A 3×3 DILATION COUNTEREXAMPLE

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Dedicated to the memory of William B. Arveson

Abstract. We define four 3×3 commuting contractions which do not dilate to commuting isometries. However they do satisfy the scalar von Neumann inequality. These matrices are all nilpotent of order 2. We also show that any three 3×3 commuting contractions which are scalar plus nilpotent of order 2 do dilate to commuting isometries.

1. Introduction

Seminal work of Sz.Nagy [**8**] showed that every contraction A has a coextension to an isometry of the form

$$S = \begin{bmatrix} A & 0 \\ * & * \end{bmatrix}.$$

This provides a simple proof of von Neumann's inequality:

$$\|p(A)\| \le \|p\|_\infty := \sup_{|z|\le 1} |p(z)|$$

for all polynomials. Indeed, this remains valid for matrices of polynomials

$$\big\| \big[p_{ij}(A)\big] \big\| \le \big\| \big[p_{ij}\big] \big\| := \sup_{|z|\le 1} \big\| \big[p_{ij}(z)\big] \big\|.$$

A decade later, Ando [**1**] showed that two commuting contractions have simultaneous coextensions to a common Hilbert space which are commuting isometries. This yields the 2-variable matrix von Neumann inequality.

However Varopoulos [**10**] showed that there exist three commuting contractions which do not satisfy von Neumann's inequality, and therefore do not have a simultaneous coextension to three commuting isometries. In the appendix, he and Kaijser provide an example with 5×5 matrices. A related example of Parrott [**6**] provides three commuting contractions which do satisfy the scalar von Neumann inequality but

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fail the matrix version. Thus they also do not dilate. A dilation theorem of Arveson [**2**] shows that there is a simultaneous coextension to commuting isometries if and only if the matrix von Neumann inequality holds. A nice treatment of this material is contained in Paulsen [**7**], and the earlier material is contained in the classic text [**9**].

Holbrook [**4**] found three 4×4 matrices which are commuting contractions yet do not coextend to commuting isometries. He also showed [**5**] that for 2×2 matrices, arbitrary commuting families of commuting contractions do have commuting isometric coextensions. See also [**3**]. Holbrook asks what the situation is for 3×3 matrices.

Sometimes these results are stated instead for unitary (power) dilations of the form

$$U_i = \begin{bmatrix} * & 0 & 0 \\ * & A_i & 0 \\ * & * & * \end{bmatrix}$$

on a Hilbert space $\mathcal{K} = \mathcal{K}_- \oplus \mathcal{H} \oplus \mathcal{K}_+$. If these unitaries commute, then the restriction of U_i to $\mathcal{H} \oplus \mathcal{K}_+$, namely the lower 2×2 corner, yields commuting isometric coextensions S_i; (and the compression to $\mathcal{K}_- \oplus \mathcal{H}$ yields commuting coisometric extensions of the A_i).

Conversely, suppose that commuting contractions A_i have coextensions to commuting isometries S_i. An old result of Ito and Brehmer [**9**, Proposition I.6.2] shows that the S_i dilate to commuting unitaries U_i so that

$$P_{\mathcal{H}} U_1^{k_1} \cdots U_s^{k_s}|_{\mathcal{H}} = A_1^{k_1} \cdots A_s^{k_s} \quad \text{for } k_i \geq 0.$$

It follows that they simultaneously have a triangular form as given above. Therefore these two formulations are equivalent.

In this note, we provide an example of four commuting contractions in the 3×3 matrices $\mathfrak{M}_3$ which cannot be coextended to commuting isometries. The question of whether there are three commuting 3×3 contractive matrices which cannot be coextended to commuting isometries remains open.

Our examples are nilpotents of order 2. We show that any finite family of commuting contractions of this form always satisfy the scalar von Neumann's inequality. It would be of interest to know whether the scalar von Neumann inequality holds for commuting 3×3 contractions. For our counterexample, we exhibit a specific matrix polynomial which shows that the matrix valued von Neumann inequality fails.

We also show that any three commuting 3×3 contractions which are of the form scalar plus nilpotent of order 2 always do have commuting isometric coextensions.

When the second author discovered these examples, he found out that the first author and Y. Zhong had a similar example from many years ago which was never published. Zhong has left academia, and could not be contacted—but he shares in the credit for this work.

