IDEALS IN A MULTIPLIER ALGEBRA ON THE BALL

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Abstract. We study the ideals of the closure of the polynomial multipliers on the Drury-Arveson space. Structural results are obtained by investigating the relation between an ideal and its weak-$*$ closure, much in the spirit of the corresponding classical facts for the disc algebra. Zero sets for multipliers are also considered and are deeply intertwined with the structure of ideals. Our approach is primarily based on duality arguments.

1. Introduction

We study the ideals and zero sets for the algebra $\mathcal{A}_d$ of multipliers on the Drury-Arveson space which are norm limits in the multiplier norm of (analytic) polynomials in d variables. Precise definitions are given in Section 2. To a great extent the theory parallels the theory for the ball algebra $\mathrm{A}(\mathbb{B}_d)$, the uniform closure of the polynomials in $\mathrm{C}(\overline{\mathbb{B}_d})$. However, as the multiplier norm is not comparable to the supremum norm over the ball, there are complications. Our results are based on duality arguments, and rely heavily on our previous contributions in [5]. Moreover, our work on zero sets have (somewhat easier) analogues in the uniform algebra case, and some of these results appear to be new in that context as well.

The initial motivation for our investigation stemmed from multivariate operator theoretic considerations surrounding the functional calculus associated to an absolutely continuous row contraction. An upcoming paper ([4]) deals with this topic. We focus here on purely function theoretic aspects.

We first review the classical theory. Let $\mathbb{B}_d \subset \mathbb{C}^d$ denote the open unit ball and let $\mathbb{S}_d$ denote its boundary, the unit sphere. The ball algebra $A(\mathbb{B}_d)$ consists of those functions φ that are holomorphic on $\mathbb{B}_d$ and continuous on $\overline{\mathbb{B}_d}$. It is a uniform algebra when equipped with the supremum norm

$$\|\varphi\|_\infty = \sup_{z\in\overline{\mathbb{B}_d}} |\varphi(z)|.$$

By the maximum modulus principle, it may also be considered as a closed subalgebra of $\mathrm{C}(\mathbb{S}_d)$ obtained as the closure of the analytic polynomials. Furthermore, let $H^\infty(\mathbb{B}_d)$ denote the algebra of bounded holomorphic functions

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on $\mathbb{B}_d$. Functions in $H^\infty(\mathbb{B}_d)$ have radial limits which are defined on $\mathbb{S}_d$ almost everywhere with respect to unique rotation invariant Borel probability measure σ. In particular, we have the isometric inclusion

$$H^\infty(\mathbb{B}_d) \subset L^\infty(\mathbb{S}_d, \sigma).$$

In fact, $H^\infty(\mathbb{B}_d)$ is a weak-$*$ closed subalgebra of $L^\infty(\mathbb{S}_d, \sigma)$, and thus it is a dual space that carries the induced weak-$*$ topology.

A natural problem is to determine the ideal structure of $\mathrm{A}(\mathbb{B}_d)$. First consider the single variable case on the unit disc $\mathbb{D} \subset \mathbb{C}$. A function $\omega \in H^\infty(\mathbb{D})$ is called inner if $|\omega| = 1$ almost everywhere on the circle $\mathbb{S}_1$. Let $X_\omega \subset \mathbb{S}_1$ denote the closure of the points across which ω cannot be continued holomorphically. This is called the support of ω (see [16]). The structure of ideals of $A(\mathbb{D})$ was unraveled by Carleson and Rudin independently [3, 21] (see also [16] for an exposition of these results).

Theorem 1.1 (Carleson, Rudin)**.** *Let $\mathcal{J} \subset A(\mathbb{D})$ be a non-trivial closed ideal. Then there exists an inner function $\omega \in H^\infty(\mathbb{D})$ and a closed subset $K \subset \mathbb{S}_1$ of Lebesgue measure 0 containing X_ω such that*

$$\mathcal{J} = \omega H^\infty(\mathbb{D}) \cap \mathcal{I}(K)$$

where $\mathcal{I}(K) \subset A(\mathbb{D})$ is the ideal of functions vanishing on K.

The general case of the ball algebra was handled much later by Hedenmalm [14]. In this higher dimensional setting, the smaller supply of inner functions available makes the description less explicit. Given a set $\mathcal{S} \subset A(\mathbb{B}_d)$, we let $Z(\mathcal{S}) \subset \overline{\mathbb{B}_d}$ be the common zero set of the functions in $\mathcal{S}$; while given a set $K \subset \overline{\mathbb{B}_d}$, we let $\mathcal{I}(K) \subset A(\mathbb{B}_d)$ be the ideal of functions vanishing on K as above. The description of ideals of $A(\mathbb{B}_d)$ reads as follows.

Theorem 1.2 (Hedenmalm)**.** *Let $\mathcal{J} \subset A(\mathbb{B}_d)$ be a closed ideal, and let $K = Z(\mathcal{J}) \cap \mathbb{S}_d$. Then*

$$\mathcal{J} = \widetilde{\mathcal{J}} \cap \mathcal{I}(K),$$

where $\widetilde{\mathcal{J}}$ denotes the weak-$*$ closure of $\mathcal{J}$ in $H^\infty(\mathbb{B}_d)$.

The proof of this result uses duality arguments. It depends on the rather precise structure of the dual space of $\mathrm{A}(\mathbb{B}_d)$ which in turn hinges on results of Henkin [15], Valskii [23] and Cole and Range [6] (for a full treatment, see [22, Chapter 9]). The Carleson-Rudin theorem contains the information that the zero set of a proper ideal intersects $\mathbb{S}_1$ in a closed set of Lebesgue measure zero. But Hedenmalm's theorem contains almost no information about zero sets.

In the present paper, we study the ideals and zero sets for the norm closed algebra $\mathcal{A}_d$ generated by the polynomial multipliers on the Drury-Arveson space. Analogues of the aforementioned results on the dual space of $\mathrm{A}(\mathbb{B}_d)$ were established by the authors in [5], and they play an equally important role in our study of $\mathcal{A}_d$. In addition to obtaining an analogue of

Hedenmalm's theorem, we show that given a zero set in the ball (which is an analytic variety), the part of the zero set on the sphere is determined up to a small set in a specific sense. Conversely, the part of the zero set on the sphere likewise determines the variety inside the ball except for a countable discrete set.

The plan of the paper is as follows. In Section 2 we introduce the necessary background and preliminaries on the Drury-Arveson space and the algebra $\mathcal{A}_d$. In Section 3, we establish several technical duality results that we require throughout the paper. We relate the property of a subset of the sphere being $\mathcal{A}_d$-totally null to the weak-$*$ closure of a certain space of measures on that set (Corollary 3.3 and Theorem 3.4). We also obtain some density and interpolation results (Corollaries 3.5, 3.6 and 3.8) as byproducts of our tools. In Section 4, we establish our structure theorem (Theorem 4.1) for ideals of $\mathcal{A}_d$, drawing a close parallel with Theorems 1.1 and 1.2. In particular, the weak-$*$ closure of an ideal plays a role. Theorem 4.3 shows that the zero set of an ideal and that of its weak-$*$ closure are the same up to an $\mathcal{A}_d$-totally null set. Finally, in Section 5 we undertake a more exhaustive study of zero sets for $\mathcal{A}_d$. We show that $\mathcal{A}_d$–totally null subsets of the sphere are necessarily zero sets (Proposition 5.1), and provide supporting evidence that the converse may also be valid (Theorem 5.4), thus connecting our work with an old unresolved question of Rudin. We also consider the problem of describing the smallest zero set for $\mathcal{A}_d$ containing a given subset X, and obtain rather definitive results in the case where X is the intersection of a zero set with the sphere (Theorem 5.9). Finally, we show that under some mild natural conditions, an interpolating sequence can be adjoined to an $\mathcal{A}_d$–totally null subset to obtain another zero set (Corollary 5.11).

2. Background and preliminaries

Let $d \geq 1$ be an integer. The *Drury-Arveson space*, denoted by H^2_d, is the reproducing kernel Hilbert space on the open unit ball $\mathbb{B}_d \subset \mathbb{C}^d$ with kernel given by

$$k(z,w) = \frac{1}{1-\langle z,w\rangle} \quad \text{for} \quad z,w \in \mathbb{B}_d.$$

This space has the complete Nevanlinna-Pick property ([20]), and furthermore it plays a central role in modern multivariate operator theory ([10, 19, 1]). It is only a supporting character in our study however, as we are mostly interested in its *multiplier algebra*, which we denote by $\mathcal{M}_d$. Recall that a function $\varphi : \mathbb{B}_d \to \mathbb{C}$ is a *multiplier* if $\varphi H^2_d \subset H^2_d$. The associated multiplication operator M_φ on H^2_d is bounded, and we define the multiplier norm as follows

$$\|\varphi\|_{\mathcal{M}_d} = \|M_\varphi\|.$$

It is basic fact that

$$\|\varphi\|_\infty \leq \|\varphi\|_{\mathcal{M}_d}$$

for every $\varphi \in \mathcal{M}_d$. While the polynomials are contained in $\mathcal{M}_d$, the inclusion

$$\mathcal{M}_d \subset H^\infty(\mathbb{B}_d)$$

is strict, and in fact the multiplier norm and supremum norm are not comparable. Throughout the paper we will focus on the norm closed subalgebra $\mathcal{A}_d$ of $\mathcal{M}_d$ generated by the polynomial multipliers. In particular, we have

$$\mathcal{A}_d \subset \mathrm{A}(\mathbb{B}_d).$$

By identifying a multiplier $\varphi \in \mathcal{M}_d$ with the multiplication operator $M_\varphi \in B(H^2_d)$, we may view $\mathcal{M}_d$ as an operator algebra on H^2_d, and this algebra is closed in the weak-$*$ topology of $B(H^2_d)$. In particular, $\mathcal{M}_d$ has a weak-$*$ topology. For sequences, this topology can easily be understood as follows: $\{\varphi_n\}_n \subset \mathcal{M}_d$ converges to $\psi \in \mathcal{M}_d$ in the weak-$*$ topology if and only if $\{\varphi_n\}_n$ is bounded in the multiplier norm and converges pointwise to ψ on $\mathbb{B}_d$.

