now give a condition sufficient for the existence of wandering vectors.

Theorem 2.8. *Let X belong to $\mathcal{X}(\sigma)$. Then the following statements are equivalent.*

i) *The representations $\sigma|_{\overline{\operatorname{Ran} X}}$ and λ are unitarily equivalent;*
ii) *$\overline{\operatorname{Ran} X} = \mathfrak{S}[w]$ for some wandering vector w;*
iii) *$X^*X = R^*R$ for some non-zero $R \in \mathfrak{R}_n = \mathfrak{L}_n'$.*

In particular, this holds if X is bounded below.

Proof. The equivalence of (i) and (ii) is clear from the definitions.

To obtain (iii) $\Rightarrow$ (i), suppose that $X^*X = R^*R$. By restricting σ to the invariant subspace $\overline{\operatorname{Ran} X}$, we may suppose that X has dense range, and that $X\xi_\varnothing$ is a cyclic vector. We now show that σ is equivalent to λ.

Since $R \in \mathfrak{R}_n$, [**13**, Corollary 2.2], shows that R factors as the product of an isometry and an outer operator in $\mathfrak{R}_n$. The equality $X^*X = R^*R$ is unchanged if the isometry is removed, so we may assume that R has dense range. Since X and R have the same positive part, there is an isometry V such that $X = VR$ and $\operatorname{Ran} V = \overline{\operatorname{Ran} X}$; whence V is unitary. Then

$$(S_iV - VL_i)R = S_i(VR) - (VR)L_i = 0.$$

Therefore V intertwines S and L and so $\sigma|_{\operatorname{Ran} V}$ is equivalent to λ.

Finally, we show (ii) $\Rightarrow$ (iii). If there is an isometry $V \in \mathcal{X}(\sigma)$ with $\operatorname{Ran} V = \overline{\operatorname{Ran} X}$, then by again restricting to this range, we may assume that V is unitary, so that σ is equivalent to λ. So $V^*S_i = L_iV^*$. Hence

$$(V^*X)L_i = V^*S_iX = L_i(V^*X);$$

whence $R := V^*X$ belongs to $\mathfrak{L}_n' = \mathfrak{R}_n$. Therefore $X = VR$ and so $X^*X = R^*R$.

Now suppose that X is bounded below. Again we may suppose that X has dense range, hence X is invertible.

Consider the Wold decomposition of S. The Cuntz part is supported on

$$\begin{aligned} \mathcal{N} &:= \bigcap_{k\geq 1} \sum_{|w|=k} \operatorname{Ran} S_w = \bigcap_{k\geq 1} \sum_{|w|=k} S_w X \ell^2(\mathbb{F}_n^+) \\ &= \bigcap_{k\geq 1} \sum_{|w|=k} X L_w \ell^2(\mathbb{F}_n^+) = X \bigcap_{k\geq 1} \sum_{|w|=k} L_w \ell^2(\mathbb{F}_n^+) = \{0\} \end{aligned} \tag{1}$$

Hence σ is a multiple of λ. Since $\operatorname{Ran} X$ has a cyclic vector $X\xi_\varnothing$, σ has multiplicity one, and thus is equivalent to λ. ∎

As an immediate corollary, we note the existence of wandering vectors is characterized by a structural property of $\mathcal{X}(\sigma)$.

Corollary 2.9. *Let σ be a representation of $\mathcal{E}_n$ on $\mathcal{H}$ with generators $S_i = \sigma(\mathfrak{s}_i)$. Then S has a wandering vector if and only if there exists $X \in \mathcal{X}(\sigma)$ such that X is bounded below.*

Proof. If $\eta \in \mathcal{H}$ is wandering for $\mathfrak{S}$, then the isometric map determined by $X\xi_w = w(S)\eta$ belongs to $\mathcal{X}(\sigma)$. The converse follows from the theorem. ∎

Remark 2.10. If one only has $X^*X \geq R^*R$ for a non-zero $R \in \mathfrak{R}_n$, one may still deduce that $\operatorname{Ran} X$ has wandering vectors. To do this, use Douglas' Lemma [**16**] to factor $R = YX$. Then argue as in Theorem 2.8 that $YS_i = L_iY$. Then with $\mathcal{N}$ as in (1), one can show that $Y\mathcal{N} = \{0\}$. Since Y has dense range, σ has a summand equivalent to λ. Moreover, since the range of an intertwiner consists of absolutely continuous vectors, the existence of this summand and Lemma 3.2 below show that the range of X is spanned by wandering vectors.

Example 2.11. There are intertwining maps whose range is not equivalent to λ. For example, consider the atomic representation of type $\pi_{z_2^\infty}$ [**13**, Example 3.2]. Then the restriction of S_2 to the *spine* $\ell^2(\mathbb{Z} \times \{0\})$ is the bilateral shift.

Observe that there is a summable sequence $(a_k)_{k\in\mathbb{Z}}$ such that $\sum_{k\in\mathbb{Z}} a_k\xi_{k,0}$ is cyclic for the bilateral shift. Indeed, Beurling's Theorem states that the (cyclic) invariant subspaces of the bilateral shift, considered as M_z on $L^2(\mathbb{T})$, have the form $L^2(E)$ for a measurable subset E of $\mathbb{T}$ or the form wH^2 where $|w| = 1$ a.e. Thus if a function g vanishes on a set of positive measure, it generates $L^2(\operatorname{supp}(g))$. On the other hand, if there is an outer function f in H^2

with $|f| = |g|$ a.e., then the cyclic subspace is wH^2 where $w = g/f$. This occurs if and only if $\log|g|$ belongs to $L^1(\mathbb{T})$. So choose a C^2 function g on $\mathbb{T}$ which vanishes at a single point in such a way that $\log|g|$ is not integrable. For example, make $g(\theta) = e^{-1/|\theta|}$ near $\theta = 0$ and smooth. Lying in C^2 guarantees that the Fourier coefficients are summable.

For each $k \in \mathbb{Z}$, there is an intertwining isometry V_k with $V_k\xi_\varnothing = \xi_{k,0}$. Then $V = \sum_{k\in\mathbb{Z}} a_k V_k$ is an intertwiner. Moreover, $V\xi_\varnothing$ is cyclic for this Cuntz representation. So V has dense range; but the representation $\pi_{z_2^\infty}$ is not equivalent to λ.

Remark 2.12. Consider the completely positive map on $\mathcal{B}(\mathcal{H})$ given by $\Phi(A) = \sum_i S_i A S_i^*$. Suppose that X intertwines S and L. Then

$$\Phi^k(XX^*) = \sum_{|w|=k} S_w XX^* S_w^* = X \sum_{|w|=k} L_w L_w^* X^* = XQ_k X^* \le XX^*.$$

Moreover, $\textsc{sot}\text{–}\lim_k \Phi^k(X^*X) = 0$. This latter condition is called *purity* by Popescu **[29]**. Under these two hypotheses, namely $\Phi(D) \le D$ and $\textsc{sot}\text{–}\lim_k \Phi^k(D) = 0$, Popescu proves the converse, that $D = X^*X$ for an intertwiner $SX = XL^{(\infty)}$ using his Poisson transform.

3. Wandering vectors and absolute continuity

In **[11]**, we showed that in the presence of summands which contain wandering vectors, the entire type L part is spanned by wandering vectors. In this section, we use the ideas of the previous section to strengthen this significantly by showing that the presence of one wandering vector implies that the type L part is spanned by wandering vectors. We then consider the various ways in which a representation can appear to be type L.

Definition 3.1. Let σ be a $*$-extendible representation of $\mathfrak{A}_n$. We say that σ is *type L* if the free semigroup algebra generated by $\sigma(\mathfrak{s}_1), \dots, \sigma(\mathfrak{s}_n)$ is type L.

A representation σ is *weak type L* if $\sigma \oplus \lambda$ is type L.

A representation σ is *weak-$*$ type L* if $\sigma^{(\infty)}$ is type L.

The representation σ of $\mathfrak{A}_n$ is *absolutely continuous* if every vector state $\psi(A) = \langle \sigma(A)x, x\rangle$ is absolutely continuous.

Notice that the restriction of a $*$-extendible representation σ of $\mathfrak{A}_n$ to the invariant subspace $\mathcal{V}_{ac}(\sigma)$ produces an absolutely continuous representation. We call this restriction the *absolutely continuous part* of σ.

Lemma 3.2. *If σ is absolutely continuous and has a wandering vector, then $\mathcal{H}$ is spanned by its wandering vectors. In particular, σ is type L.*

Proof. Let η be a wandering vector in $\mathcal{H}$, and set $\mathcal{H}_0 = \mathfrak{S}[\eta]$. Let V be the isometry in $\mathcal{X}(\sigma)$ mapping $\ell^2(\mathbb{F}_n^+)$ onto $\mathcal{H}_0$.

By Theorem 2.7, every vector $x \in \mathcal{H}$ is in the range of some intertwining map $X \in \mathcal{X}(\sigma)$. We may assume that $\|X\| = 1/2$. Then $V \pm X$ are intertwiners which are bounded below. By Theorem 2.8, the ranges of these two intertwiners are the ranges of isometric intertwiners, and thus are spanned by wandering vectors. But the range of X is contained in the sum of the ranges of $V \pm X$; and hence x is contained in the span of all wandering vectors. ∎

Corollary 3.3. *If σ is any representation of $\mathcal{E}_n$ such that $\sigma(\mathfrak{A}_n)$ has a wandering vector, then the span of the wandering vectors for $\sigma(\mathfrak{A}_n)$ is $\mathcal{V}_{ac}(\sigma)$.*

Proof. Any wandering vector is an absolutely continuous vector, so simply restrict σ to the $\sigma(\mathfrak{A}_n)$-invariant subspace consisting of absolutely continuous vectors and apply the lemma. ∎

We now delineate the various type L forms, and their relationships as we know today. There are no known examples of absolutely continuous representations without wandering vectors.

Theorem 3.4. *Consider the following conditions for a $*$-extendible representation σ of $\mathfrak{A}_n$:*

(1a) *σ is absolutely continuous*
(1b) *$\sigma \oplus \lambda$ is type L (i.e. σ is weak type L)*
(1c) *$\sigma \oplus \tau$ is type L for any (all) type L representation τ.*
(2a) *$\sigma^{(\infty)}$ is type L (i.e. σ is weak-$*$ type L)*
(2b) *σ is absolutely continuous and $\sigma^{(\infty)}$ has a wandering vector*
(2c) *$\sigma^{(\infty)}$ is spanned by wandering vectors*
(3a) *σ is type L*
(3b) *σ is absolutely continuous and $\sigma^{(k)}$ has a wandering vector for some finite k*
(3c) *$\sigma^{(k)}$ is spanned by wandering vectors for some finite k*
(4a) *σ is absolutely continuous and has a wandering vector*
(4b) *σ is spanned by wandering vectors*

Then properties with the same numeral are equivalent, and larger numbers imply smaller.