2. The Example

Let $\mathcal{H} = \mathbb{C}^3$ have an orthonormal basis f, e_1, e_2. For $i = 1, 2$, pick $\theta_i \in (0, \pi/2)$ and set $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$. Define

$$A_1 = e_1 f^* = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_2 = e_2 f^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$A_3 = (c_1 e_1 + s_1 e_2) f^* = \begin{bmatrix} 0 & 0 & 0 \\ c_1 & 0 & 0 \\ s_1 & 0 & 0 \end{bmatrix} \qquad A_4 = (c_2 e_1 + i s_2 e_2) f^* = \begin{bmatrix} 0 & 0 & 0 \\ c_2 & 0 & 0 \\ i s_2 & 0 & 0 \end{bmatrix}.$$

Observe that $A_i A_j = 0$ for all $1 \le i, j \le 4$.

Theorem 2.1. *The matrices A_1, A_2, A_3 and A_4 do not have simultaneous coextensions to commuting isometries.*

Proof. Suppose that the A_i coextend to commuting isometries

$$S_i = \begin{bmatrix} A_i & 0 \\ D_i & V_i \end{bmatrix} \quad \text{for } i = 1, 2, 3, 4$$

on a Hilbert space $\mathcal{K}$. We may assume that $\mathcal{K}$ is the smallest invariant subspace for the S_i containing $\mathcal{H}$. That is, if we write $S^k = S_1^{k_1} S_2^{k_2} S_3^{k_3} S_4^{k_4}$ for $k = (k_1, k_2, k_3, k_4) \in \mathbb{N}^4$, then

$$\mathcal{K} = \bigvee_{k \in \mathbb{N}^4} S^k \mathcal{H}.$$

Observe that since $\|A_i f\| = 1$, we have

$$S_i f = A_i f \quad \text{for } i = 1, 2, 3, 4.$$

Hence

$$(c_1 S_1 + s_1 S_2 - S_3) f = 0.$$

Therefore

$$0 = S^k (c_1 S_1 + s_1 S_2 - S_3) f = (c_1 S_1 + s_1 S_2 - S_3) S^k f.$$

Since $S_i f = e_i$ for $i = 1, 2$,

$$\ker(c_1 S_1 + s_1 S_2 - S_3) \supset \bigvee_{k \in \mathbb{N}^4} S^k \operatorname{span}\{f, S_1 f, S_2 f\} = \mathcal{K}.$$

So

$$S_3 = c_1 S_1 + s_1 S_2.$$

Similarly,

$$S_4 = c_2 S_1 + i s_2 S_2.$$

Since S_3 is an isometry,

$$\begin{aligned} I &= S_3^* S_3 \\ &= c_1^2 S_1^* S_1 + s_1^2 S_2^* S_2 + c_1 s_1 (S_2^* S_1 + S_1^* S_2) \\ &= I + c_1 s_1 (S_2^* S_1 + S_1^* S_2). \end{aligned}$$

It follows that

$$S_2^* S_1 + S_1^* S_2 = 0.$$

Likewise, using the fact that S_4 is an isometry,

$$0 = (iS_2)^* S_1 + S_1^*(iS_2) = -i(S_2^* S_1 - S_1^* S_2).$$

Therefore

$$S_2^* S_1 = 0.$$

This implies that S_1 and S_2 have pairwise orthogonal ranges, and therefore do not commute. This contradiction establishes the result. ∎

Remark 2.2. It is possible to coextend any three of these operators to commuting isometries. This will follow from Theorem 4.3. Indeed, the construction given there will simultaneously coextend A_1, A_2 and all matrices $A_j = (c_j e_1 + s_j e_2) f^*$ such that $c_j \bar{s}_j$ have a common argument. So this example could not be given using real matrices.

Remark 2.3. It is not even possible to find commuting isometries of the form

$$S_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}.$$

As in our proof above, we may suppose that $\mathcal{H}$ is a cyclic subspace. Exactly the same argument shows that

$$S_3 = c_1 S_1 + s_1 S_2 \quad \text{and} \quad S_4 = c_2 S_1 + i s_2 S_2.$$

Hence $S_2^* S_1 = 0$, which contradicts commutativity.

3. Von Neumann's Inequality

It is also of interest to find commuting contractions in $\mathfrak{M}_3$ that fail the scalar von Neumann inequality. Apparently computer searches for such examples have been unsuccessful. We show that our example does satisfy the scalar von Neumann inequality. So it does not settle this question. We provide an explicit matrix polynomial which fails the matrix von Neumann inequality.