As mentioned in the introduction, several of our results have proofs that are based on duality arguments in $\mathcal{A}_d$. Because of the inequality

$$\|\varphi\|_\infty \leq \|\varphi\|_{\mathcal{M}_d}, \quad \varphi \in \mathcal{A}_d,$$

every regular Borel measure on the unit sphere $\mathbb{S}_d$ gives rise to a continuous linear functional on $\mathcal{A}_d$ via integration. Denote by $M(\mathbb{S}_d)$ the space of such measures. Let $\mu \in M(\mathbb{S}_d)$ and let $\Phi_\mu \in \mathcal{A}_d^*$ be the induced integration functional on $\mathcal{A}_d$. We say that μ is *$\mathcal{A}_d$-Henkin* if Φ_μ extends to a weak-$*$ continuous linear functional on $\mathcal{M}_d$. At the other extreme, we say that μ is *$\mathcal{A}_d$-totally singular* if it is singular with respect to every $\mathcal{A}_d$-Henkin measure. These definitions are direct analogues of the the classical ones for the ball algebra [22, Chapter 9]. The point mass δ_ζ is $\mathcal{A}_d$-totally singular for every $\zeta \in \mathbb{S}_d$ by [5, Proposition 6.1]. Given a Borel subset $E \subset \mathbb{S}_d$, we let $\mathrm{TS}_{\mathcal{A}_d}(E)$ denote the space of all $\mathcal{A}_d$-totally singular measures μ which are concentrated on E, in the sense that $\mu(A \cap E) = \mu(A)$ for every Borel set A.

A Borel subset $K \subset \mathbb{S}_d$ is called *$\mathcal{A}_d$-totally null* if $|\mu|(K) = 0$ for every $\mathcal{A}_d$-Henkin measure μ. Equivalently, K is $\mathcal{A}_d$-totally null if every measure supported on it is $\mathcal{A}_d$-totally singular ([5, Corollary 5.6]). On such sets, we can achieve "peak interpolation" using functions in $\mathcal{A}_d$ with additional control on the multiplier norm of the interpolating function. The following deep result ([5, Theorem 9.5]) is analogous to a classical theorem of Bishop for uniform algebras [2].

Theorem 2.1. *Let $K \subset \mathbb{S}_d$ be a closed $\mathcal{A}_d$-totally null subset and let $\varepsilon > 0$. Then for every $f \in \mathrm{C}(K)$, there exists $\varphi \in \mathcal{A}_d$ such that*

(i) $\varphi|_K = f$,
(ii) $|\varphi(\zeta)| < \|f\|_K$ *for every* $\zeta \in \overline{\mathbb{B}_d} \setminus K$, *and*
(iii) $\|\varphi\|_{\mathcal{M}_d} \leq (1+\varepsilon)\|f\|_K$.

It is not known whether ε can be chosen to be zero in the statement above.

Note that since the multiplier norm and supremum norm are not comparable, not all continuous linear functionals on $\mathcal{A}_d$ are of the form Φ_μ for some $\mu \in M(\mathbb{S}_d)$. The following result is a summary of [5, Corollary 4.3, Theorem 4.4] and it paints a more complete picture.

Theorem 2.2. *The dual space $\mathcal{A}_d^*$ can be completely isometrically identified with $\mathcal{M}_{d*} \oplus_1 \mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d)$. In particular, any functional $\Psi \in \mathcal{A}_d^*$ can be decomposed as $\Psi = \Psi_a + \Phi_{\mu_s}$ where $\Psi_a \in \mathcal{M}_{d*}$, $\mu_s \in \mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d)$ and*

$$\|\Psi\|_{\mathcal{A}_d^*} = \|\Psi_a\|_{\mathcal{M}_{d*}} + \|\mu_s\|_{M(\mathbb{S}_d)}.$$

Here, $\mathcal{M}_{d*}$ denotes the Banach space whose dual is $\mathcal{M}_d$. It is known to be unique in a rather strong sense [17].

Because of Theorem 2.2, to understand $\mathcal{A}_d^*$ it is sufficient to understand both pieces $\mathcal{M}_{d*}$ and $\mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d)$ separately. One property of these spaces we will require is that they are closed under absolute continuity: they form *bands*.

Theorem 2.3. *Let $\mu, \nu \in M(\mathbb{S}_d)$ be such that μ is absolutely continuous with respect to ν. If ν is $\mathcal{A}_d$-totally singular then so is μ. If ν is $\mathcal{A}_d$-Henkin then so is μ.*

Let us now briefly discuss the space $\mathcal{M}_{d*}$, which is more difficult to grasp than $\mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d)$. By definition, it contains Φ_μ whenever μ is an $\mathcal{A}_d$-Henkin measure. However, since the norm on $\mathcal{A}_d$ is not comparable to the supremum norm over the ball, it must contain functionals which are not given as integration against some measure on the sphere. For every element $\Psi \in \mathcal{M}_{d*}$ and every $\varepsilon > 0$, there are functions $f, g \in H_d^2$ such that

$$\Psi(\varphi) = \langle M_\varphi f, g\rangle_{H_d^2}, \quad \varphi \in \mathcal{A}_d$$

and

$$\|f\|_{H_d^2}\, \|g\|_{H_d^2} < \|\Psi\|_{\mathcal{A}_d^*} + \varepsilon.$$

These facts follow from [8, Theorem 2.10], but we will not require them explicitly.

Throughout the paper, if X is a Banach space and $E \subset X$, we put

$$E^\perp = \{\Lambda \in X^* : \Lambda(x) = 0 \quad \text{for all} \quad x \in E\}$$

while if $F \subset X^*$ we put

$$F_\perp = \{x \in X : \Lambda(x) = 0 \quad \text{for all} \quad \Lambda \in F\}.$$

Standard separation arguments using the Hahn-Banach theorem show that if $E \subset X$ is a norm closed subspace and $F \subset X^*$ is weak-$*$ closed subspace, then

$$(E^\perp)_\perp = E \quad \text{and} \quad (F_\perp)^\perp = F.$$

For our purposes, we will need the following F.&M. Riesz type result, which is [5, Theorem 4.7]. If $E \subset \mathcal{A}_d$, we let $\widetilde{E} \subset \mathcal{M}_d$ denote the closure of E in the weak-$*$ topology of $\mathcal{M}_d$.

Theorem 2.4. *Let $\mathcal{J} \subset \mathcal{A}_d$ be a closed ideal and let $\Psi \in \mathcal{J}^\perp$. Suppose that $\Psi = \Psi_a + \Phi_{\mu_s}$ where $\Psi_a \in \mathcal{M}_{d*}$ and $\mu_s \in \mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d)$. Then, $\Psi_a \in \widetilde{\mathcal{J}}_\perp$ and $\mu_s \in \mathrm{TS}_{\mathcal{A}_d}(Z(\mathcal{J}) \cap \mathbb{S}_d)$, where $Z(\mathcal{J})$ is the common zero set of the functions in $\mathcal{J}$.*

3. Duality and ideals in $\mathcal{A}_d$

In this section we prove duality results relating ideals and $\mathcal{A}_d$-totally singular measures that will be used throughout the paper. First, let us establish some terminology and notation.

If $\mathcal{S} \subset \mathcal{A}_d$ is a set, we let $Z(\mathcal{S}) \subset \overline{\mathbb{B}_d}$ denote the common zero set of the functions in $\mathcal{S}$. Observe that if $\langle \mathcal{S} \rangle \subset \mathcal{A}_d$ denotes the closed ideal generated by S, then $Z(\mathcal{S}) = Z(\langle \mathcal{S} \rangle)$. Furthermore, if $X \subset \overline{\mathbb{B}_d}$ is a set, we let $\mathcal{I}(X) \subset \mathcal{A}_d$ denote the closed ideal of functions vanishing on X.

It is easily verified that for any sets $\mathcal{S} \subset \mathcal{A}_d$ and $X \subset \overline{\mathbb{B}_d}$, we have

$$X \subset Z(\mathcal{I}(X)) \quad \text{and} \quad \langle \mathcal{S} \rangle \subset \mathcal{I}(Z(\mathcal{S})).$$

In particular, we have that

$$Z(\mathcal{I}(Z(\mathcal{S}))) = Z(\mathcal{S}) \quad \text{and} \quad \mathcal{I}(Z(\mathcal{I}(X))) = \mathcal{I}(X). \tag{1}$$

We start with a basic observation.