Proof. (1a) $\Rightarrow$ (1b). If σ is absolutely continuous, then $\sigma \oplus \lambda$ is absolutely continuous and has a wandering vector. Thus by Lemma 3.2, $\sigma \oplus \lambda$ is spanned by its wandering vectors, and so is type L.

(1b) $\Rightarrow$ (1a). Since $\sigma \oplus \lambda$ is type L and has a wandering vector, Lemma 3.2 shows that $\sigma \oplus \lambda$ is spanned by its wandering vectors. Thus $\sigma \oplus \lambda$ is absolutely continuous, and hence so is σ.

(1a) $\Rightarrow$ (1c): If τ is any type L representation, there is an integer p so that $\tau^{(p)}$ has a wandering vector. Thus $(\sigma \oplus \tau)^{(p)}$ is absolutely continuous and has a wandering vector, and so is also type L. However being type L is not affected by finite ampliations, as this has no effect on the WOT-closure. So $\sigma \oplus \tau$ is type L.

(1c) $\Rightarrow$ (1a): If $\sigma \oplus \tau$ is type L for some type L representation τ, then $\sigma \oplus \tau$ is absolutely continuous. By considering vectors of the form $x \oplus 0$, we find that σ is absolutely continuous. So (1a), (1b), and (1c) are all equivalent.

If σ is weak-$*$ type L, then $\sigma^{(\infty)}$ has a finite ampliation which is spanned by wandering vectors. But of course this ampliation is equivalent to $\sigma^{(\infty)}$, so (2a) implies (2c). Clearly, if (2c) holds, then every vector in $\mathcal{H}^{(\infty)}$ is absolutely continuous, so in particular, σ is an absolutely continuous representation; thus (2b) holds. If σ is absolutely continuous and $\sigma^{(\infty)}$ has a wandering vector, then $\mathcal{H}^{(\infty)}$ is spanned by wandering vectors and thus $\sigma^{(\infty)}$ is type L. So (2a), (2b), and (2c) are all equivalent and imply (1).

The equivalence of (3a), (3b) and (3c) follows from [**11**, Corollary 1.9], and evidently implies (2).

By Lemma 3.2, (4a) and (4b) are equivalent and clearly imply (3). ∎

It is worthwhile examining the various weaker notions of type L in light of the Structure Theorem for Free Semigroup Algebras [**11**]. Let σ be a representation of $\mathcal{E}_n$ and let $\mathfrak{S}$ and

$\mathfrak{W}$ denote the corresponding free semigroup algebra and von Neumann algebra respectively. Then there is a projection P in $\mathfrak{S}$ characterized as the largest projection in $\mathfrak{S}$ for which $P\mathfrak{S}P$ is self-adjoint. Then $\mathfrak{S} = \mathfrak{W}P + \mathfrak{S}P^\perp$, $P^\perp\mathcal{H}$ is invariant for $\mathfrak{S}$ and $\mathfrak{S}P^\perp$ is type L. We wish to break this down a bit more.

Definition 3.5. A representation σ of $\mathcal{E}_n$ or $\mathcal{O}_n$ is of *von Neumann type* if the corresponding free semigroup algebra $\mathfrak{S}$ is a von Neumann algebra. If σ has no summand of either type L or von Neumann type, say that it is of *dilation type*. We also will say that σ is weak-$*$ of some type if $\sigma^{(\infty)}$ is of that type.

A very recent result of Charles Read [**30**] shows that there can indeed be representations of von Neumann type.

The reason for the nomenclature *dilation type* is that after all summands of von Neumann type and type L are removed, the remainder must have a non-zero projection P prescribed by the structure theorem such that $P\mathcal{H}$ is cyclic and $P^\perp\mathcal{H}$ is cyclic for $\mathfrak{S}^*$. For these algebras, the type L corner must be a multiple of λ. To see this, consider the subspace $\mathcal{W} = \big(\sum_i S_i P\mathcal{H}\big) \ominus P\mathcal{H}$. This is a wandering subspace for the type L part. It is necessarily non-zero, for otherwise $\mathfrak{S}$ would be a von Neumann algebra. Moreover, $\mathcal{W}$ is cyclic for the type L corner because of the cyclicity of $P\mathcal{H}$. Hence the type L part is equivalent to $\lambda^{(\dim \mathcal{W})}$. This is an observation that was, unfortunately, overlooked in [**11**]. Hence one sees that the compressions $A_i = PS_i|_{P\mathcal{H}}$ form a row contraction with S_i as their minimal isometric dilation (in the sense of Frahzo–Bunce–Popescu). We record the most useful part of this for future reference.

Proposition 3.6. *If σ is dilation type, then it has wandering vectors. In particular, dilation type and weak-$*$ dilation type coincide.*

Proof. The first statement was proven in the preamble. Once one has a wandering vector, the span of the wandering vectors includes all of the absolutely continuous vectors, which includes the weak-$*$ type L part. ∎

We can now clarify the exceptional case in which there may be pathology.

Proposition 3.7. *Let σ be a $*$-extendible representation of $\mathfrak{A}_n$. If the type L and absolutely continuous parts do not coincide, then σ is of von Neumann type, and decomposes as $\sigma \simeq \sigma_a \oplus \sigma_s$ where σ_a is absolutely continuous and σ_s has no absolutely continuous part.*

Proof. Decompose $\sigma \simeq \sigma_v \oplus \sigma_d \oplus \sigma_l$ into its von Neumann, dilation and type L parts. By Proposition 3.6, if there is a dilation part, then there are wandering vectors. So by Corollary 3.3, the type L and absolutely continuous parts coincide. Likewise if there is a type L part, the equivalence of (1a) and (1b) in Theorem 3.4 shows that the type L and absolutely continuous parts will coincide. So σ is necessarily of von Neumann type.

Since $\mathcal{V}_{ac}(\sigma)$ is invariant for $\mathfrak{S}_\sigma$, and $\mathfrak{S}_\sigma$ is a von Neumann algebra, $\mathcal{V}_{ac}(\sigma)$ is a reducing subspace for $\mathfrak{S}_\sigma$. This gives the desired decomposition $\sigma \simeq \sigma_a \oplus \sigma_s$. ∎

Definition 3.8. Call a $*$-extendible representation σ of $\mathfrak{A}_n$ *regular* if the absolutely continuous and type L parts of σ coincide.

Remark 3.9. Proposition 3.7 shows that the only pathology that can occur in the various weak type L possibilities is due to a lack of wandering vectors.

It is conceivable that a representation is type L but has no wandering vectors. Such an algebra is reductive and nonselfadjoint. There is no operator algebra known to have this property. So the (unlikely) existence of such an algebra would yield a counterexample to a well-known variant of the invariant subspace problem.

A $*$-extendible representation σ which is weak-$*$ type L but not type L must be von Neumann type by the preceding proposition. But then $\overline{\sigma(\mathfrak{A}_n)}^{\text{w-*}}$ would be a weak-$*$ closed subalgebra isomorphic to $\mathfrak{L}_n$ which is WOT-dense in a von Neumann algebra. We have no free semigroup algebra example of this type of behaviour. However, Loebl and Muhly [**22**] have constructed an operator algebra which is weak-$*$ closed and nonselfadjoint, but with the WOT-closure equal to a von Neumann algebra. Therefore it is conceivable that such a free semigroup algebra could exist.

Finally, one could imagine that σ is of weak-$*$ von Neumann type but absolutely continuous.

Clearing up the question of whether any of these possibilities can actually occur remains one of the central questions in the area. We conjecture that every representation is regular. Indeed, we would go further and speculate that type L representations always have wandering vectors.

4. Reflexivity and hyper-reflexivity

In this section, we establish two reflexivity results that extend previous work in light of the previous section.

Theorem 4.1. *If $\mathfrak{S}$ is a free semigroup algebra which has a wandering vector, then it is reflexive.*

Proof. By [**11**, Proposition 5.3], $\mathfrak{S}$ is reflexive if and only if the restriction to its type L part is reflexive. Thus, without loss of generality, we may assume that $\mathfrak{S}$ is type L. Since $\mathfrak{S}$ is type L and has a wandering vector, Lemma 3.2 shows that $\mathcal{H}$ is spanned by wandering vectors. Let $W \subseteq \mathcal{H}$ be the set of all unit wandering vectors. For each $\alpha \in W$, let $\mathcal{H}_\alpha = \mathfrak{S}[\alpha]$ and let $V_\alpha : \ell^2(\mathbb{F}_n^+) \to \mathcal{H}_L$ be the intertwining isometry which sends ξ_w to $S_w\alpha$. Then the invariant subspaces $\mathcal{H}_\alpha$ span $\mathcal{H}$ and each restriction $\mathfrak{S}|_{\mathcal{H}_\alpha}$ is unitarily equivalent to $\mathfrak{L}_n$ via V_α.

If $T \in \operatorname{Alg}\operatorname{Lat}\mathfrak{S}$, then $\mathcal{H}_\alpha$ is invariant for T. Since $\mathfrak{L}_n$ is reflexive, there is an element $B_\alpha \in \mathfrak{L}_n$ so that $T|_{\mathcal{H}_\alpha} = V_\alpha B_\alpha V_\alpha^*$. For each $\alpha \in W$, there is an element $A_\alpha \in \mathfrak{S}$ so that $A_\alpha|_{\mathcal{H}_\alpha} = V_\alpha B_\alpha V_\alpha^*$. Fix an element $\alpha_0 \in W$, let $V_0 = V_{\alpha_0}$ and $A_0 = A_{\alpha_0}$. We shall show that $T = A_0$. By replacing T with $T - A_0$, we may assume that $T|_{\mathcal{H}_0} = 0$, so that our task is to show $T = 0$.