Lemma 3.1. *If $A_1, \dots, A_n \in \mathfrak{M}_3$ are commuting nilpotents of order 2, then there is a vector f and vectors $v_i \in (\mathbb{C}f)^\perp$ so that either $A_i = v_i f^*$ for $1 \le i \le n$ or $A_i = f v_i^*$ for $1 \le i \le n$.*

Proof. Observe that a nilpotent $A \in \mathfrak{M}_3$ of order 2 must have rank one. Thus it can be expressed as $A = vf^*$ where f is a unit vector. And since

$$0 = A^2 = \langle v, f\rangle A,$$

we have that $\langle v, f\rangle = 0$. If two such non-zero operators $A_i = v_i f_i^*$ commute, then

$$\langle v_2, f_1\rangle v_1 f_2^* = (v_1 f_1^*)(v_2 f_2^*) = (v_2 f_2^*)(v_1 f_1^*) = \langle v_1, f_2\rangle v_2 f_1^*.$$

So either

$$\langle v_2, f_1\rangle = \langle v_1, f_2\rangle = 0$$

or $v_2 f_2^*$ is a multiple of $v_1 f_1^*$, in which case this identity remains true. So

$$\operatorname{span}\{v_1, v_2\} \perp \operatorname{span}\{f_1, f_2\}.$$

Furthermore, if n non-zero operators $v_i f_i^*$ commute, then the pairwise relations yield

$$\operatorname{span}\{v_1, \dots, v_n\} \perp \operatorname{span}\{f_1, \dots, f_n\}.$$

Therefore one of these subspaces is 1-dimensional. By taking adjoints if necessary, we may suppose that $\operatorname{span}\{f_1, \dots, f_n\}$ is one dimensional. So after a scalar change, we have $A_i = v_i f^*$ for $i = 1, \dots, n$; and each v_i belongs to $(\mathbb{C}f)^\perp$. ∎

Proposition 3.2. *Any finite number of commuting contractions A_1, $\dots$, A_n in $\mathfrak{M}_3$ of the form scalar plus order 2 nilpotent satisfy the scalar von Neumann inequality.*

Proof. We first suppose that each A_i is a nilpotent of order 2. By Lemma 3.1, we may suppose that there is a unit vector f and vectors $v_i \in (\mathbb{C}f)^\perp$ of norm at most 1 so that $A_i = v_i f^*$. Since $A_i A_j = 0$ for all $1 \le i, j \le n$, we need concern ourselves only with the linear part of a polynomial.

Consider a polynomial

$$p(z) = c + \sum_{i=1}^{n} a_i z_i + q(z)$$

where $q(z)$ consists of higher order terms. Without loss of generality, we may suppose that some $a_i \neq 0$. Let λ_i be scalars of modulus 1 so that $a_i = \lambda_i |a_i|$. Set

$$B = \Big(\sum_{i=1}^{n} |a_i| \Big)^{-1} \sum_{i=1}^{n} a_i A_i.$$

Clearly $\|B\| \leq 1$. Then

$$\begin{aligned} p(A_1, \dots, A_n) &= cI + \sum_{i=1}^{n} a_i A_i = cI + \sum_{i=1}^{n} |a_i| B \\ &= p(\overline{\lambda}_1 B, \dots, \overline{\lambda}_n B) = q(B), \end{aligned}$$

where $q(x) = p(\overline{\lambda}_1 x, \dots, \overline{\lambda}_n x)$. Hence by von Neumann's inequality,

$$\|p(A_1, \dots, A_n)\| = \|q(B)\| \leq \|q\|_\infty \leq \|p\|_\infty.$$

Now suppose that $A_i = \lambda_i I + N_i$ where $N_i^2 = 0$. If $N_i \neq 0$, the fact that $\|A_i\| \leq 1$ implies that $|\lambda_i| < 1$. If some $A_i = \lambda_i I$ with $|\lambda_i| = 1$, replace it with $A_i' = (1 - \varepsilon)\lambda_i I$ for $\varepsilon > 0$ small. If we establish the inequality for this new n-tuple, we recover the case we desire by letting ε tend to 0.