Proposition 3.1. *Let $E \subset \mathbb{S}_d$ be a Borel set. Then*

$$\mathrm{TS}_{\mathcal{A}_d}(E)_\perp = \mathcal{I}(E).$$

Proof. Obviously we have $\mathcal{I}(E) \subset \mathrm{TS}_{\mathcal{A}_d}(E)_\perp$. Conversely, let $\zeta \in E$. Then $\delta_\zeta \in \mathrm{TS}_{\mathcal{A}_d}(E)$ so that $\varphi \in \mathrm{TS}_{\mathcal{A}_d}(E)_\perp$ implies $\varphi(\zeta) = 0$. Hence

$$\mathrm{TS}_{\mathcal{A}_d}(E)_\perp \subset \mathcal{I}(E). \qquad \blacksquare$$

The following result complements this proposition in a special case. The proof is inspired by a clever argument from [14].

Theorem 3.2. *Let $\mathcal{J} \subset \mathcal{A}_d$ be a closed ideal. Then*

$$\mathcal{J}^\perp \cap \mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d) \ = \ \mathcal{I}(Z(\mathcal{J}))^\perp \cap \mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d) \ = \ \mathrm{TS}_{\mathcal{A}_d}(Z(\mathcal{J}) \cap \mathbb{S}_d).$$

Proof. It is clear that

$$\mathrm{TS}_{\mathcal{A}_d}(Z(\mathcal{J}) \cap \mathbb{S}_d) \ \subset \ \mathcal{I}(Z(\mathcal{J}))^\perp \cap \mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d) \ \subset \ \mathcal{J}^\perp \cap \mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d).$$

Take μ in $\mathcal{J}^\perp \cap \mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d)$. Note then that $\varphi\mu$ annihilates $\mathcal{A}_d$ for all φ in $\mathcal{J}$. Hence, $\varphi\mu$ is trivially an $\mathcal{A}_d$-Henkin measure. On the other hand, $\varphi\mu$ belongs to $\mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d)$ by Theorem 2.3; so that $\varphi\mu = 0$. This shows that the support of μ is contained in the zero set of φ. Repeating this argument for every $\varphi \in \mathcal{J}$ shows that μ is supported on $Z(\mathcal{J}) \cap \mathbb{S}_d$, whence μ belongs to $\mathrm{TS}_{\mathcal{A}_d}(Z(\mathcal{J}) \cap \mathbb{S}_d)$. $\blacksquare$

Next, we explore some consequences of the previous observations.

Corollary 3.3. *Let $\mathcal{J} \subset \mathcal{A}_d$ be a closed ideal, and let $K = Z(\mathcal{J}) \cap \mathbb{S}_d$. If $\mathrm{TS}_{\mathcal{A}_d}(K)$ is closed in the weak-$*$ topology of $\mathcal{A}_d^*$, then K is $\mathcal{A}_d$-totally null.*

Proof. We have that

$$\mathrm{TS}_{\mathcal{A}_d}(K) = (\mathrm{TS}_{\mathcal{A}_d}(K)_\perp)^\perp$$

whence Proposition 3.1 implies that

$$\mathcal{I}(K)^\perp \subset \mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d).$$

Any $\mathcal{A}_d$-Henkin measure supported on K lies in $\mathcal{I}(K)^\perp$, and by the previous inclusion it must be the zero measure. We conclude that K is $\mathcal{A}_d$-totally null. ∎

In particular, taking $\mathcal{J} = \{0\}$ shows that $\mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d)$ is not closed in the weak-$*$ topology of $\mathcal{A}_d^*$ (in fact, it is weak-$*$ dense as was observed in [5, Proposition 6.3]). The general problem of determining when $Z(\mathcal{J}) \cap \mathbb{S}_d$ is $\mathcal{A}_d$-totally null is a difficult one and we return to it in Section 5. For now, we show that the converse of Corollary 3.3 holds as well. In fact, the statement is true for general $\mathcal{A}_d$-totally null closed subsets.

Theorem 3.4. *Let $\mathcal{J} \subset \mathcal{A}_d$ be a closed ideal. Let $E \subset \mathbb{S}_d$ be an $\mathcal{A}_d$-totally null Borel set disjoint from $Z(\mathcal{J})$ such that $\overline{E} \subset Z(\mathcal{J}) \cup E$. Then, $\mathcal{J}^\perp + \mathrm{TS}_{\mathcal{A}_d}(E)$ is weak-$*$ closed.*

Proof. Put $X = Z(\mathcal{J})$ and $X_0 = X \cap \mathbb{S}_d$. By the Krein-Smulyan theorem, it suffices to show that for every net $\{\Psi_\alpha\}_\alpha \subset \mathcal{J}^\perp + \mathrm{TS}_{\mathcal{A}_d}(E)$ with $\|\Psi_\alpha\| \leq 1$ which converges to $\Psi \in \mathcal{A}_d^*$ in the weak-$*$ topology, we have that Ψ belongs to $\mathcal{J}^\perp + \mathrm{TS}_{\mathcal{A}_d}(E)$. Now, for each α we can write

$$\Psi_\alpha = \Theta_\alpha + \Phi_{\mu_\alpha}$$

for some $\Theta_\alpha \in \mathcal{J}^\perp$ and $\mu_\alpha \in \mathrm{TS}_{\mathcal{A}_d}(E)$. Furthemore, by Theorem 2.4 we have

$$\mathcal{J}^\perp = \widetilde{\mathcal{J}}_\perp \oplus_1 \mathrm{TS}_{\mathcal{A}_d}(X_0);$$

so that for every α we may write

$$\Theta_\alpha = \Lambda_\alpha \oplus \Phi_{\nu_\alpha}$$

for some $\Lambda_\alpha \in \widetilde{\mathcal{J}}_\perp$ and $\nu_\alpha \in \mathrm{TS}_{\mathcal{A}_d}(X_0)$. Thus,

$$\Psi_\alpha = \Lambda_\alpha \oplus \Phi_{\mu_\alpha + \nu_\alpha}.$$

Since the measures μ_α and ν_α are concentrated on disjoint Borel sets, we have that

$$\|\mu_\alpha\| \leq \|\mu_\alpha + \nu_\alpha\| \leq \|\Psi_\alpha\| \leq 1.$$

Therefore, upon passing to a subnet we may suppose that $\{\mu_\alpha\}_\alpha$ converges to a measure μ concentrated on $\overline{E} \subset X \cup E$ in the weak-$*$ topology of $M(\overline{E})$. Hence, $\{\Theta_\alpha\}_\alpha$ converges to $\Psi - \Phi_\mu$ in the weak-$*$ topology of $\mathcal{A}_d^*$. Since $\mathcal{J}^\perp \subset \mathcal{A}_d^*$ is weak-$*$ closed we see that $\Psi - \Phi_\mu \in \mathcal{J}^\perp$. Finally, note that we can decompose $\mu = \mu_1 + \mu_2$ where μ_1 is concentrated on E and μ_2 is concentrated on X_0. Since E is assumed to be $\mathcal{A}_d$-totally null, we have $\mu_1 \in \mathrm{TS}_{\mathcal{A}_d}(E)$; while $\Phi_{\mu_2} \in \mathcal{J}^\perp$ since $X_0 \subset X = Z(\mathcal{J})$. Hence

$$\Psi - \Phi_\mu + \Phi_{\mu_2} \in \mathcal{J}^\perp$$

and

$$\Psi = (\Psi - \Phi_\mu + \Phi_{\mu_2}) + \Phi_{\mu_1} \in \mathcal{J}^\perp + \mathrm{TS}_{\mathcal{A}_d}(E).$$

Thus $\mathcal{J}^\perp + \mathrm{TS}_{\mathcal{A}_d}(E)$ is closed in the weak-$*$ topology of $\mathcal{A}_d^*$. ∎

Note that upon taking $\mathcal{J} = \mathcal{A}_d$ in the theorem above, we see that $\mathrm{TS}_{\mathcal{A}_d}(E)$ is closed in the weak-$*$ topology of $\mathcal{A}_d^*$ whenever $E \subset \mathbb{S}_d$ is a closed $\mathcal{A}_d$-totally null subset. We obtain the following density result as an immediate consequence.

Corollary 3.5. *Let $K \subset \mathbb{S}_d$ be a closed set. Then, K is $\mathcal{A}_d$-totally null if and only if $\mathcal{I}(K)$ is weak-$*$ dense in $\mathcal{M}_d$.*

Proof. Let $\mathcal{J}$ be the weak-$*$ closure of $\mathcal{I}(K)$ in $\mathcal{M}_d$ and note that

$$\mathcal{J}_\perp = \mathcal{M}_{d*} \cap \mathcal{I}(K)^\perp.$$

Assume first that $\mathcal{I}(K)$ is weak-$*$ dense in $\mathcal{M}_d$, that is $\mathcal{J} = \mathcal{M}_d$. Any $\mathcal{A}_d$-Henkin measure supported on K lies in $\mathcal{I}(K)^\perp \cap \mathcal{M}_{d*} = \mathcal{J}_\perp = \{0\}$; so that K is $\mathcal{A}_d$-totally null.