Given $\alpha \in W$, the operator $X = V_0 + .5V_\alpha$ is an intertwining map between S and L which is bounded below. Moreover, $\mathcal{M} := \operatorname{Ran} X$ is closed and invariant for $\mathfrak{S}$; hence $\mathcal{M}$ is also invariant for T. But

$$TX\xi_\varnothing = TV_0\xi_\varnothing + .5TV_\alpha\xi_\varnothing = .5A_\alpha V_\alpha\xi_\varnothing =: y$$

belongs to $\mathcal{H}_\alpha \cap \mathcal{M}$. This implies that there is a vector $\zeta \in \ell^2(\mathbb{F}_n^+)$ so that $y = X\zeta = V_0\zeta + .5V_\alpha\zeta$ belongs to $\mathcal{H}_\alpha$, and thus $V_0\zeta$ lies in $\mathcal{H}_0 \cap \mathcal{H}_\alpha$. If $\zeta = 0$ then $y = 0$, so that A_α has the non-zero vector $V_\alpha\xi_\varnothing$ in its kernel. Otherwise $V_0\zeta$ is a non-zero vector in $\mathcal{H}_0 \cap \mathcal{H}_\alpha$ and $A_\alpha V_0\zeta = TV_0\zeta = 0$. Therefore, $A_\alpha|_{\mathcal{H}_\alpha}$ has non-trivial kernel. Hence B_α is an element of $\mathfrak{L}_n$ with non-trivial kernel. Since non-zero elements of $\mathfrak{L}_n$ are injective [**13**, Theorem 1.7], we deduce that $B_\alpha = 0$. Hence $0 = A_\alpha|_{\mathcal{H}_\alpha} = T|_{\mathcal{H}_\alpha}$. Since $\bigvee_{\alpha\in W} H_\alpha = \mathcal{H}$, we conclude that $T = 0$ as desired. ∎

Recall that an operator algebra $\mathfrak{A}$ is hyper-reflexive if there is a constant C so that

$$\operatorname{dist}(T,\mathfrak{A}) \le C\beta_{\mathfrak{A}}(T) := C \sup_{P\in\operatorname{Lat}\mathfrak{A}} \|P^\perp T P\|.$$

The known families of hyper-reflexive algebras are fairly small. It includes nest algebras [**3**] with constant 1, the analytic Toeplitz algebra [**9**] and the free semigroup algebras $\mathfrak{L}_n$ [**13**]. Bercovici [**4**] obtained distance constant 3 for all algebras having property $\mathcal{X}_{0,1}$ and also showed that an operator algebra $\mathfrak{A}$ has property $\mathcal{X}_{0,1}$ whenever its commutant contains two isometries with orthogonal ranges. In particular, $\mathfrak{L}_n$ has property $\mathcal{X}_{0,1}$ when $n \ge 2$. Bercovici's results significantly increased the known class of hyper-reflexive algebras.

There is a long-standing open question about whether all von Neumann algebras are hyper-reflexive, which is equivalent to whether every derivation is inner [**8**]. The missing cases are von Neumann algebras whose commutant are certain intractable type II_1 algebras. This could include certain type II_∞ representations of $\mathcal{O}_n$, and hence would apply in our context. So for the next result, we restrict ourselves to the type L case.

Theorem 4.2. *If $\mathfrak{S}$ is a type L free semigroup algebra which has a wandering vector, then $\mathfrak{S}$ is hyper-reflexive.*

Before giving the proof, we pause for the following remark.

Remark 4.3. If $\mathfrak{S}$ is type L and has a wandering vector, then by [**13**] it has property $\mathbb{A}_1$ and by [**15**] it even has property $\mathbb{A}_{\aleph_0}$. In particular, Theorem 4.2 together with a result from [**19**] implies that every weak-$*$ closed subspace of a type L free semigroup algebra with a wandering vector is also hyper-reflexive. Even though $\mathcal{X}_{0,1}$ is only a bit stronger than $\mathbb{A}_{\aleph_0}$, we were unable to show that $\mathfrak{S}$ has it. So we are unable to apply Bercovici's argument. Thus, the proof which follows uses methods reminiscent of those used in [**13**].

If $\mathfrak{S}$ is type L and has no wandering vector, then as noted in Remark 3.9, the algebra will be nonselfadjoint and reductive. In particular, it is not reflexive.

Proof. Let $T \in \mathcal{B}(\mathcal{H})$, and set $\beta(T) = \sup_{P\in\operatorname{Lat}\mathfrak{S}} \|P^\perp TP\|$. Let x_0 be a wandering vector of $\mathfrak{S}$. Then $\mathfrak{S}|_{\mathfrak{S}[x_0]} \simeq \mathfrak{L}_n$. Since $\mathfrak{L}_n$ is hyper-reflexive with constant 3, there exists an $A \in \mathfrak{S}$ with $\|(T-A)|_{\mathfrak{S}[x_0]}\| \le 3\beta(T)$. By replacing T with $T - A$, we can assume that $\|T|_{\mathfrak{S}[x_0]}\| \le 3\beta(T)$.

Let x be a wandering vector with $x \ne x_0$ and let V be the isometric intertwiner from $\ell^2(\mathbb{F}_n^+)$ onto $\mathfrak{S}[x]$ satisfying $V\xi_w = S_w x$. We shall show that

$$\big\|T|_{\mathfrak{S}[x]}\big\| \le 26\beta(T). \tag{2}$$

Let $x_i = S_i x_0$, for $i = 1, 2$. For $i = 0, 1, 2$, define isometric intertwiners V_i from $\ell^2(\mathbb{F}_n^+)$ to $\mathcal{H}$ by $V_i\xi_w = S_w x_i$ for $w \in \mathbb{F}_n^+$.

For $i = 1, 2$, set $T_i = V_i + rV$ where $0 < r < 1/\sqrt{2}$, and define $\mathcal{N}_i = \operatorname{Ran} T_i$. We claim that $\mathcal{N}_1$ and $\mathcal{N}_2$ are at a positive angle to each other; so that $\mathcal{N}_1 \cap \mathcal{N}_2 = \{0\}$ and $\mathcal{N}_1 + \mathcal{N}_2$ is closed. Indeed, using $\delta := 1 - r\sqrt{2} > 0$,

$$\begin{aligned}\|T_1\xi - T_2\eta\| &\ge \|V_1\xi - V_2\eta\| - r\|V(\xi-\eta)\| \\ &\ge \|\xi\oplus\eta\| - r(\|\xi\| + \|\eta\|) \quad \ge \delta\|\xi\oplus\eta\|.\end{aligned}$$

So the natural map of $\mathcal{N}_1 \oplus \mathcal{N}_2$ onto $\mathcal{N}_1 + \mathcal{N}_2$ is an isomorphism.

Observe next that for any $w \in \mathbb{F}_n^+$, we have

$$\begin{aligned}\sum_{|w|=k} S_w(\mathcal{N}_1 + \mathcal{N}_2) &= \sum_{|w|=k} S_w T_1 \ell^2(\mathbb{F}_n^+) + S_w T_2 \ell^2(\mathbb{F}_n^+) \\ &= T_1 \sum_{|w|=k} L_w \ell^2(\mathbb{F}_n^+) + T_2 \sum_{|w|=k} L_w \ell^2(\mathbb{F}_n^+).\end{aligned}$$

Therefore,

$$\lim_{k\to\infty} \sum_{|w|=k} S_w(\mathcal{N}_1 + \mathcal{N}_2) = 0.$$

As $\sum_{j=1}^n S_j \mathcal{N}_i$ has co-dimension one in $\mathcal{N}_i$, we find that.

$$\dim \Big(\mathcal{N}_1 + \mathcal{N}_2 - \sum_{j=1}^n S_j(\mathcal{N}_1 + \mathcal{N}_2) \Big) = 2.$$

By the Wold decomposition, we deduce that $\mathfrak{S}|_{\mathcal{N}_1+\mathcal{N}_2} \simeq \mathfrak{L}_n^{(2)}$. This algebra is hyper-reflexive with distance constant 3. So there is an element $A \in \mathfrak{S}$ such that $\|(T-A)|_{\mathcal{N}_1+\mathcal{N}_2}\| \leq 3\beta(T)$.

Note that $\mathcal{M} := \mathfrak{S}[x_1 - x_2] = \mathrm{Ran}(T_1 - T_2) \subset \mathcal{N}_1 + \mathcal{N}_2$; also, $\mathcal{M}$ is a cyclic subspace of $\mathfrak{S}[x_0]$. Since $\|T|_{\mathcal{M}}\| \leq 3\beta(T)$, $\|A|_{\mathcal{M}}\| \leq 6\beta(T)$. As $\mathfrak{S}$ is type L, $\|A\| = \|A|_{\mathcal{M}}\|$, so $\|T|_{\mathcal{N}_1+\mathcal{N}_2}\| \leq 9\beta(T)$. We now improve this to an estimate of $\|T|_{\mathfrak{S}[x]}\|$.

Suppose that y is a unit vector in $\mathfrak{S}[x]$. Observe that

$$T_1(V^* y) = V_1 V^* y + r y$$

lies in $\mathcal{N}_1 \subset \mathcal{N}_1 + \mathcal{N}_2$. So $\|T(V_1 V^* y + r y)\| \leq 9(1+r)\beta(T)$. As $V_1 V^* y$ is a unit vector in $\mathfrak{S}[x_0]$, $\|T V_1 V^* y\| \leq 3\beta(T)$. Hence

$$\|Ty\| \leq r^{-1}(12 + 9r)\beta(T).$$

Choosing r sufficiently close to $1/\sqrt{2}$ yields that $\|T|_{\mathfrak{S}[x]}\| \leq 26\beta(T)$, so (2) holds.

We now can estimate $\|T\|$. Fix any unit vector $y \in \mathcal{H}$, and let $\mathfrak{T}$ be the free semigroup algebra generated by $S_i \oplus L_i$. Since $\mathfrak{S}$ is type L, by [**11**, Theorem 1.6] there is a vector $\zeta \in \ell^2(\mathbb{F}_n^+)$ with $\|\zeta\| < \varepsilon$ such that $\mathfrak{T}[y \oplus \zeta]$ is a subspace of $\mathcal{H} \oplus \ell^2(\mathbb{F}_n^+)$ which is generated by a wandering vector. Hence $\mathfrak{T}[y \oplus \zeta]$ is the range of an isometry W' from $\ell^2(\mathbb{F}_n^+)$ to $\mathcal{H} \oplus \ell^2(\mathbb{F}_n^+)$ intertwining L_i with $S_i \oplus L_i$. Then $W'' := P_{\mathcal{H}} W'$ is a contraction in $\mathcal{B}(\ell^2(\mathbb{F}_n^+), \mathcal{H})$ satisfying $S_i W'' = W'' L_i$. Moreover, there is a vector $\xi \in \ell^2(\mathbb{F}_n^+)$ of norm $(1+\varepsilon^2)^{1/2}$ such that $W''\xi = y$. Identify $\mathfrak{S}[x_0]$ with $\ell^2(\mathbb{F}_n^+)$ via the isometry $V_0 \in \mathcal{B}(\ell^2(\mathbb{F}_n^+), \mathcal{H})$, and set $W := W'' V_0^* \in \mathcal{B}(\mathfrak{S}[x_0], \mathcal{H})$ and $w := V_0 \xi$.