Define Möbius maps

$$b_i(z) = \frac{z - \lambda_i}{1 - \overline{\lambda}_i z} \quad \text{for } 1 \leq i \leq n.$$

Then

$$B_i := b_i(A_i) = (1 - |\lambda_i|^2)^{-1} N_i$$

are commuting nilpotents of order 2, and $A_i = b_i^{-1}(B_i)$. Moreover, B_i are contractions by the one variable von Neumann inequality (or by direct computation). If p is a polynomial, then

$$q(z) = p(b_1^{-1}(z_1), \dots, b_n^{-1}(z_n))$$

is a function in the polydisc algebra $\mathrm{A}(\mathbb{D}^n)$ of the same norm as p because each b_i^{-1} takes the circle $\mathbb{T}$ onto itself. Hence

$$\|p(A_1, \dots, A_n)\| = \|q(B_1, \dots, B_n)\| \leq \|q\|_\infty = \|p\|_\infty.$$

Therefore the scalar von Neumann inequality is satisfied. ∎

If the matrix von Neumann inequality holds for a 4-tuple of matrices $A_1,\dots,A_4$, then the canonical map from $\mathrm{A}(\mathbb{D}^4)$ into $\mathcal{A} = \mathrm{Alg}(\{A_i\})$ taking z_i to A_i for $1 \le i \le 4$ is completely contractive. Therefore Arveson's Dilation Theorem [**2**] shows that the 4-tuple does dilate to commuting unitaries. Hence the restriction to the common invariant subspace generated by the original space $\mathcal{H}$ yields a coextension to commuting isometries. Hence there is a matrix polynomial which shows that this inequality fails for our example in Theorem 2.1. We will exhibit one.

Recall the notation of Theorem 2.1. Apply the Gram-Schmidt process to the vectors u_1 and u_2 to get orthogonal unit vectors f_1 and f_2, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ c_1 \\ c_2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ s_1 \\ is_2 \end{bmatrix}, \quad f_1 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}, \quad \text{and} \quad f_2 = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}.$$

Proposition 3.3. *Let*

$$p(z) = \begin{bmatrix} \bar\alpha_1 z_1 + \bar\alpha_2 z_2 + \bar\alpha_3 z_3 + \bar\alpha_4 z_4 \\ \bar\beta_1 z_1 + \bar\beta_2 z_2 + \bar\beta_3 z_3 + \bar\beta_4 z_4 \end{bmatrix}.$$

Then

$$\|p\|_\infty = \sup_{|z_i|=1} \|p(z)\| < 2 = \|p(A_1, A_2, A_3, A_4)\|.$$

So the matrix von Neumann inequality fails for (A_1, A_2, A_3, A_4).

Proof. Let $z = (z_1, z_2, z_3, z_4)^t$ with $|z_i| = 1$. Since f_1 and f_2 are orthonormal,

$$\|p(z)\|^2 = |\langle z, f_1\rangle|^2 + |\langle z, f_2\rangle|^2 \le \|z\|^2 = 2.$$

This inequality is strict unless $z \in \mathrm{span}\{f_1, f_2\} = \mathrm{span}\{u_1, u_2\}$. By compactness, the norm of p is exactly 2 only if this value is obtained. However we claim that no z with $|z_i| = 1$ lies in this subspace. Indeed, suppose that

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ c_1 \\ c_2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ s_1 \\ is_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ ac_1 + bs_1 \\ ac_2 + ibs_2 \end{bmatrix}$$

We require

$$1 = |a| = |b| = |ac_1 + bs_1| = |ac_2 + ibs_2|.$$

Arguing as in the proof of Theorem 2.1, we find that $\mathrm{Re}(\bar a b) = 0 = \mathrm{Re}(i\bar a b)$. Thus $\bar a b = 0$, contradicting $|a| = |b| = 1$. So $\|p\| < 2$.