Assume conversely that K is $\mathcal{A}_d$-totally null. Now, we have that

$$\mathrm{TS}_{\mathcal{A}_d}(K)_\perp = \mathcal{I}(K)$$

by Proposition 3.1. By Theorem 3.4 we see that

$$\mathcal{I}(K)^\perp = (\mathrm{TS}_{\mathcal{A}_d}(K)_\perp)^\perp = \mathrm{TS}_{\mathcal{A}_d}(K)$$

is disjoint from $\mathcal{M}_{d*}$; whence $\mathcal{J}_\perp = \{0\}$ and $\mathcal{J} = (\mathcal{J}_\perp)^\perp = \mathcal{M}_d$. ∎

In fact a stronger density statement holds.

Corollary 3.6. *Let $K \subset \mathbb{S}_d$ be a closed $\mathcal{A}_d$-totally null subset. Then the unit ball of $\mathcal{I}(K)$ is weak-$*$ dense in the unit ball of $\mathcal{M}_d$.*

Proof. As observed in the previous proof, we have

$$\mathcal{I}(K)^\perp = \mathrm{TS}_{\mathcal{A}_d}(K) \subset \mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d).$$

Therefore, by Theorem 2.2 we conclude that

$$\mathcal{I}(K)^* = \mathcal{A}_d^*/\mathcal{I}(K)^\perp = \mathcal{M}_{d*} \oplus_1 (\mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d)/\,\mathrm{TS}_{\mathcal{A}_d}(K)).$$

Consequently we obtain that

$$\mathcal{I}(K)^{**} = \mathcal{M}_d \oplus \mathrm{TS}_{\mathcal{A}_d}(K)^\perp.$$

The desired result is now a consequence of Goldstine's theorem. ∎

We close this section with another interesting density result that depends on duality techniques. The other tool we need for its proof is the following, which is of independent interest. Given $\lambda \in \overline{\mathbb{B}_d}$, we denote by $\tau_\lambda \in \mathcal{A}_d^*$ the functional of evaluation at λ.

Theorem 3.7. *Let $K \subset \mathbb{S}_d$ be a closed $\mathcal{A}_d$-totally null subset and let $\lambda \in \mathbb{B}_d$. Then,*

$$\mathcal{I}(K \cup \{\lambda\})^\perp = \mathrm{TS}_{\mathcal{A}_d}(K) \oplus_1 \mathbb{C}\tau_\lambda$$

and there exists a surjective isometric isomorphism

$$\rho : \mathcal{A}_d/\mathcal{I}(K \cup \{\lambda\}) \to \mathrm{C}(K \cup \{\lambda\})$$

given by the restriction map

$$\rho(\varphi + \mathcal{I}(K \cup \{\lambda\})) = \varphi|_{K \cup \{\lambda\}} \quad \text{for} \quad \varphi \in \mathcal{A}_d.$$

Proof. By Proposition 3.1, we have that

$$\mathrm{TS}_{\mathcal{A}_d}(K)_\perp = \mathcal{I}(K).$$

Invoking Theorem 3.4, we see that $\mathrm{TS}_{\mathcal{A}_d}(K)$ is closed in the weak-$*$ topology of $\mathcal{A}_d^*$ and thus

$$\mathrm{TS}_{\mathcal{A}_d}(K) + \mathbb{C}\tau_\lambda$$

is closed in the weak-$*$ topology of $\mathcal{A}_d^*$. Moreover,

$$\begin{aligned}(\mathrm{TS}_{\mathcal{A}_d}(K) + \mathbb{C}\tau_\lambda)_\perp &= \mathrm{TS}_{\mathcal{A}_d}(K)_\perp \cap (\mathbb{C}\tau_\lambda)_\perp \\ &= \mathcal{I}(K) \cap \mathcal{I}(\{\lambda\}) \\ &= \mathcal{I}(K \cup \{\lambda\}).\end{aligned}$$

Hence

$$\mathcal{I}(K \cup \{\lambda\})^\perp = \mathrm{TS}_{\mathcal{A}_d}(K) + \mathbb{C}\tau_\lambda = \mathrm{TS}_{\mathcal{A}_d}(K) \oplus_1 \mathbb{C}\tau_\lambda$$

where the last equality follows from Theorem 2.2, since $\lambda \in \mathbb{B}_d$ and therefore $\tau_\lambda \in \mathcal{M}_{d*}$. The map ρ is clearly a well-defined homomorphism, so we need only show that it is isometric and surjective.

Recall now that $\delta_\zeta \in \mathrm{TS}_{\mathcal{A}_d}(K)$ for every $\zeta \in K$, so that

$$\sup_{z \in K \cup \{\lambda\}} |\varphi(z)| \leq \sup \left\{ |\Psi(\varphi)| : \Psi \in \mathcal{I}(K \cup \{\lambda\})^\perp, \|\Psi\| \leq 1 \right\}$$

whence

$$\sup_{z \in K \cup \{\lambda\}} |\varphi(z)| \leq \|\varphi + \mathcal{I}(K \cup \{\lambda\})\|_{\mathcal{A}_d/\mathcal{I}(K \cup \{\lambda\}}$$

for every $\varphi \in \mathcal{A}_d$.

Conversely, if $\Psi \in \mathcal{I}(K \cup \{\lambda\})^\perp$ with $\|\Psi\| \leq 1$, we can write $\Psi = c\tau_\lambda + \Phi_\mu$ with $c \in \mathbb{C}, \mu \in \mathrm{TS}_{\mathcal{A}_d}(K)$ and

$$1 \geq \|c\tau_\lambda + \Phi_\mu\| = \|c\tau_\lambda\| + \|\Phi_\mu\| = |c| + \|\mu\|.$$

Hence,

$$|\Psi(\varphi)| \leq |c||\varphi(\lambda)| + \|\mu\| \sup_{z \in K} |\varphi(z)| \leq \sup_{z \in K \cup \{\lambda\}} |\varphi(z)|.$$

Therefore

$$\|\varphi + \mathcal{I}(K \cup \{\lambda\})\| \leq \sup_{z \in K \cup \{\lambda\}} |\varphi(z)|$$

for every $\varphi \in \mathcal{A}_d$, and equality is established; so the map ρ is isometric.

To see that ρ is surjective, observe that the dual space of $\mathcal{A}_d/\mathcal{I}(K \cup \{\lambda\})$ is naturally identified with

$$\mathcal{I}(K \cup \{\lambda\})^\perp = \mathrm{TS}_{\mathcal{A}_d}(K) \oplus_1 \mathbb{C}\tau_\lambda.$$

However since K is $\mathcal{A}_d$-totally null, $M(K) = \mathrm{TS}_{\mathcal{A}_d}(K)$ and thus

$$\mathcal{A}_d/\mathcal{I}(K \cup \{\lambda\})^* = M(K \cup \{\lambda\}) = \mathrm{C}(K \cup \{\lambda\})^*.$$

It follows from the Hahn-Banach Theorem that

$$\rho(\mathcal{A}_d/\mathcal{I}(K \cup \{\lambda\})) = \mathrm{C}(K \cup \{\lambda\}).$$ ∎

As a consequence, we obtain the following interpolation result for functions in $\mathcal{A}_d$.

Corollary 3.8. *Let $K \subset \mathbb{S}_d$ be a closed $\mathcal{A}_d$-totally null subset and $\lambda \in \mathbb{B}_d$. Then the restriction of the open unit ball of $\mathcal{A}_d$ to $K \cup \{\lambda\}$ coincides with the open unit ball of $\mathrm{C}(K \cup \{\lambda\})$.*

Proof. Let $q : \mathcal{A}_d \to \mathcal{A}_d/\mathcal{I}(K \cup \{\lambda\})$ be the quotient map. By Theorem 3.7, the restriction map

$$\rho : \mathcal{A}_d/\mathcal{I}(K \cup \{\lambda\}) \to \mathrm{C}(K \cup \{\lambda\})$$

is an isometric isomorphism. In particular, the open unit ball of $\rho(q(\mathcal{A}_d))$ coincides with the open unit ball of $\mathrm{C}(K \cup \{\lambda\})$. Let $f \in \mathrm{C}(K \cup \{\lambda\})$ such that $\|f\| < 1$. Then, there is $\varphi \in \mathcal{A}_d$ such that $\varphi|_{K\cup\{\lambda\}} = f$ and $\|q(\varphi)\| = \|f\|$. By definition of the quotient norm we can find $\psi \in \mathcal{A}_d$ such that $\|\psi\| < 1$ and $\psi|_{K\cup\{\lambda\}} = f$, which completes the proof. ∎

It would be interesting to know if a full analogue of Theorem 2.1 holds for $K \cup \{\lambda\}$. Furthermore, it would be of interest to understand what happens when $\{\lambda\}$ is replaced by an arbitrary finite set $F \subset \mathbb{B}_d$. The restriction of $\mathcal{A}_d$ (or even $\mathrm{A}(\mathbb{B}_d)$) to a two point set $F = \{\lambda, \mu\} \subset \mathbb{B}_d$ cannot be isometric to $\mathrm{C}(F)$ because of Schwarz's lemma. The complete Nevanlinna-Pick property governs the norm of a multiplier that achieves specific values at these two points. This is closely related to questions about interpolating sequences. See Theorem 5.10 for a related result.