Let J be the inclusion map of $\mathfrak{S}[x_0]$ into $\mathcal{H}$. For $|t| < 1$, consider $V_t = J + tW$. This is an intertwining map which is bounded below, and thus by Theorem 2.8, there is a wandering vector x_t of $\mathfrak{S}$ so that $\mathrm{Ran}(V_t) = \mathfrak{S}[x_t]$. So

$$\|T(w + ty)\| \leq 26\beta(T)\|w + ty\|.$$

Since $\|Tw\| \leq 3\beta(T)\|w\|$, if we let t increase to 1 and ε decrease to 0, we obtain $\|Ty\| \leq 55\beta(T)$. So $\|T\| \leq 55\beta(T)$. Thus, $\mathfrak{S}$ is hyper-reflexive with constant at most 55. ∎

The following proposition is complementary to [**11**, Proposition 2.10] showing that if $\mathfrak{S}$ is of Cuntz type, then $\mathfrak{S}'' = \mathfrak{W}$ is a von Neumann algebra.

Proposition 4.4. *Let $\mathfrak{S}$ be a free semigroup algebra acting on a Hilbert space $\mathcal{H}$ which is not of Cuntz type. Then $\mathfrak{S}'' = \mathfrak{S}$.*

Proof. Since $\mathfrak{S}$ is not Cuntz type, by the Wold decomposition, it has a direct summand equivalent to $\mathfrak{L}_n$. That is, we may decompose the generators $S_1, ..., S_n$ as $S_i = T_i \oplus L_i$ on $\mathcal{H} = H_1 \oplus \ell^2(\mathbb{F}_n^+)$.

Let $\mathfrak{W}$ be the von Neumann algebra generated by $\mathfrak{S}$. By the Structure Theorem [**11**, Theorem 2.6], there is a largest projection P in $\mathfrak{S}$ such that $P\mathfrak{S}P$ is self-adjoint and $\mathfrak{S} = \mathfrak{W}P + P^\perp \mathfrak{S} P^\perp$. Now $\mathfrak{S}'' \subset \mathfrak{W}'' = \mathfrak{W}$, so $\mathfrak{S}P \subset \mathfrak{S}''P \subset \mathfrak{W}P = \mathfrak{S}P$; whence $\mathfrak{S}''P = \mathfrak{S}P$.

By Theorem 3.2, $P^\perp\mathcal{H}$ is spanned by wandering vectors. For any wandering vector x_α, let V_α be the canonical intertwining isometry from $\ell^2(\mathbb{F}_n^+)$ into $\mathcal{H}$ defined by $V_\alpha \xi_w = S_w x_\alpha$ for $w \in \mathbb{F}_n^+$. If we select $x_0 = 0 \oplus \xi_\varnothing$, then V_0 maps onto the free summand. It is easy to check that $V_\alpha V_0^*$ commutes with $\mathfrak{S}$.

Let $A \in \mathfrak{S}''$. Then since $0 \oplus I$ commutes with $\mathfrak{S}$, A must have the form $A = A_1 \oplus A_2$. Moreover, $A_2 \in \mathfrak{L}_n'' = \mathfrak{L}_n$ by [**13**]. There is an element $B \in \mathfrak{S}$ such that $B = B_1 \oplus A_2$. Subtracting this from A, we may suppose that $A = A_1 \oplus 0$. Then

$$Ax_\alpha = A(V_\alpha V_0^*)x_0 = (V_\alpha V_0^*)Ax_0 = 0.$$

Thus $AP^\perp = 0$. As above, AP lies in $\mathfrak{S}$, whence A belongs to $\mathfrak{S}$. ∎

5. A Kaplansky Density Theorem

Kaplansky's famous density theorem states that if σ is a $*$-representation of a C*-algebra $\mathfrak{A}$, then the unit ball of $\sigma(\mathfrak{A})$ is WOT-dense in the ball of the von Neumann algebra $\mathfrak{W} = \overline{\sigma(\mathfrak{A})}^{\text{w-}*} = \overline{\sigma(\mathfrak{A})}^{\text{WOT}}$. In general, there is no analogue of this for operator algebras which are not self-adjoint. Indeed, it is possible to construct many examples of pathology [**32**]. On the other hand, the density theorem is such a useful fact that it is worth seeking such a result whenever possible. In this section, we establish a density theorem for regular representations of $\mathfrak{A}_n$.

Consider the following "proof" of the Kaplansky density theorem. Consider the C*-algebra $\mathcal{A}$ sitting inside its double dual $\mathcal{A}^{**}$, which is identified with the universal enveloping von Neumann algebra $\mathcal{W}_u$ of $\mathcal{A}$. Any representation σ of $\mathcal{A}$ extends uniquely to a normal representation $\overline{\sigma}$ of $\mathcal{W}_u$ onto $\mathcal{W} = \sigma(\mathcal{A})''$. Because this is a surjective $*$-homomorphism of C*-algebras, it is a complete quotient map. In particular, any element of the open ball of $\mathcal{W}$ is the image of an element in the ball of $\mathcal{W}_u$. Now by Goldstine's Theorem, every element of the ball of $\mathcal{A}^{**}$ is the weak-$*$ limit of a net in the ball of $\mathcal{A}$. Mapping this down into $\mathcal{W}$ by $\overline{\sigma}$ yields the result.

We call this a "proof" because the usual argument that $\mathcal{W}_u$ is isometrically isomorphic to $\mathcal{A}^{**}$ requires the Kaplansky density theorem. Indeed, each state on $\mathcal{A}$ extends to vector state on $\mathcal{W}_u$. But the fact that all functionals on $\mathcal{A}$ have the same norm on $\mathcal{W}_u$ follows from knowing that the unit ball is weak-$*$ dense in the ball of $\mathcal{W}_u$. It seems quite likely that the use of Kaplansky's density theorem could be avoided, making this argument legitimate.

Nevertheless, we can use this argument to decide when such a result holds in our context. Moreover, in the C*-algebra context, Kaplansky's theorem extends easily to matrices over the algebra because they are also C*-algebras. In our case, it follows from the proof.

The double dual of $\mathfrak{A}_n$ may be regarded as a free semigroup algebra, in the following way. We shall use it as a tool in the proof of the Kaplansky density theorem, and we pause to highlight some of its features.

Definition 5.1. Regard $\mathfrak{A}_n$ as a subalgebra of $\mathcal{E}_n$. Then the second dual $\mathfrak{A}_n^{**}$ is naturally identified with a weak-$*$ closed subalgebra of $\mathcal{E}_n^{**}$. This will be called the *universal free semigroup algebra*. That this is a free semigroup algebra will follow from the discussion below. We shall denote its structure projection by P_u.

Denote by j the natural inclusion of a Banach space into its double dual. Then $j(\mathfrak{A}_n)$ generates $\mathcal{E}_n^{**}$ as a von Neumann algebra.

If σ is a $*$-representation of $\mathcal{E}_n$ on a Hilbert space $\mathcal{H}$, then σ has a unique extension to a normal $*$-representation $\overline{\sigma}$ of $\mathcal{E}_n^{**}$ on the same Hilbert space $\mathcal{H}$. Moreover, $\overline{\sigma}(\mathcal{E}_n^{**})$ is the von Neumann algebra $\sigma(\mathcal{E}_n)''$ generated by $\sigma(\mathcal{E}_n)$.

Fix once and for all a universal representation π_u of $\mathcal{E}_n$ acting on the Hilbert space $\mathcal{H}_u$ with the property that π_u has infinite multiplicity, i.e. $\pi_u \simeq \pi_u^{(\infty)}$. This is done to ensure that the WOT and weak-$*$ topologies coincide on the universal von Neumann algebra $\mathfrak{W}_u = \pi_u(\mathcal{E}_n)''$. Then $\overline{\pi_u}$ is a $*$-isomorphism of $\mathcal{E}_n^{**}$ onto $\mathfrak{W}_u$. This carries $\mathfrak{A}_n^{**}$ onto the weak-$*$ closed subalgebra closure $\mathfrak{S}_u$ of $\pi_u(\mathfrak{A}_n)$. This coincides with the WOT-closure, and thus this is a free semigroup algebra. Hence $\mathfrak{A}_n^{**}$ is a free semigroup algebra.

Since π_u has infinite multiplicity and contains a copy of λ, its type L part is spanned by wandering vectors. So by Theorem 3.4, the range of $\overline{\pi_u}(P_u^\perp)$ is $\mathcal{V}_{ac}(\pi_u)$.

Proposition 5.2. *Let σ be a representation of $\mathcal{E}_n$ and let $P_u \in \mathcal{A}_n^{**}$ be the the universal structure projection. Then $\overline{\sigma}(P_u^\perp)$ is the projection onto $\mathcal{V}_{ac}(\sigma)$.*

Proof. Consider the kernel of $\overline{\sigma}$. There is a central projection $Q_\sigma \in \mathcal{E}_n^{**}$ such that $\ker \overline{\sigma} = Q_\sigma \mathcal{E}_n^{**}$. Moreover, we may regard $\mathcal{H}$ as a closed subspace of $\mathcal{H}_u$ and $\overline{\sigma}$ as given by multiplication by $Q_\sigma^\perp$, namely $\overline{\sigma}(X) = Q_\sigma^\perp X|_{\mathcal{H}}$ for any $X \in \mathcal{E}_n^{**}$.

Let M be the range of $\overline{\sigma}(P_u^\perp)$ and let $x \in M$. Then $x \in Q_\sigma^\perp P_u^\perp \mathcal{H}_u$, so x belongs to $\mathcal{V}_{ac}(\pi_u)$. Thus for any $A \in \mathfrak{A}_n$,

$$\langle \sigma(A)x, x\rangle = \langle \pi_u(j(A))Q_\sigma^\perp P_u^\perp x, Q_\sigma^\perp P_u^\perp\rangle.$$

As the range of $P_u^\perp$ consists of absolutely continuous vectors, we see that this is an absolutely continuous functional, so $x \in \mathcal{V}_{ac}(\sigma)$.