4. Description of the ideals of $\mathcal{A}_d$

In this short section, we give a structure theorem for ideals of $\mathcal{A}_d$. The result is a natural analogue of Theorems 1.1 and 1.2. Recall that if $\mathcal{J} \subset \mathcal{A}_d$, then $\widetilde{\mathcal{J}} \subset \mathcal{M}_d$ denotes its closure in the weak-$*$ topology of $\mathcal{M}_d$.

Theorem 4.1. *Let $\mathcal{J} \subset \mathcal{A}_d$ be a closed ideal, and let $K = Z(\mathcal{J}) \cap \mathbb{S}_d$. Then*

$$\mathcal{J} = \mathcal{I}(K) \cap \widetilde{\mathcal{J}}.$$

Proof. By Theorem 2.4, we have that

$$\mathcal{J}^\perp = \widetilde{\mathcal{J}}_\perp \oplus_1 \mathrm{TS}_{\mathcal{A}_d}(K).$$

Note that $\widetilde{\mathcal{J}}_\perp \subset \mathcal{M}_{d*} \subset \mathcal{A}_d^*$, so $(\widetilde{\mathcal{J}}_\perp)_\perp \subset \mathcal{A}_d$. A straightforward verification shows that

$$(\widetilde{\mathcal{J}}_\perp)_\perp = \widetilde{\mathcal{J}} \cap \mathcal{A}_d.$$

Moreover,

$$\mathcal{I}(K) = \mathrm{TS}_{\mathcal{A}_d}(K)_\perp$$

by Proposition 3.1. We conclude that

$$\begin{aligned} \mathcal{J} = (\mathcal{J}^\perp)_\perp &= (\widetilde{\mathcal{J}}_\perp)_\perp \cap \mathrm{TS}_{\mathcal{A}_d}(K)_\perp \\ &= (\widetilde{\mathcal{J}} \cap \mathcal{A}_d) \cap \mathcal{I}(K) \\ &= \widetilde{\mathcal{J}} \cap \mathcal{I}(K). \end{aligned}$$

■

As a consequence, we obtain the following.

Corollary 4.2. *Let $\mathcal{J} \subset \mathcal{A}_d$ be a closed ideal, and let $\widetilde{\mathcal{J}} \subset \mathcal{M}_d$ be its weak-$*$ closure. Then, the following statements are equivalent:*

(i) $\widetilde{\mathcal{J}} \cap \mathcal{A}_d = \mathcal{J}$,
(ii) $Z(\widetilde{\mathcal{J}} \cap \mathcal{A}_d) = Z(\mathcal{J})$, *and*
(iii) $Z(\widetilde{\mathcal{J}} \cap \mathcal{A}_d) \cap \mathbb{S}_d = Z(\mathcal{J}) \cap \mathbb{S}_d$.

Proof. The only non-trivial statement is that (iii) implies (i). Assume that

$$Z(\widetilde{\mathcal{J}} \cap \mathcal{A}_d) \cap \mathbb{S}_d = Z(\mathcal{J}) \cap \mathbb{S}_d$$

and denote this set by X. Note that

$$\mathcal{J} \subset \widetilde{\mathcal{J}} \cap \mathcal{A}_d \subset \widetilde{\mathcal{J}};$$

whence $\widetilde{\mathcal{J}}$ is the weak-$*$ closure of $\widetilde{\mathcal{J}} \cap \mathcal{A}_d$. By Theorem 4.1, we may write

$$\mathcal{J} = \widetilde{\mathcal{J}} \cap \mathcal{I}(X) = (\widetilde{\widetilde{\mathcal{J}} \cap \mathcal{A}_d}) \cap \mathcal{I}(X) = \widetilde{\mathcal{J}} \cap \mathcal{A}_d,$$

and the proof is complete. ■

Consequently we see that an equality between ideals can be detected by an equality between zero sets. Remarkably, the set $Z(\mathcal{J}) \setminus Z(\widetilde{\mathcal{J}} \cap \mathcal{A}_d)$ is always rather small.

Theorem 4.3. *Let $\mathcal{J} \subset \mathcal{A}_d$ be a closed ideal and let $\widetilde{\mathcal{J}} \subset \mathcal{M}_d$ be its weak-$*$ closure. Then, $Z(\mathcal{J}) \setminus Z(\widetilde{\mathcal{J}} \cap \mathcal{A}_d)$ is an $\mathcal{A}_d$-totally null subset of $\mathbb{S}_d$.*

Proof. Let $X = Z(\mathcal{J})$ and $Y = Z(\widetilde{\mathcal{J}} \cap \mathcal{A}_d)$. Put $K = X \setminus Y$. If φ is in $\widetilde{\mathcal{J}}$, then there exists a sequence $\{\psi_n\}_n \subset \mathcal{J}$ which converges to φ in the weak-$*$ topology of $\mathcal{M}_d$. In particular, it follows that $\varphi(\lambda) = 0$ for every $\lambda \in Z(\mathcal{J}) \cap \mathbb{B}_d$; whence

$$Z(\mathcal{J}) \cap \mathbb{B}_d = Z(\widetilde{\mathcal{J}} \cap \mathcal{A}_d) \cap \mathbb{B}_d.$$

We conclude that $K \subset \mathbb{S}_d$.

Let η be an $\mathcal{A}_d$-Henkin measure. We claim that $|\eta|(K) = 0$. Note that the restriction of the measure $|\eta|$ to K is also $\mathcal{A}_d$-Henkin because it is absolutely continuous with respect to η (Theorem 2.3). Call this measure μ. Let

$\zeta_0 \in K$ and choose a function $\varphi \in \widetilde{\mathcal{J}} \cap \mathcal{A}_d$ such that $\varphi(\zeta_0) \neq 0$. There exists a sequence $\{\psi_n\}_n \subset \mathcal{J}$ converging to φ in the weak-$*$ topology of $\mathcal{M}_d$. Observe that $\overline{\varphi}\mu$ is $\mathcal{A}_d$-Henkin by Theorem 2.3 so we find

$$\int_K |\varphi|^2 \, d\mu = \lim_{n\to\infty} \int_K \psi_n \overline{\varphi} \, d\mu = 0$$

since each ψ_n vanishes on X. In particular,

$$\mu(\{\zeta \in K : \varphi(\zeta) \neq 0\}) = 0.$$

Since φ is continuous and $\varphi(\zeta_0) \neq 0$, this shows that ζ_0 does not lie in the support of μ. As $\zeta_0 \in K$ was arbitrary, we conclude that $|\eta|(K) = 0$. Since η was an arbitrary $\mathcal{A}_d$-Henkin measure, K is $\mathcal{A}_d$-totally null. ∎

5. Zero sets for $\mathcal{A}_d$

In this section we will be interested in *zero sets* for $\mathcal{A}_d$, that is closed sets $X \subset \overline{\mathbb{B}_d}$ such that there is a set $\mathcal{S} \subset \mathcal{A}_d$ for which $X = Z(\mathcal{S})$. We focus in particular on the size of zero sets, and on the intersection of zero sets with the sphere. In the case of the ball algebra, it is a classical fact that a closed subset of the sphere is the zero set of a single function in $\mathrm{A}(\mathbb{B}_d)$ if and only if it is $\mathrm{A}(\mathbb{B}_d)$–totally null (see [22, Chapter 10] for the appropriate definition and statement). In $\mathcal{A}_d$, we can establish one of these implications.

Proposition 5.1. *Let $K \subset \mathbb{S}_d$ be a closed $\mathcal{A}_d$-totally null subset. Then there is a function $\varphi \in \mathcal{A}_d$ such that $Z(\varphi) = K$.*

Proof. By Theorem 2.1, there exists $\psi \in \mathcal{A}_d$ such that

$$\psi|_K = 1 \quad \text{and} \quad |\psi(\zeta)| < 1 \ \text{ for } \zeta \in \mathbb{S}_d \setminus K.$$

Hence, we see that for $z \in \overline{\mathbb{B}_d}$, we have $\psi(z) = 1$ if and only if $z \in K$. If we let $\varphi = 1 - \psi$, then $Z(\varphi) = K$. ∎

We do not know whether the converse holds.

Question 5.2. *Let $\varphi \in \mathcal{A}_d$ such that $Z(\varphi)$ is contained in $\mathbb{S}_d$. Must $Z(\varphi)$ be $\mathcal{A}_d$-totally null?*

More generally, an old unresolved question of Rudin ([22, page 415]) asks whether the zero set of an ideal of $\mathrm{A}(\mathbb{B}_d)$ which is contained in the sphere is necessarily $\mathrm{A}(\mathbb{B}_d)$–totally null (equivalently, the zero set of a *single* function in $\mathrm{A}(\mathbb{B}_d)$). One can formulate the corresponding problem in $\mathcal{A}_d$.

Question 5.3. *Let X be a zero set for $\mathcal{A}_d$ that is contained in $\mathbb{S}_d$. Must X be $\mathcal{A}_d$-totally null?*

Since Rudin's problem has been around for a long time, we expect this question to be very difficult to answer. Nevertheless, Corollaries 3.3, 3.5 and Theorem 3.4 provide a different approach to the problem. Moreover, note that if $X \subset \mathbb{S}_d$ is a zero set, then putting $\mathcal{J} = \mathcal{I}(X)$, we have $X = Z(\mathcal{J})$ by (1). Therefore

$$X = \big(Z(\widetilde{\mathcal{J}} \cap \mathcal{A}_d) \cap \mathbb{S}_d\big) \;\cup\; \big(Z(\mathcal{J}) \setminus Z(\widetilde{\mathcal{J}} \cap \mathcal{A}_d)\big) \tag{2}$$

since

$$Z(\widetilde{\mathcal{J}} \cap \mathcal{A}_d) \cap \mathbb{B}_d = Z(\mathcal{J}) \cap \mathbb{B}_d$$

as was shown in the proof of Theorem 4.3. Note that the union in (2) is disjoint, and that the second member in the right-hand side of the equality is $\mathcal{A}_d$-totally null by Theorem 4.3. In particular, X is $\mathcal{A}_d$-totally null if and only if $Z(\widetilde{\mathcal{J}} \cap \mathcal{A}_d) \cap \mathbb{S}_d$ is. This observation may help in answering Question 5.3, as the ideal $\widetilde{\mathcal{J}} \cap \mathcal{A}_d$ may be easier to handle given our understanding of the weak-$*$ closed ideals of $\mathcal{M}_d$ (see [9, 12, 18]).

Next, we establish a partial result which may be regarded as offering supporting evidence that indeed, the properties for a closed set $X \subset \mathbb{S}_d$ of being the zero set for an ideal of $\mathcal{A}_d$, of being the zero set of a single function in $\mathcal{A}_d$, and of being $\mathcal{A}_d$-totally null are equivalent.

Theorem 5.4. *Let $\mathcal{J} \subset \mathcal{A}_d$ be a closed ideal. Let $E \subset \mathbb{S}_d$ be an $\mathcal{A}_d$-totally null Borel set such that $\overline{E} \subset Z(\mathcal{J}) \cup E$. If we put $\mathcal{J}' = \mathcal{J} \cap \mathcal{I}(E)$, then*

$$Z(\mathcal{J}') = Z(\mathcal{J}) \cup E.$$

Proof. Put $X = Z(\mathcal{J})$. We may suppose that E is disjoint from X by replacing it with $E \setminus X$ if necessary. It is clear that $X \cup E \subset Z(\mathcal{J}')$, so we need only show the reverse inclusion.

Observe that

$$(\mathcal{J}^{\perp} + \mathrm{TS}_{\mathcal{A}_d}(E))_{\perp} = (\mathcal{J}^{\perp})_{\perp} \cap \mathrm{TS}_{\mathcal{A}_d}(E)_{\perp} = \mathcal{J} \cap \mathcal{I}(E) = \mathcal{J}'$$

by Proposition 3.1. By Theorem 3.4, we see that $\mathcal{J}^{\perp} + \mathrm{TS}_{\mathcal{A}_d}(E)$ is closed in the weak-$*$ topology of $\mathcal{A}_d^*$, whence

$$\mathcal{J}'^{\perp} = \mathcal{J}^{\perp} + \mathrm{TS}_{\mathcal{A}_d}(E) = \widetilde{\mathcal{J}}_{\perp} \oplus_1 (\mathrm{TS}_{\mathcal{A}_d}(X \cap \mathbb{S}_d) + \mathrm{TS}_{\mathcal{A}_d}(E))$$

by Theorem 2.4. Let now $\lambda \in Z(\mathcal{J}')$. If $\lambda \in \mathbb{B}_d$, then the functional τ_λ of evaluation at λ lies in $\mathcal{J}'^{\perp} \cap \mathcal{M}_{d*} = \widetilde{\mathcal{J}}_{\perp} \subset \mathcal{J}^{\perp}$. In particular, we must have $\lambda \in Z(\mathcal{J}) = X$. If $\lambda \in \mathbb{S}_d$, then τ_λ is given by integration against δ_λ, so that

$$\delta_\lambda \in \mathrm{TS}_{\mathcal{A}_d}(\mathbb{S}_d) \cap \mathcal{J}'^{\perp} = \mathrm{TS}_{\mathcal{A}_d}(X \cap \mathbb{S}_d) + \mathrm{TS}_{\mathcal{A}_d}(E);$$

whence $\lambda \in (X \cap \mathbb{S}_d) \cup E$. We conclude that

$$Z(\mathcal{J}') \subset X \cup E. \qquad \blacksquare$$

An immediate consequence of this result is the following.

Corollary 5.5. *If $X \subset \overline{\mathbb{B}_d}$ is a zero set for $\mathcal{A}_d$ and E is an $\mathcal{A}_d$-totally null Borel set such that $\overline{E} \subset X \cup E$, then $X \cup E$ is a zero set for $\mathcal{A}_d$.*

In relation with Question 5.3, one may wonder to what extent a zero set for $\mathcal{A}_d$ is determined by its intersection with the sphere. In order to address this issue, we need a standard lemma.

Lemma 5.6. *Let $X \subset \overline{\mathbb{B}_d}$ be a closed subset. Then, the character space of $\mathcal{A}_d/\mathcal{I}(X)$ can be canonically identified with $Z(\mathcal{I}(X))$. Moreover, under this identification the Gelfand transform*

$$\Gamma : \mathcal{A}_d/\mathcal{I}(X) \to \mathrm{C}(Z(\mathcal{I}(X)))$$

is given by the restriction map

$$\Gamma(\varphi + \mathcal{I}(X)) = \varphi|_{Z(\mathcal{I}(X))} \quad \textit{for} \quad \varphi \in \mathcal{A}_d.$$

Proof. Let

$$q : \mathcal{A}_d \to \mathcal{A}_d/\mathcal{I}(X)$$

be the quotient map. As this map is surjective, the adjoint map q^* is an injective map from $(\mathcal{A}_d/\mathcal{I}(X))^*$ to $\mathcal{A}_d^*$ which takes characters to characters. Let χ be a character of $\mathcal{A}_d/\mathcal{I}(X)$. Then $q^*(\chi) = \chi \circ q$ is a character of $\mathcal{A}_d$. Hence there is a $\lambda \in \overline{\mathbb{B}_d}$ so that $\chi \circ q = \tau_\lambda$, where τ_λ is evaluation at λ. Observe that

$$\varphi(\lambda) = (\chi \circ q)(\varphi) = 0 \quad \text{for all} \quad \varphi \in \mathcal{I}(X);$$

whence $\lambda \in Z(\mathcal{I}(X))$.

Conversely, let $\lambda \in Z(\mathcal{I}(X))$. Then, $\mathcal{I}(X) \subset \ker \tau_\lambda$; so there exists a character χ_λ of $\mathcal{A}_d/\mathcal{I}(X)$ with $\chi_\lambda \circ q = \tau_\lambda$. Hence the character space of $\mathcal{A}_d/\mathcal{I}(X)$ is canonically identified with $Z(\mathcal{I}(X))$. The claim about the Gelfand transform follows immediately. ∎

Recall that given a commutative Banach algebra $\mathcal{A}$, its *Shilov boundary* is the smallest closed subset Σ of the character space $\Delta_{\mathcal{A}}$ with the property that

$$\max_{\chi \in \Delta_{\mathcal{A}}} |\chi(a)| = \max_{\chi \in \Sigma} |\chi(a)|$$

for every $a \in \mathcal{A}$. The next lemma establishes a simple topological property of the Shilov boundary for some quotients of $\mathcal{A}_d$.

Lemma 5.7. *Let $X \subset \overline{\mathbb{B}_d}$ and let Σ_X denote the Shilov boundary of the algebra $\mathcal{A}_d/\mathcal{I}(X)$. Then, Σ_X contains $Z(\mathcal{I}(X)) \cap \mathbb{S}_d$ and $\Sigma_X \cap \mathbb{B}_d$ consists of all isolated points of $Z(\mathcal{I}(X)) \cap \mathbb{B}_d$. Moreover, if z is an isolated point of $Z(\mathcal{I}(X)) \cap \mathbb{B}_d$, then there is a $\varphi \in \mathcal{A}_d$ such that*

$$\varphi(z) = 1 \quad \textit{and} \quad \varphi|_{Z(I(X)) \setminus \{z\}} = 0.$$

Proof. By virtue of Lemma 5.6 we see that the character space of $\mathcal{A}_d/\mathcal{I}(X)$ is $Z(\mathcal{I}(X))$ and that the Gelfand transform is the restriction of a function in $\mathcal{A}_d$ to that set. For $\zeta \in Z(\mathcal{I}(X)) \cap \mathbb{S}_d$ let

$$\theta_\zeta(z) = \frac{1 + \langle z, \zeta \rangle_{\mathbb{C}^d}}{2}, \quad z \in \mathbb{B}_d.$$

Then, $\theta_\zeta \in \mathcal{A}_d$ and

$$|\theta_\zeta(\lambda)| < |\theta_\zeta(\zeta)| = 1$$

whenever $\lambda \in \overline{\mathbb{B}_d} \setminus \{\zeta\}$; whence $\zeta \in \Sigma_X$. Thus, $Z(\mathcal{I}(X)) \cap \mathbb{S}_d \subset \Sigma_X$.