Conversely, if $x \in \mathcal{V}_{ac}(\sigma)$, then there exists an intertwiner $X \in \mathcal{X}(\sigma)$ and $\zeta \in \ell^2(\mathbb{F}_n^+)$ so that $x = X\zeta$. Observe that $Q_\sigma^\perp X$ belongs to $\mathcal{X}(\pi_u)$, hence $x \in \mathcal{V}_{ac}(\pi_u)$. Since the absolutely continuous part of π_u coincides with the type L part of π_u, we conclude that $x \in P_u^\perp \mathcal{H}_u \cap Q_\sigma^\perp \mathcal{H}_u$ and therefore $\overline{\sigma}(P_u^\perp)x = x$. ∎

Since the type L part of a representation σ is contained in the absolutely continuous part, it follows that $\overline{\sigma}(P_u^\perp) \geq P_\sigma^\perp$. Notice that by the previous result, σ is regular if and only if $\overline{\sigma}(P_u^\perp) = P_\sigma^\perp$, where P_σ is the structure projection for $\mathfrak{S}_\sigma$.

Proposition 5.3. *Let σ be a regular $*$-representation of $\mathcal{E}_n$. Then $\overline{\sigma(\mathfrak{A}_n)}^{\text{WOT}} = \overline{\sigma(\mathfrak{A}_n)}^{w\text{-}*}$ and $\overline{\sigma}(\mathfrak{A}_n^{**}) = \overline{\sigma(\mathfrak{A}_n)}^{w\text{-}*}$.*

Proof. Let $\mathfrak{T} := \overline{\sigma(\mathfrak{A}_n)}^{\text{w-}*}$, $\mathfrak{S} := \overline{\sigma(\mathfrak{A}_n)}^{\text{WOT}}$ and let $\mathfrak{W}$ be the von-Neumann algebra generated by $\sigma(\mathfrak{A}_n)$. Let $P_{\mathfrak{T}}$ and $P_{\mathfrak{S}}$ be the structure projections for $\overline{\sigma(\mathfrak{A}_n)}^{\text{w-}*}$ and $\overline{\sigma(\mathfrak{A}_n)}^{\text{WOT}}$ respectively.

Then $P_{\mathfrak{T}}^\perp \geq P_{\mathfrak{S}}^\perp$. Since the absolutely continuous part of σ contains the range of $P_{\mathfrak{T}}^\perp$, the regularity of σ yields that $P_{\mathfrak{T}} = P_{\mathfrak{S}} = \overline{\sigma}(P_u)$. Hence $\mathfrak{T} = \mathfrak{W}P + \mathfrak{T}P^\perp$ and $\mathfrak{S} = \mathfrak{W}P + \mathfrak{S}P^\perp$. Moreover both $\mathfrak{T}P^\perp$ and $\mathfrak{S}P^\perp$ are canonically isomorphic to $\mathfrak{L}_n$ and the isomorphisms agree on $\sigma(\mathfrak{A}_n)$. Hence they are equal. For typographical ease, write $P = P_{\mathfrak{T}} = P_{\mathfrak{S}}$.

Given $X \in \mathfrak{S}$, find $X' \in \mathcal{E}_n^{**}$ such that $\overline{\sigma}(X') = X$. We may suppose that $X' = Q_\sigma^\perp X'$. This determines X' uniquely, and $\overline{\sigma}$ is injective on $Q_\sigma^\perp \mathcal{E}_n^{**}$. By Proposition 5.2 and the regularity of σ, $\overline{\sigma}(P_u) = P$. So $\overline{\sigma}(P_u X' P_u^\perp) = PXP^\perp = 0$, whence $P_u X' P_u^\perp = 0$. To see that X' belongs to $\mathfrak{A}_n^{**}$, it remains to show that $P_u^\perp X' P_u^\perp$ lies in $\mathfrak{A}_n^{**} P_u^\perp$, which is type L. But $\mathfrak{A}_n^{**} P_u^\perp$ and $\mathfrak{S}P^\perp$ are both canonically isometrically isomorphic to $\mathfrak{L}_n$, from which it is clear that $\overline{\sigma}|_{\mathfrak{A}_n^{**} P_u^\perp}$ is an isomorphism onto $\mathfrak{S}P^\perp$. ∎

We can now prove our Kaplansky-type theorem.

Theorem 5.4. *Let σ be a regular $*$-representation of $\mathcal{E}_n$. Then the unit ball of $\sigma(\mathfrak{A}_n)$ is weak-$*$ dense in the unit ball of $\overline{\sigma(\mathfrak{A}_n)}^{w\text{-}*}$, and the same holds for $\mathfrak{M}_k(\sigma(\mathfrak{A}_n))$.*

Proof. Let $\mathfrak{S} := \overline{\sigma(\mathfrak{A}_n)}^{\text{w-}*} = \overline{\sigma(\mathfrak{A}_n)}^{\text{WOT}}$. We first show that

$$\ker \overline{\sigma}|_{\mathfrak{A}_n^{**}} = \mathfrak{A}_n^{**} Q_\sigma P_u. \tag{3}$$

To see this, notice that $\overline{\sigma}|_{\mathfrak{A}_n^{**} P_u^\perp}$ is an isometric map of the type L part of $\mathfrak{A}_n^{**}$ onto the type L part of $\mathfrak{S}$, that is, $\overline{\sigma}$ maps $\mathfrak{A}_n P_u^\perp$ isometrically onto $\mathfrak{S}P_\sigma^\perp$. Therefore, if $X \in \mathfrak{A}_n^{**}$ and $\overline{\sigma}(X) = 0$, then $\overline{\sigma}(X)P_\sigma^\perp = 0$, so that $XP_u^\perp = 0$. As $X \in \ker\overline{\sigma}$, we find $X \in \mathfrak{A}_n^{**} Q_\sigma P_u$. The reverse inequality is clear, so (3) holds.

Next we show that $\overline{\sigma}|_{\mathfrak{A}_n^{**}}$ is a complete quotient map onto $\mathfrak{S}$. For $X \in \mathfrak{A}_n^{**}$, we have

$$\begin{aligned} \operatorname{dist}(X, \ker\overline{\sigma}|_{\mathfrak{A}_n^{**}}) &\leq \|X - XQ_\sigma P_u\| \\ &= \left\|XQ_\sigma^\perp + XP_u^\perp Q_\sigma\right\| \\ &= \max\{\left\|XQ_\sigma^\perp\right\|, \left\|XP_u^\perp Q_\sigma\right\|\} \\ &\leq \max\{\left\|XQ_\sigma^\perp\right\|, \left\|XP_u^\perp\right\|\} \\ &= \max\{\|\overline{\sigma}(X)\|, \left\|\overline{\sigma}(X)P_\sigma^\perp\right\|\} \\ &= \|\overline{\sigma}(X)\|. \end{aligned}$$

The reverse inequality is clear, so that $\|\overline{\sigma}(X)\| = \operatorname{dist}(X, \ker\overline{\sigma}|_{\mathfrak{A}_n^{**}})$. By tensoring Q_σ and P_u with the identity operator on a k-dimensional Hilbert space, the same argument holds for $X \in M_k(\mathfrak{A}_n^{**})$ and the map $\sigma_k := \sigma \otimes I_{\mathbb{C}^k}$. Thus $\overline{\sigma}|_{\mathfrak{A}_n^{**}}$ is a complete contraction.

Consider any element T of the open unit ball of $\mathfrak{S}$. Since the map of $\mathfrak{A}_n^{**}$ onto $\mathfrak{S}$ is a complete quotient map, there is a contraction $T_u \in \mathfrak{A}_n^{**}$ which maps onto T. By Goldstine's Theorem, the unit ball of a Banach space is weak-$*$ dense in the ball of its double dual. So select a net A_λ in the ball of $\mathfrak{A}_n$ so that $j(A_\lambda)$ converges weak-$*$ to T_u. Then evidently $\sigma(A_\lambda)$ converges weak-$*$ (and thus WOT) to T. If one wants $\|A_\lambda\| \leq \|T\|$, a routine modification will achieve this.

Because $\overline{\sigma}|_{\mathfrak{A}_n^{**}}$ is a complete contraction, the same argument persists for matrices over the algebra as well. ∎

Lemma 5.5. *If σ is absolutely continuous and $\mathfrak{S}$ satisfies Kaplansky's Theorem with a constant, then σ is type L.*

Proof. As σ is absolutely continuous, $\sigma \oplus \lambda$ is type L. Let τ denote the weak-$*$ continuous homomorphism of $\mathfrak{L}_n$ into $\mathfrak{S}$ obtained from the isomorphism of $\mathfrak{L}_n$ with $\mathfrak{S}_{\sigma\oplus\lambda}$ followed by the projection onto the first summand.

Note that if L is an isometry in $\mathfrak{L}_n$, then $(\sigma \oplus \lambda)(L)$ is an isometry [**11**, Theorem 4.1]. Hence $\tau(L)$ is an isometry as well.

Consider $\ker\tau$. This is a weak-$*$ closed two-sided ideal in $\mathfrak{L}_n$. If this ideal is non-zero, then the range of the ideal is spanned by the ranges of isometries in the ideal [**14**]. In particular, the kernel would contain these isometries, contrary to the previous paragraph. Hence τ is injective.

Let C be the constant in the density theorem for $\mathfrak{S}$. If $T \in \mathfrak{S}$ and $\|T\| \le 1/C$, then there is a net A_i in the unit ball of $\mathfrak{A}_n$ such that $\sigma(A_i)$ converges weak-$*$ to T. Drop to a subnet if necessary so that the net $\lambda(A_i)$ converges weak-$*$ to an element $A \in \mathfrak{L}_n$. Then $(\sigma \oplus \lambda)(A_i)$ converges weak-$*$ to $T \oplus A$. Hence $\tau(A) = T$. That means that τ is surjective, and hence is an isomorphism.

Now if σ is not type L, then it is von Neumann type by Proposition 3.7; and hence contains proper projections. But $\mathfrak{L}_n$ contains no proper idempotents [**13**]; so this is impossible. Therefore σ must be type L. ∎

Theorem 5.6. *For a representation σ of $\mathcal{E}_n$, the following statements are equivalent.*

1. *The unit ball of $\sigma(\mathfrak{A}_n)$ is* WOT*-dense in the ball of $\mathfrak{S} = \overline{\sigma(\mathfrak{A}_n)}^{\text{WOT}}$. i.e. Kaplansky's density theorem holds.*
2. *The* WOT*-closure of the unit ball of $\sigma(\mathfrak{A}_n)$ in $\mathfrak{S} = \overline{\sigma(\mathfrak{A}_n)}^{\text{WOT}}$ has interior. i.e. Kaplansky's density theorem holds with a constant.*
3. *σ is regular.*

Proof. (3) implies (1) follows from Theorem 5.4. That (1) implies (2) is obvious, so suppose (2) holds. If σ is not regular, then it is von Neumann type by Proposition 3.7; and $\sigma \simeq \sigma_a \oplus \sigma_s$. Since Kaplansky holds with a constant, this persists for σ_a because the WOT-closure does not change by dropping σ_s, it being the full von Neumann algebra already. This contradicts Lemma 5.5. ∎

Definition 5.7. A functional φ on $\mathfrak{A}_n$ is *singular* if it annihilates the type L part of $\mathfrak{A}_n^{**}$.