Next, let z be a non-isolated point of $Z(\mathcal{I}(X)) \cap \mathbb{B}_d$. Since $Z(\mathcal{I}(X)) \cap \mathbb{B}_d$ is a complex analytic variety, there is an irreducible subvariety V containing z which has dimension at least 1 (see [24, Theorem 3.2B]). By the maximum modulus principle (see [13, Theorem III.B.16]), we see that $\overline{V} \cap \mathbb{S}_d$ is non-empty and

$$|\varphi(z)| \leq \max_{\lambda \in \overline{V} \cap \mathbb{S}_d} |\varphi(\lambda)|$$

for every $\varphi \in \mathcal{A}_d$. In particular, $\Sigma_X \subset \overline{\mathbb{B}_d} \setminus (V \cap \mathbb{B}_d)$ and z does not belong to Σ_X.

Finally suppose that z is an isolated point of $Z(\mathcal{I}(X)) \cap \mathbb{B}_d$. By the Shilov idempotent theorem (see [11, Corollary III.6.53]), there is an element $b \in \mathcal{A}_d/\mathcal{I}(X)$ such that

$$\Gamma(b)(z) = 1 \quad \text{and} \quad \Gamma(b)|_{Z(I(X))\setminus\{z\}} = 0.$$

In particular, there is $\varphi \in \mathcal{A}_d$ such that

$$\varphi(z) = 1 \quad \text{and} \quad \varphi|_{Z(I(X))\setminus\{z\}} = 0.$$

Therefore z belongs to the Shilov boundary. ∎

Recall also that a *representing measure* for a point $\lambda \in \overline{\mathbb{B}_d}$ is a regular Borel measure μ on $\overline{\mathbb{B}_d}$ satisfying

$$\varphi(\lambda) = \int \varphi \, d\mu$$

for every $\varphi \in \mathcal{A}_d$. We show that for non-isolated points of a zero set inside the ball, there always exists a representing measure supported on the part of that zero set lying in the sphere.

Lemma 5.8. *Let $X \subset \overline{\mathbb{B}_d}$ and let $\lambda \in Z(\mathcal{I}(X)) \cap \mathbb{B}_d$ be a non-isolated point. Then, there is a positive representing measure for λ that is supported on $Z(\mathcal{I}(X)) \cap \mathbb{S}_d$.*

Proof. The closure of the Gelfand transform of $\mathcal{A}_d/\mathcal{I}(X)$ can be considered as a uniform algebra on its Shilov boundary Σ_X. Thus, there is a positive representing measure μ for the non-isolated point $\lambda \in Z(\mathcal{I}(X)) \cap \mathbb{B}_d$ supported on Σ_X. Let $z \in Z(\mathcal{I}(X)) \cap \mathbb{B}_d$ be an isolated point. In particular, $z \neq \lambda$. By Lemma 5.7, there is $\varphi \in \mathcal{A}_d$ such that $\varphi(z) = 1$ while φ vanishes on $Z(\mathcal{I}(X)) \setminus \{z\}$. We find

$$0 = \varphi(\lambda) = \int_{\Sigma_X} \varphi d\mu = \mu(\{z\})$$

and thus z does not belong to the support of μ. By Lemma 5.7 again, we conclude that μ must be supported on $Z(\mathcal{I}(X)) \cap \mathbb{S}_d$. ∎

It turns out that a zero set for $\mathcal{A}_d$ is completely determined by its intersection with the sphere, except for isolated points in the ball. This is the content of the next result.

Theorem 5.9. *Let X be a zero set for $\mathcal{A}_d$ and let $X_0 = X \cap \mathbb{S}_d$. Define $\widehat{X_0}$ to be the set consisting of X_0 together with all points in $\mathbb{B}_d$ which have a representing measure supported on X_0. Then, $\widehat{X_0}$ is the smallest zero set for $\mathcal{A}_d$ containing X_0, and $X \setminus \widehat{X_0}$ is a countable discrete set.*

Proof. Let Y be the smallest zero set for $\mathcal{A}_d$ containing X_0 and put $\mathcal{B} = \mathcal{A}_d/\mathcal{I}(Y)$. Note that $Y \cap \mathbb{S}_d = X_0$ since X is a zero set. By Lemma 5.6, the character space of $\mathcal{B}$ is canonically identified with

$$Z(\mathcal{I}(Y)) = Y,$$

since Y is a zero set (use (1)). Moreover, under this identification, the Gelfand transform $\Gamma : \mathcal{B} \to \mathrm{C}(Y)$ is given as $\Gamma(q(\varphi)) = \varphi|_Y$ for every $\varphi \in \mathcal{A}_d$, where $q : \mathcal{A}_d \to \mathcal{B}$ is the quotient map.

Let $\Sigma \subset Y$ be the Shilov boundary of $\mathcal{B}$. By Lemma 5.7, Σ consists of X_0 together with any isolated points of $Y \cap \mathbb{B}_d$. However if z is an isolated point of $Y \cap \mathbb{B}_d$, then the same lemma provides a function $\varphi \in \mathcal{A}_d$ such that

$$\varphi(z) = 1 \quad \text{and} \quad \varphi|_{Y \setminus \{z\}} = 0.$$

Therefore $Y \cap Z(\varphi) = Y \setminus \{z\}$ is a zero set containing X_0 that is smaller than Y. This contradiction establishes that Y has no isolated points in the ball. We conclude that X_0 is the Shilov boundary of $\mathcal{B}$.

In particular, the uniform closure $\mathcal{C} = \overline{\Gamma(\mathcal{B})} \subset \mathrm{C}(Y)$ is a uniform algebra with Shilov boundary given by X_0. Hence, $\mathcal{C}$ can be considered as a subalgebra of $\mathrm{C}(X_0)$. It follows that the evaluation functional at any point of $Y \cap \mathbb{B}_d$ has a representing measure supported on X_0. We conclude that $Y \subset \widehat{X_0}$.

Note now that if $\lambda \in \overline{\mathbb{B}_d}$ has a representing measure supported on X_0, then clearly any $\varphi \in \mathcal{A}_d$ vanishing on X_0 must also satisfy $\varphi(\lambda) = 0$. Hence $\widehat{X_0} \subset Z(\mathcal{I}(X_0))$. Since Y is a zero set containing X_0, we also have that

$$\widehat{X_0} \subset Z(\mathcal{I}(X_0)) \subset Z(\mathcal{I}(Y)) = Y.$$

So in fact $Y = \widehat{X_0}$ (here again we used (1)). Finally, $X \setminus \widehat{X_0}$ must consist of isolated points by Lemma 5.8, and thus it is countable and discrete. ∎

Note that one cannot replace X_0 by an arbitrary closed subset of $\mathbb{S}_d$ in the previous theorem. For example, if $X_0 \subset \mathbb{T}$ is a proper subset with positive Lebesgue measure, then it is a classical fact that the smallest zero set for $\mathcal{A}_1 = \mathrm{A}(\mathbb{D})$ containing X_0 is $\overline{\mathbb{D}}$, which differs from $\widehat{X_0}$.

As another remark, we mention that the proof technique used above applies verbatim in the uniform algebra setting, and appears to be new there as well. Let Ω be a bounded, strictly pseudo-convex domain in $\mathbb{C}^n$ which is the interior of its closure. Let $B = \overline{\Omega} \setminus \Omega$. Consider the Banach algebra $A(\Omega)$ consisting of functions in $\mathrm{C}(\overline{\Omega})$ which are analytic on Ω. It is known that the character space of $A(\Omega)$ can be identified with $\overline{\Omega}$ via point evaluation. We can prove the following:

Let X be a zero set for $A(\Omega)$ and let $X_0 = X \cap B$. Define $\widehat{X_0}$ to be the set consisting of X_0 together with all points in Ω which have a representing measure supported on X_0. Then, $\widehat{X_0}$ is the smallest zero set for $A(\Omega)$ containing X_0, and $X \setminus \widehat{X_0}$ is a countable discrete set.

We now explore a related issue, and show that given a closed $\mathcal{A}_d$-totally null subset $K \subset \mathbb{S}_d$ (which is a zero set by Proposition 5.1), we can find a countably infinite set $\Lambda \subset \mathbb{B}_d$ such that $\Lambda \cup K$ is also a zero set for $\mathcal{A}_d$. For this purpose we will use the notion of an *interpolating sequence.*

Recall that $\Lambda = \{z_n\}_n \subset \mathbb{B}_d$ is an interpolating sequence for $\mathcal{M}_d$ if the restriction map $\rho_\Lambda : \mathcal{M}_d \to \ell^\infty$ defined as

$$\rho_\Lambda(\varphi) = \varphi|_\Lambda = \{\varphi(z_n)\}_n \quad \text{for} \quad \varphi \in \mathcal{M}_d$$

is surjective. This map is obviously contractive and the norm of the inverse of the induced isomorphism of $\mathcal{M}_d/\ker(\rho_\Lambda)$ onto ℓ^∞ is called the *interpolation constant.* Except in the single variable case of the unit disc, there is no known characterization of interpolating sequences for $\mathcal{M}_d$. On the other hand, such sequences are known to be plentiful. For example, [7, Proposition 9.1] shows that any sequence in $\mathbb{B}_d$ converging to the boundary contains an interpolating subsequence.