Proposition 5.8. *For a functional φ on $\mathfrak{A}_n$ of norm 1, the following are equivalent:*

1. *φ is singular.*
2. *There is a regular representation σ of $\mathcal{E}_n$ and vectors $x, y \in \mathcal{H}_\sigma$ with $x = P_\sigma x$ such that $\varphi(A) = \langle \sigma(A)x, y\rangle$.*
3. $\lim\limits_{k\to\infty} \|\varphi|_{\mathfrak{A}_{n,0}^k}\| = 1$.

If φ extends to a state on $\mathcal{E}_n$ (i.e. $\varphi(I) = 1$), then (3) *is equivalent to*

(3′) $\|\varphi|_{\mathfrak{A}_{n,0}}\| = 1$.

Proof. If $\varphi \in \mathfrak{A}^*$, it is a weak-$*$ continuous functional on $\mathfrak{A}_n^{**}$, so we may represent it as a vector functional on $\mathcal{H}_u$, say $\varphi(A) = \langle \pi_u(A)x, y\rangle$. Since φ annihilates the type L part, it does not change the functional to replace x by $P_u x$. So (1) implies (2).

If (2) holds, then for every $A \in \mathfrak{A}_n$ we have $\varphi(A) = \langle \overline{\sigma}(j(A)P_u)x, y\rangle$, which clearly annihilates the type L part of $\mathfrak{A}^{**}$. Thus (2) implies (1).

If (1) holds, then $\varphi(j(A)P_u^\perp) = 0$, so $\varphi(j(A)) = \varphi(j(A)P_u)$. Now $\mathfrak{A}_n^{**}P_u = \bigcap_{k\geq 1}(\mathfrak{A}_{n,0}^{**})^k$, so that $\|\varphi|_{(\mathfrak{A}_{n,0}^{**})^k}\| = 1$ for all $k \geq 1$. It is easy to see that $(\mathfrak{A}_{n,0}^{**})^k = (\mathfrak{A}_{n,0}^k)^{**}$. By basic functional analysis, a functional on a Banach space X has the same norm on the second dual. Therefore $\|\varphi|_{\mathfrak{A}_{n,0}^k}\| = 1$ for all $k \geq 1$.

If (3) holds, then there is a sequence A_k in the ball of $\mathfrak{A}_{n,0}^k$ so that $\lim\limits_{k\to\infty}\|\varphi(A_k)\| = 1$. Dropping to a weak-$*$ convergent subnet, we may assume that this subnet converges to an element $A \in \bigcap_{k\geq 1}(\mathfrak{A}_{n,0}^{**})^k = \mathfrak{A}_n^{**}P_u$. Thus $\|\varphi|_{\mathfrak{A}_n^{**}P_u}\| = 1$.

We claim that $\varphi|_{\mathfrak{A}_n^{**}P_u^\perp} = 0$. If not, there is a norm one element $B = BP_u^\perp$ with $\varphi(B) = \varepsilon > 0$. Then

$$\|A + \varepsilon B\| = \|AA^* + \varepsilon^2 BB^*\|^{1/2} \leq (1+\varepsilon^2)^{1/2}.$$

But $\varphi(A+\varepsilon B) = 1+\varepsilon^2$, and thus $\|\varphi\| > 1$. This contradiction shows that φ annihilates the type L part, and thus is singular. So (3) implies (1).

Clearly (3) implies (3′). Conversely, if φ extends to a state on $\mathcal{E}_n$, it may be regarded as a normal state on $\mathcal{E}_n^{**}$ and hence represented as $\varphi(A) = \langle \sigma(A)\xi, \xi\rangle$ where σ is obtained from the GNS construction. If $A \in \mathfrak{A}_{n,0}^{**}$ satisfies $1 = \|A\| = \varphi(A)$, then $\sigma(A)\xi = \xi$ is an eigenvalue. Therefore $\varphi(A^k) = 1$ for all $k \geq 1$, showing that $\|\varphi|_{(\mathfrak{A}_{n,0}^{**})^k}\| = 1$ for all $k \geq 1$. Arguing as above establishes (3). ∎

Here is a version of the Jordan decomposition.

Proposition 5.9. *Every functional φ on $\mathfrak{A}_n$ splits uniquely as the sum of an absolutely continuous functional φ_a and a singular one φ_s. Moreover*

$$\|\varphi\| \leq \|\varphi_a\| + \|\varphi_s\| \leq \sqrt{2}\|\varphi\|$$

and these inequalities are sharp.

Proof. Set $\varphi_a(A) = \varphi(\pi_u(A)P_u^\perp)$ and $\varphi_s = \varphi(\pi_u(A)P_u)$. Clearly this is the desired decomposition. For uniqueness, suppose that ψ is both singular and absolutely continuous. Then $\|\psi\| = \lim\limits_{k\to\infty}\|\psi|_{\mathfrak{A}_{n,0}^k}\| = 0$.

Regard $\mathfrak{A}_n$ as a subalgebra of $\mathcal{E}_n$ and extend φ to a linear functional (again called φ) on $\mathcal{E}_n$ with the same norm. Then (using the GNS construction and the polar decomposition of functionals on a C*-algebra) there exists a $*$-representation σ of $\mathcal{E}_n$ on a Hilbert space $\mathcal{H}_\sigma$ and vectors $x, y \in \mathcal{H}_\sigma$ with $\|x\|\,\|y\| = \|\varphi\|$ so that for every $A \in \mathcal{E}_n$, $\varphi(A) = \langle\sigma(A)x, y\rangle$. Therefore for $A \in \mathfrak{A}_n$, we have $\varphi_a(A) = \langle\sigma(A)\overline{\sigma}(P_u^\perp)x, y\rangle$ and $\varphi_s(A) = \langle\sigma(A)\overline{\sigma}(P_u)x, y\rangle$. Hence

$$\begin{aligned}\|\varphi_a\| + \|\varphi_s\| &\leq \|\overline{\sigma}(P^\perp)x\|\,\|y\| + \|\overline{\sigma}(P_u)x\|\,\|y\| \\ &\leq \sqrt{2}\big(\|\overline{\sigma}(P^\perp)x\|^2 + \|\overline{\sigma}(P)x\|^2\big)^{1/2}\|y\| = \sqrt{2}\|\varphi\|.\end{aligned}$$

The example following will show that the $\sqrt{2}$ is sharp. ∎

Example 5.10. Consider the atomic representation $\sigma_{1,1}$ on $\mathbb{C}\xi_* \oplus \ell^2(\mathbb{F}_n^+)$ given by $S_1\xi_* = \xi_*$ and $S_2\xi_* = \xi_\varnothing$; and $S_i|_{\ell^2(\mathbb{F}_n^+)} = L_i$. Set

$$\varphi(A) = \langle\sigma_{1,1}(A)(\xi_* + \xi_\varnothing)/\sqrt{2}, \xi_\varnothing\rangle.$$

Then $\mathfrak{S}_\sigma$ contains $A = \xi_\varnothing\xi_*^*/\sqrt{2} + (I - \xi_*\xi_*^*)/\sqrt{2}$ and $\varphi(A) = 1$. So we see that $\|\varphi\| = 1$.

On the other hand,

$$\varphi_s(A) = \langle \sigma_{1,1}(A)\xi_*/\sqrt{2}, \xi_\varnothing \rangle \quad \text{and} \quad \varphi_a(A) = \langle \sigma_{1,1}(A)\xi_\varnothing/\sqrt{2}, \xi_\varnothing \rangle$$

both have norm $1/\sqrt{2}$. So $\|\varphi_s\| + \|\varphi_a\| = \sqrt{2}$.

Question 5.11. Let S be the unilateral shift and consider the representation of $\mathfrak{A}_2$ obtained from the minimal isometric dilation of $A_1 = S/\sqrt{2}$ and $A_2 = (S+P_0)^*/\sqrt{2}$. The weak-$*$ closed self-adjoint algebra generated by A_1 and A_2 is all of $\mathcal{B}(\mathcal{H})$. Therefore this representation is either dilation type with $P\mathfrak{S}P = \mathcal{B}(\mathcal{H})$ or it is type L, depending on whether the functional $\varphi = e_0 e_0^*$ is singular or absolutely continuous. To check. it suffices to determine whether φ has norm 1 or less on $\mathfrak{A}_{n,0}$. We would like to know which it is.

Question 5.12. Charles Read has given an example of a representation of $\mathfrak{A}_2$ such that $\mathcal{B}(\mathcal{H}) = \overline{\sigma(\mathfrak{A}_2)}^{\text{w-*}}$. Is Read's example singular or absolutely continuous? Again it suffices to take any convenient state on $\mathcal{B}(\mathcal{H})$ and estimate its norm on $\mathfrak{A}_{2,0}$ as equal to 1 or strictly less.

We provide an example of how the density theorem can be used to establish an interpolation result for finitely correlated presentations. Such representations are obtained from a row contraction of matrices $A = \begin{bmatrix} A_1 & \dots & A_n \end{bmatrix} \in M_{1,n}(M_k(\mathbb{C}))$ by taking the minimal isometric dilation [**17, 7, 23**]. These representations were classified in [**12**]. The structure projection P has range equal to the span of all $\{A_i^*\}$ invariant subspaces on which A is isometric. In particular, it is finite rank. Also, the type L part is a finite multiple, say α, of the left regular representation. Thus elements of the free semigroup algebra $\mathfrak{S}$ have the form $\begin{bmatrix} X & 0 \\ Y & Z^{(\alpha)} \end{bmatrix}$ where X and Y lie in $P\mathfrak{W}P$ and $P^\perp\mathfrak{W}P$ respectively, and $Z \in \mathfrak{L}_n$, where $\mathfrak{W}$ is the von Neumann algebra generated by $\mathfrak{S}$.

Theorem 5.13. *Let σ be a finitely correlated representation. If $A \in \mathfrak{S}_\sigma$ has $\|A\| < 1$ and $k \in \mathbb{N}$, then there is an operator $B \in \mathfrak{A}_n$ so that $\sigma(B)P = AP$ and the Fourier series of B up to level k agree with the coefficients of $AP^\perp$.*

Proof. Fix $\varepsilon < 1 - \|A\|$. Let $Q_k \in \mathcal{B}(\ell^2(\mathbb{F}_n^+))$ denote the projection onto $\operatorname{span}\{\xi_w : |w| \le k\}$.

Identify $\mathfrak{S}P^\perp$ with $\mathfrak{L}_n^{(\alpha)}$ and find $C \in \mathfrak{L}_n$ so that $AP^\perp = C^{(\alpha)}$.

Since the weak and strong operator topologies have the same closed convex sets, the density theorem implies that there exists a sequence $\{L_k\}$ in $\mathfrak{L}_n$ so that $\|L_k\| < 1 - \varepsilon$ and $A = \textsc{sot}\lim \sigma(L_k)$. Recalling that P and Q_k are finite rank, we conclude that there exists $B_1 \in \mathfrak{A}_n$ so that

$$\|(A - \sigma(B_1))P\| + \|Q_k(C - B_1)\| < \varepsilon/2.$$

By [**15**, Corollary 3.7], there is an element $C_1 \in \mathfrak{L}_n$ so that $Q_k C_1 = Q_k(C - B_1)$ and $\|C_1\| = \|Q_k(C - B_1)\|$. Hence the element of $\mathfrak{S}$ defined by $A_1 = (A - \sigma(B_1))P + C_1^{(\alpha)}P^\perp$ satisfies $\|A_1\| < \varepsilon/2$.

Now choose $B_2 \in \mathfrak{A}_n$ so that $\|B_2\| < \varepsilon/2$ and

$$\|(A_1 - \sigma(B_2))P\| + \|Q_k(C_1 - B_2)\| < \varepsilon/4.$$

Proceed as above to define $C_2 \in \mathfrak{L}_n$ so that $Q_k C_2 = Q_k(C_1 - B_2)$ and $\|C_2\| = \|Q_k(C_1 - B_2)\|$; and then define $A_2 = (A_1 - \sigma(B_2))P + C_2^{(\alpha)}P^\perp$ satisfying $\|A_2\| < \varepsilon/4$.

Proceeding recursively, we define B_j for $j \geq 1$ so that $B = \sum_{j\geq 1} B_k$ is the desired approximant. ∎

6. Constructive examples of Kaplansky

In this section, we give a couple of examples where we were able to construct the approximating sequences more explicitly. We concentrate on exhibiting the structure projection P as a limit of contractions. It is then easy to see that the whole left ideal $\mathfrak{W}P$ has the same property by applying the C*-algebra Kaplansky theorem. We do not have an easy argument to show that one can extend this to the type L part without increasing the constant.

Proposition 6.1. *Let $\mathfrak{S}$ be the free semigroup algebra generated by isometries $S_1, \dots, S_n$; and let $\mathfrak{A}$ be the norm closed algebra that they generate. Let $P \in \mathfrak{S}$ be the projection given by the Structure Theorem. If $P \neq I$ is the* WOT*-limit of a sequence in $\mathfrak{A}$ of norm at most r, then the* SOT*-closure of the r-ball of $\mathfrak{A}_0^k$ contains $\mathfrak{S}P$ for all $k \geq 0$.*

Proof. Since $P \neq I$, $\mathfrak{S}$ has a type L part. Let Φ be the canonical surjection of $\mathfrak{S}$ onto $\mathfrak{L}_n$ with $\Phi(S_i) = L_i$ [**11**, Theorem 1.1]. Recall that the kernel of Φ is $\bigcap_{k=1}^{\infty} \mathfrak{S}_0^k = \mathfrak{W}P$. Since the weak and strong operator topologies have the same closed convex sets, we may suppose that the sequence in $\mathfrak{A}$ converges to P strongly. In particular, the restriction of this sequence to the type L part converges strongly to 0. Hence the Fourier coefficients each converge to 0. Thus a minor modification yields a sequence $A_k \in \mathfrak{A}_0^k$ of norm at most r converging SOT to P.

If T lies in the unit ball of $\mathfrak{W}P$, then by the usual Kaplansky density theorem, there is a sequence B_k in the unit ball of $\mathrm{C}^*(S)$ which converges SOT to T. We may assume that B_k are polynomials in S_i, S_i^* for $1 \leq i \leq n$ of total degree at most k. Then observe that $B_k A_{2k}$ lies in $\mathfrak{A}_{n,0}^k$, and converges SOT to $TP = T$. ∎

Our first example is a special class of finitely correlated representations which are obtained from dilating multiples of unitary matrices.

Theorem 6.2. *Suppose that U_i for $1 \leq i \leq n$ are unitary matrices in $\mathcal{B}(\mathcal{V})$, where $\mathcal{V}$ has finite dimension d, and that α_i are non-zero scalars so that $\sum_{i=1}^n |\alpha_i|^2 = 1$. Let S_i be the joint isometric dilation of $A_i = \alpha_i U_i$ to a Hilbert space $\mathcal{H}$. Let $\mathfrak{S}$ be the free semigroup algebra that they generate; and let $\mathfrak{A}$ denote the norm-closed algebra. Then the projection $P = P_{\mathcal{V}}$ is the projection that occurs in the Structure Theorem, and there is a sequence of contractions in $\mathfrak{A}$ which converges* SOT *to P.*

Lemma 6.3. *If $\mathcal{U}$ is a set of unitary matrices in $\mathfrak{M}_d$, then the closure of the set of all non-empty words in elements of $\mathcal{U}$ is a subgroup of the unitary group $\mathfrak{U}$ and the algebra generated by $\mathcal{U}$ is a C*-algebra.*

Proof. The closure $\mathcal{G}$ of words in $\mathcal{U}$ is multiplicative and compact. Any unitary matrix U is diagonalizable with finite spectrum. A routine pigeonhole argument shows that there is a sequence U^{n_i} which converges to I, and thus $U^{n_i - 1}$ converges to U^{-1}. It follows that $\mathcal{G}$ is a group. It is immediate that the algebra generated by $\mathcal{U}$ contains $\mathcal{U}^*$ and thus is self-adjoint. ∎

Proof of Theorem 6.2. From the Lemma, we see that the algebra generated by $\{A_i^*\}$ is self-adjoint, and thus the space $\mathcal{V}$ is the span of its minimal A_i^* invariant subspaces. From the

Structure Theorem for finitely correlated representations **[12]**, we deduce that $P = P_{\mathcal{V}}$ is a projection in $\mathfrak{S}$ and that $\mathfrak{S} = \mathfrak{W}P + \mathfrak{S}P^\perp$, where $\mathfrak{W}$ is the von Neumann algebra generated by $\mathfrak{S}$, $P^\perp\mathcal{H}$ is invariant, and $\mathfrak{S}P^\perp$ is a (finite) ampliation of $\mathfrak{L}_n$.

Consider the space $\mathfrak{X}$ consisting of all infinite words $x = i_1 i_2 i_3 \dots$ where $1 \le i_j \le n$ for $j \ge 1$. This is a Cantor set in the product topology. Put the product measure μ on $\mathfrak{X}$ obtained from the measure on $\{1, \dots, n\}$ which assigns mass $|\alpha_i|^2$ to i.

Fix $\varepsilon > 0$. Since the closed semigroup $\mathcal{G}$ generated by $\{U_i\}$ is a compact group by Lemma 6.3, one may choose a finite set $\mathcal{S}$ of non-empty words which form an ε-net (in the operator norm). Let N denote the maximum length of these words. Then we have the following consequence: given any word $w = i_1 \dots i_k$, there is a word $v = j_1 \dots j_l$ in $\mathcal{S}$ with $l \le N$ so that $U_{wv} = U_{i_1} \dots U_{i_k} U_{j_1} \dots U_{j_l}$ satisfies $\|U_{wv} - I\| < \varepsilon$.

Recursively determine a set $\mathcal{W}$ of words so that S_w have pairwise orthogonal ranges and $\|U_w - I\| < \varepsilon$ for $w \in \mathcal{W}$ as follows: start at an arbitrary level k_0 and take all words w with $|w| = k_0$ such that $\|U_w - I\| < \varepsilon$. If a set of words of length at most k has been selected, add to $\mathcal{W}$ those words of length $k+1$ which have ranges orthogonal to those already selected and satisfy $\|U_w - I\| < \varepsilon$.

We claim that $\text{SOT–}\sum_{w\in\mathcal{W}} S_w S_w^* = I$. The argument is probabilistic. Let $\delta = \min\{|\alpha_i|^2\}$. Associate to w the subset X_w of all infinite words in $\mathfrak{X}$ with w as an initial segment. By construction, the sets X_w are pairwise disjoint clopen sets for $w \in \mathcal{W}$ with measure $|\alpha_w|^2$, where we set $\alpha_w = \prod_{t=1}^k \alpha_{i_t}$. Verifying our claim is equivalent to showing that $\bigcup_{w\in\mathcal{W}} X_w$ has measure 1. Consider the complement Y_k of $\bigcup_{w\in\mathcal{W},\ |w|\le k} X_w$. This is the union of certain sets X_w for words w of length k. For each such word, there is a word $v \in \mathcal{S}$ so that $\|U_{wv} - I\| < \varepsilon$. Now S_{wv} has range contained in the range of S_w, which is orthogonal to the ranges of words in $\mathcal{W}$ up to level k. It follows from the construction of $\mathcal{W}$ that there will be a word $w' \in \mathcal{W}$ so that w' divides wv. As a consequence, Y_{k+N} has measure smaller than Y_k by a factor of at most $1 - \delta^N$ because for each interval X_w in Y_k, there is an interval X_{wv} which is in the complement, and its measure is at least $\delta^N \mu(X_w)$. Therefore $\lim\limits_{k\to\infty} \mu(Y_k) = 0$.