We first establish a result which is of independent interest.

Theorem 5.10. *Let $\Lambda = \{z_n\}_n \subset \mathbb{B}_d$ be an interpolating sequence for $\mathcal{M}_d$ with interpolation constant $\gamma > 0$. Let $K \subset \mathbb{S}_d$ be a closed $\mathcal{A}_d$-totally null subset containing $\overline{\Lambda} \cap \mathbb{S}_d$. Then the restriction map $\rho : \mathcal{A}_d \to \mathrm{C}(K \cup \Lambda)$ is surjective.*

Proof. We first show that ρ takes the ball of radius γ in $\mathcal{I}(K)$ onto a dense subset of the unit ball of

$$\{f \in \mathrm{C}(K \cup \Lambda) : f|_K = 0\}.$$

Note that since $\overline{\Lambda} \cap \mathbb{S}_d \subset K$, we can identify elements of that space with sequences in $c_0(\Lambda)$. Fix $\varepsilon > 0$, $N \geq 1$ and $\mathbf{a} = \{a_n\}_n \in c_0(\Lambda)$ such that $\|\mathbf{a}\| \leq 1$ and $a_n = 0$ for $n > N$. Since Λ is an interpolating sequence, there is $\psi \in \mathcal{M}_d$ with $\|\psi\| \leq \gamma$ such that $\psi|_\Lambda = \mathbf{a}$.

Because evaluation at a point in the open unit ball is a weak-$*$ continuous functional on $\mathcal{M}_d$, we may use Corollary 3.6 to find $\varphi_1 \in \mathcal{I}(K)$ such that $\|\varphi_1\| \leq \gamma$ and

$$|\varphi_1(z_n) - a_n| = |\varphi_1(z_n) - \psi(z_n)| < \varepsilon/2 \quad \text{for} \quad 1 \leq n \leq N.$$

Put $N_1 = N$. Note that since $\varphi_1 \in \mathcal{I}(K)$ and $\overline{\Lambda} \cap \mathbb{S}_d \subset K$, we must have

$$\lim_{n\to\infty} \varphi_1(z_n) = 0.$$

In particular, there is $N_2 > N_1$ such that

$$|\varphi_1(z_n) - a_n| = |\varphi_1(z_n)| < \varepsilon/2 \quad \text{for} \quad n > N_2.$$

Proceeding recursively, we obtain a sequence of functions $\{\varphi_k\}_k \subset \mathcal{I}(K)$ and a strictly increasing sequence of positive integers $\{N_k\}_k$ such that $\|\varphi_k\| \leq \gamma$ and

$$|\varphi_k(z_n) - a_n| < \varepsilon/2 \quad \text{for} \quad 1 \leq n \leq N_k \quad \text{and} \quad n > N_{k+1}$$

for each $k \geq 1$.

Now select a positive integer M so large that $2\gamma < M\varepsilon$. Define

$$\varphi = \frac{1}{M}\sum_{k=1}^{M} \varphi_k.$$

It is clear that $\varphi \in \mathcal{I}(K)$ and $\|\varphi\| \leq \gamma$. Moreover

$$|\varphi(z_n) - a_n| < \frac{\varepsilon}{2} \quad \text{for} \quad 1 \leq n \leq N \quad \text{and} \quad n > N_{M+1}$$

because this is valid for each $\varphi_k, 1 \leq k \leq M$. We claim that this inequality also holds for $N < n \leq N_{M+1}$. Indeed, in that case there is $1 \leq j \leq M$ such that $N_j < n \leq N_{j+1}$. Note that if $k \leq j-1$ then $n > N_{k+1}$ so that

$$|\varphi_k(z_n)| = |\varphi_k(z_n) - a_n| < \varepsilon/2$$

while if $k \geq j+1$ then $n \leq N_k$ and

$$|\varphi_k(z_n)| = |\varphi_k(z_n) - a_n| < \varepsilon/2$$

as well. Therefore we can estimate

$$|\varphi(z_n)| \leq \frac{1}{M}\sum_{k \neq j} |\varphi_k(z_n)| + \frac{|\varphi_j(z_n)|}{M} < \frac{\varepsilon}{2} + \frac{\gamma}{M} < \varepsilon \quad \text{for} \quad N < n < N_{M+1}.$$

Hence $\|\rho(\varphi) - \mathbf{a}\| < \varepsilon$. Since ε, N and $\mathbf{a}$ were arbitrary, this establishes our claim that the image under ρ of the ball of radius γ in $\mathcal{I}(K)$ is dense in the unit ball of

$$\{f \in \mathrm{C}(K \cup \Lambda) : f|_K = 0\}.$$

In turn, arguing as in [5, Lemma 5.9] we see that the image under ρ of the open ball of radius γ in $\mathcal{I}(K)$ contains the open unit ball of the space above. Moreover, by Theorem 2.1 (or the more elementary result [5, Proposition 5.10]), the restriction of the open unit ball of $\mathcal{A}_d$ to K coincides with the open unit ball of $\mathrm{C}(K)$. Fix now $f \in \mathrm{C}(K \cup \Lambda)$ with $\|f\| < 1$. Pick $\theta \in \mathcal{A}_d$ such that $\|\theta\| < 1$ and $\theta|_K = f|_K$. Then $g = f - \theta|_{K\cup\Lambda} \in \mathrm{C}(K \cup \Lambda)$ vanishes on K and $\|g\| < 2$. Choose $\varphi \in \mathcal{I}(K)$ with $\|\varphi\| < 2\gamma$ so that $\varphi|_{K\cup\Lambda} = g$. Evidently we have $\rho(\theta + \varphi) = f$ and we conclude that ρ is surjective. ∎

Since $\ker \rho = \mathcal{I}(K\cup\Lambda)$, one consequence of this result is that $\mathcal{A}_d/\mathcal{I}(K\cup\Lambda)$ is isomorphic to $\mathrm{C}(K \cup \Lambda)$. The proof actually shows that the inverse of the isomorphism has norm at most $2\gamma+1$. We also single out a nice consequence for zero sets.

Corollary 5.11. *If $\Lambda \subset \mathbb{B}_d$ is an interpolating sequence for $\mathcal{M}_d$ and $K \subset \mathbb{S}_d$ is a closed $\mathcal{A}_d$-totally null subset containing $\overline{\Lambda} \cap \mathbb{S}_d$, then $K \cup \Lambda$ is a zero set for $\mathcal{A}_d$.*

Proof. By Lemma 5.6, the character space of $\mathcal{A}_d/\mathcal{I}(K\cup\Lambda)$ is naturally identified with $Z(\mathcal{I}(K\cup\Lambda))$. On the other hand, we have that $\mathcal{A}_d/\mathcal{I}(K\cup\Lambda)$ and $\mathrm{C}(K\cup\Lambda)$ are isomorphic as Banach algebras via the restriction map by virtue of Theorem 5.10. It is easily verified that this implies that

$$Z(\mathcal{I}(K\cup\Lambda)) = K\cup\Lambda. \qquad \blacksquare$$

Finally, let us go back to the problem of describing the smallest zero set for $\mathcal{A}_d$ containing an arbitrary subset $X\subset\mathbb{S}_d$. It appears to be much more difficult to obtain a precise picture in the general case. One basic fact we can prove in general is the following.

Proposition 5.12. *Let $X\subset\overline{\mathbb{B}_d}$. Then, X intersects every connected component of $Z(\mathcal{I}(X))$.*

Proof. By Lemma 5.6, we see that the character space of $\mathcal{A}_d/\mathcal{I}(X)$ is $Z(\mathcal{I}(X))$ and that the Gelfand transform is simply the restriction of a function in $\mathcal{A}_d$ to that set. Assume that there is a connected component U of $Z(\mathcal{I}(X))$ which is disjoint from X. By the Shilov idempotent theorem, there is a $\varphi\in\mathcal{A}_d$ whose restriction to $Z(\mathcal{I}(X))$ coincides with the characteristic function of U. In particular, we see that $\varphi(z)=0$ for every $z\in X$ while U is disjoint from $Z(\varphi)$. We conclude that $\varphi\in\mathcal{I}(X)$, whence

$$X\subset Z(\varphi)\cap Z(\mathcal{I}(X))\subset Z(\mathcal{I}(X))\setminus U.$$

This is absurd since $Z(\mathcal{I}(X))$ is the smallest zero set containing X. $\blacksquare$

We finish with some (mostly) unsupported speculation concerning the description of $Z(\mathcal{I}(X))$ for an arbitrary set $X\subset\mathbb{S}_d$. Let Y be the closure of the set consisting of X together with all points $\lambda\in\mathbb{B}_d$ having a positive representing measure μ such that $\mu(X)>0$. Must we have $Y=Z(\mathcal{I}(X))$? It is not difficult to verify that a positive answer to this question in the case of $\mathrm{A}(\mathbb{B}_d)$ would yield a positive solution to Rudin's problem about the zero sets of ideals which are contained in the sphere (Question 5.3).

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