Choose a finite set $\mathcal{W}_0 \subset \mathcal{W}$ so that

$$r := \mu\Big(\bigcup_{w\in\mathcal{W}_0} X_w\Big) > 1 - \varepsilon.$$

Define $T = \sum_{w\in\mathcal{W}_0} \overline{\alpha_w} S_w$ in $\mathfrak{A}$. Note that

$$\|T\|^2 = \sum_{w\in\mathcal{W}_0} |\alpha_w|^2 = r < 1.$$

Observe that $PS_w = \alpha_w U_w$. Define a state τ on $\mathcal{B}(\mathcal{H})$ as the normalized trace of the compression to $\mathcal{V}$. Since $\|U_w - I\| < \varepsilon$, it follows that $|\tau(U_w) - 1| < \varepsilon$. Compute

$$\begin{aligned} \operatorname{Re}\tau(T) &= \sum_{w\in\mathcal{W}_0} \overline{\alpha_w} \operatorname{Re}\tau(S_w) = \sum_{w\in\mathcal{W}_0} |\alpha_w|^2 \operatorname{Re}\tau(U_w) \\ &\ge \sum_{w\in\mathcal{W}_0} |\alpha_w|^2 (1-\varepsilon) = r(1-\varepsilon) > (1-\varepsilon)^2. \end{aligned}$$

By taking $\varepsilon = 1/k$ and $k_0 = k$ in the construction above, we obtain a sequence T_k of polynomials $T_k \in \mathfrak{A}_0^k$ which are contractions and $\lim\limits_{k\to\infty} \tau(T_k) = 1$. It follows that there is a subsequence which converges WOT to a limit $T \in \mathfrak{S}$ which lies in $\bigcap_{k\ge1} \mathfrak{S}_0^k = \mathfrak{W}P$. Moreover

$\|T\| \leq 1$ and $\tau(T) = 1$. The only contraction in $\mathcal{B}(\mathcal{V})$ with trace 1 if the identity, and therefore the compression $PT = P$. As T is contractive, we deduce that $P^\perp T = P^\perp TP = 0$, whence $T = P$. As the SOT and WOT-closures of the balls are the same, there is a sequence in the convex hull of the T_k's which converges to P strongly. ∎

Our second constructive example is the set of atomic representations introduced in [**13**]. To analyze these, we will need some of Voiculescu's theory of free probability.

Theorem 6.4. *If $\mathfrak{S}$ is an atomic free semigroup algebra, then the structure projection is a* SOT*-limit of contractive polynomials in the generators.*

It is convenient for our calculation to deal with certain norm estimates in the free group von Neumann algebra. We thank Andu Nica for showing us how to handle this free probability machinery.

Lemma 6.5. *Let p and q be free proper projections of trace $\alpha \leq 1/2$ in a finite von Neumann algebra $(\mathfrak{M}, \tau)$. Then $\|pqp\| = 4\alpha(1-\alpha)$.*

Proof. Given $a \in \mathfrak{M}$, form the power series $M_a(z) = \displaystyle\sum_{n\geq 1} \tau(a^n) z^n$. In particular, $M_p(z) = M_q(z) = \alpha z(1-z)^{-1}$. Voiculescu's S-transform is given by $S_a(\zeta) = (1+\zeta)\zeta^{-1} M_a^{<-1>}(\zeta)$ where $M_a^{<-1>}$ denotes the inverse of M_a under composition. So

$$S_p(\zeta) = S_q(\zeta) = \frac{1+\zeta}{\zeta} \frac{\zeta}{\alpha+\zeta} = \frac{1+\zeta}{\alpha+\zeta}.$$

By [**31**, Theorem 2.6], the S-transform is multiplicative on free pairs. Hence

$$S_{pq} = \frac{(1+\zeta)^2}{(\alpha+\zeta)^2} = \frac{1+\zeta}{\zeta} M_{pq}^{<-1>}(\zeta).$$

Observe that $\tau((pqp)^n) = \tau((pq)^n)$ and so $M_{pq} = M_{pqp}$. Thus

$$M_{pqp}^{<-1>}(\zeta) = \frac{\zeta(1+\zeta)}{(\alpha+\zeta)^2}.$$

We obtain the quadratic equation

$$(1-z)M_{pqp}^2(z) + (1-2\alpha z)M_{pqp}(z) - \alpha^2 z = 0.$$

Solving, we obtain

$$M_{pqp}(z) = \frac{2\alpha z - 1 + \sqrt{1-4\alpha(1-\alpha)z}}{2(1-z)}$$

Since $M_{pqp}(0) = 0$, we must choose an appropriate branch of the function $(1-4\alpha(1-\alpha)z)^{1/2}$. This may be defined on the complement of the line segment

$$\{z : \operatorname{Re} z = 1/4\alpha(1-\alpha) \text{ and } \operatorname{Im} z \leq 0\}$$

and takes positive real values on real numbers $x < 1/4\alpha(1-\alpha)$. An easy calculation shows that the singularity at $z = 1$ is removable.

The power series for M_{pqp} converges on the largest disk on which it is analytic. The branch point occurring at $z = 1/4\alpha(1-\alpha)$ is the only obstruction, and thus the radius of convergence is $1/4\alpha(1-\alpha)$. On the other hand, from Hadamard's formula, the reciprocal of the radius of convergence is

$$4\alpha(1-\alpha) = \limsup_{k\to\infty} \tau((pqp)^k)^{1/k} = \|pqp\|. \qquad \blacksquare$$

Corollary 6.6. *Let U_i for $1 \le i \le n$ denote the generators of the free group von Neumann algebra, and let P_i be spectral projections for U_i for sets of measure at most $\alpha \le 1/2$. Then $\|\sum_{i=1}^n P_i\| \le 1 + 2n^2\sqrt{\alpha}$.*

Proof. By Lemma 6.5, we have $\|P_iP_j\| = \|P_iP_jP_i\|^{1/2} \le 2\sqrt{\alpha}$ for $i \ne j$. If $\|\sum_{i=1}^n P_i\| = 1+x$, then

$$(1+x)^2 = \|\sum_{i=1}^n P_i + \sum_{i\ne j} P_iP_j\| \le 1 + x + n(n-1)2\sqrt{\alpha}.$$

Hence $x \le 2n^2\sqrt{\alpha}$ as claimed. $\blacksquare$

Recall from [**13**] the atomic representation $\sigma_{u,\lambda}$ determined by a primitive word $u = i_1 \dots i_d$ in $\mathbb{F}_n^+$ and a scalar λ in $\mathbb{T}$. Define a Hilbert space $\mathcal{H}_u \simeq \mathbb{C}^d \oplus \ell^2(\mathbb{F}_n^+)^{d(n-1)}$ with orthonormal basis $\zeta_1, \dots, \zeta_d$ for $\mathbb{C}^d$ and index the copies of $\ell^2(\mathbb{F}_n^+)$ by (s,j), where $1 \le s \le d$, $1 \le j \le n$ and $j \ne i_s$, with basis $\{\xi_{s,j,w} : w \in \mathbb{F}_n^+\}$. Define a representation $\sigma_{u,\lambda}$ of $\mathbb{F}_n^+$ and isometries $S_i = \sigma_{u,\lambda}(i)$ by

$$\begin{aligned} S_i\zeta_s &= \zeta_{s-1} && \text{if} \quad i = i_s,\ s > 1 \\ S_i\zeta_1 &= \lambda\zeta_d && \text{if} \quad i = i_1 \\ S_i\zeta_s &= \xi_{s,i,\varnothing} && \text{if} \quad i \ne i_s \\ S_i\xi_{s,j,w} &= \xi_{s,j,iw} && \text{for all} \quad i,s,j,w \end{aligned}$$

For our purposes, we need to observe that the vectors $\zeta_1, \dots, \zeta_d$ form a ring which is cyclically permuted by the appropriate generators S_{i_s}; and all other basis vectors are wandering. The projection P in the structure theorem is the projection onto $\mathbb{C}^d$.

Lemma 6.7. *Let u be a primitive word and let $\lambda \in \mathbb{T}$. Let $\mathfrak{S}$ be the atomic free semigroup algebra corresponding to the representation $\sigma_{u,\lambda}$. Then the projection P from the Structure Theorem is the limit of contractive polynomials in the generators.*

Proof. Let u_s denote the cyclic permutations of u for $1 \le s \le d$ satisfying $S_{u_s}\zeta_s = \lambda\zeta_s$. As in the proof [**13**, Lemma 3.7] of the classification of atomic representations, there is a sequence of the form $A_{k,s} = p_k(S_{u_s})$ which converge SOT to the projections $\zeta_s\zeta_s^*$ where $p_k(x) = x^kq_k(x)$ are polynomials with $\|p_k\|_\infty = 1 = p_k(\lambda)$.

It is routine to choose such polynomials p_k with the added stipulation that there is an open set $\mathcal{V}_k$ of measure $1/k$ containing the point λ so that $\|p_k\chi_{\mathbb{T}\setminus\mathcal{V}_k}\| < 1/k$. We now consider the elements $A_k = \sum_{s=1}^d A_{k,s}$. Clearly the sequence A_k converges SOT to the projection P. So it suffices to establish that $\lim\limits_{k\to\infty} \|A_k\| = 1$.

As $\mathfrak{A}$ has a unique operator algebra structure independent of the representation of $\mathcal{O}_n$, polynomials in the isometries S_{u_s} may be replaced by the corresponding polynomials in the generators L_s of the left regular representation for the free semigroup $\mathbb{F}_d^+$. The isometries L_s have pairwise orthogonal ranges for distinct s, and consequently the operators $A_{k,s}$ have orthogonal ranges. Hence the norm of A_k equals the norm of the column operator with entries $A_{k,s}$. Now it is evident that the left regular representation of $\mathbb{F}_d^+$ may be obtained as the restriction of the left regular representation of the free group $\mathbb{F}_d$ to an invariant subspace. So the norm is increased if L_s are replaced by the generators U_s of the free group von Neumann algebra. Thus the norm of A_k is dominated by the column vector with entries $p_k(U_s)$.

Let Q_s denote the spectral projection of U_s for the open set $\mathcal{V}_k$. Then

$$\begin{bmatrix} p_k(U_1) \\ \vdots \\ p_k(U_d) \end{bmatrix} = \operatorname{diag}\{p_k(U_1), \dots, p_k(U_d)\} \begin{bmatrix} Q_1 \\ \vdots \\ Q_d \end{bmatrix} + \begin{bmatrix} p_k(U_1)Q_1^\perp \\ \vdots \\ p_k(U_d)Q_d^\perp \end{bmatrix}.$$

Since $\|p_k(U_s)\| = 1$, the norm of the first term is at most

$$\|Q_1 + \cdots + Q_d\|^{1/2} < 1 + 2d^2/\sqrt{k}$$

by Corollary 6.6. The second term is dominated by

$$\sqrt{d}\, \|p_k \chi_{\mathbb{T}\setminus\mathcal{V}_k}\|_\infty < \sqrt{d}/k.$$

Hence the norms converge to 1 as claimed. ∎

It is now only a technical exercise to show how one may use similar arguments to combine sequences corresponding to finitely many points on the circle for a given word u, and to deal with finitely many such words at once. The inclusion of summands of type L such as the atomic representations of inductive type does not affect things since these sequences are already converging strongly to 0 on the wandering subspaces of these atomic representations. Details are omitted.

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