Now we compute $\|p(A_1, A_2, A_3, A_4)\|$. Clearly it suffices to consider the $2,1$ and $3,1$ entries. That is

$$p\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}c_1\\s_1\end{bmatrix}, \begin{bmatrix}c_2\\is_2\end{bmatrix} \right) = \begin{bmatrix} \bar\alpha_1 + \bar\alpha_3 c_1 + \bar\alpha_4 c_2 \\ \bar\alpha_2 + \bar\alpha_3 s_1 + \bar\alpha_4 is_2 \\ \bar\beta_1 + \bar\beta_3 c_1 + \bar\beta_4 c_2 \\ \bar\beta_2 + \bar\beta_3 s_1 + \bar\beta_4 is_2 \end{bmatrix} = \begin{bmatrix} \langle u_1, f_1\rangle \\ \langle u_2, f_1\rangle \\ \langle u_1, f_2\rangle \\ \langle u_2, f_2\rangle \end{bmatrix}$$

Therefore

$$\|p(A_1, A_2, A_3, A_4)\|^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} |\langle u_i, f_j\rangle|^2 = \sum_{i=1}^{2} \|u_i\|^2 = 4. \qquad \blacksquare$$

4. Dilating Three 3×3 Nilpotents of Order 2

The purpose of this section is to prove a positive dilation result for certain triples of 3×3 commuting contractions. First we need a couple of lemmas.

Lemma 4.1. *Suppose that three commuting contractions A_1, A_2 and A_3 have coextensions S_1, S_2 and S_3 which are commuting isometries. Then if $|a_i| \le 1$ for $i = 1, 2, 3$, the operators $a_i A_i$ also have coextensions which are commuting isometries.*

Proof. Observe that $a_i A_i$ coextend to commuting contractions $a_i S_i$. If $|a_1| < 1$, let $d_i = (1 - |a_i|^2)^{1/2}$ and coextend $a_1 S_1$ to

$$T_1 = \begin{bmatrix} a_1 S_1 & 0 & 0 & 0 & \dots \\ d_1 S_1 & 0 & 0 & 0 & \dots \\ 0 & S_1 & 0 & 0 & \dots \\ 0 & 0 & S_1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

and simultaneously for $i = 2, 3$, coextend $a_i S_i$ to

$$a_i T_i = a_i S_i \otimes I = \begin{bmatrix} a_1 S_i & 0 & 0 & 0 & \dots \\ 0 & a_1 S_i & 0 & 0 & \dots \\ 0 & 0 & a_1 S_i & 0 & \dots \\ 0 & 0 & 0 & a_1 S_i & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

It is easy to see that these contractions commute, and T_i are isometries.

Now dilate $a_2 T_2$ to an isometry, and dilate T_1 and T_3 to commuting isometries in the same manner. Finally, repeat a third time to dilate the third term to an isometry. $\blacksquare$

Lemma 4.2. *Given three unit vectors v_1, v_2, v_3 in $\mathbb{C}^2$, there exist three commuting unitaries U_1, U_2, U_3 in $\mathfrak{M}_2$ such that*

$$U_i v_j = U_j v_i \quad \textit{for } 1 \le i, j \le 3.$$

Proof. We may choose an orthonormal basis for $\mathbb{C}^2$ in which $v_1 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. We set $U_1 = I_2$. If $v_2 = e^{i\theta} v_1$, let $U_2 = e^{i\theta} I_2$ and choose U_3 to be any unitary matrix such that $U_3 v_1 = v_3$. This works.

Otherwise, v_1 and v_2 are linearly independent. Write $v_3 = a v_1 + b v_2$. For convenience, we may multiply v_3 and v_2 by scalars of modulus 1 so that a and b are real. This does not affect the problem. Write

$$v_2 = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} \alpha_3 \\ \beta_3 \end{bmatrix}, \quad U_2 = \begin{bmatrix} \bar\beta_2 & \alpha_2 \\ -\bar\alpha_2 & \beta_2 \end{bmatrix} \text{ and } U_3 = \begin{bmatrix} \bar\beta_3 & \alpha_3 \\ -\bar\alpha_3 & \beta_3 \end{bmatrix}.$$

A simple calculation shows that

$$aU_1 + bU_2 = \begin{bmatrix} a + b\bar\beta_2 & b\alpha_2 \\ -b\bar\alpha_2 & a + b\beta_2 \end{bmatrix} = \begin{bmatrix} \bar\beta_3 & \alpha_3 \\ -\bar\alpha_3 & \beta_3 \end{bmatrix} = U_3.$$

This shows that the unitary matrices commute. By construction,

$$U_i v_1 = v_i = U_1 v_i \quad \text{for } i = 2, 3.$$

Finally observe that

$$U_3 v_2 = (aU_1 + bU_2) v_2 = a v_2 + b U_2 v_2 = U_2 (a v_1 + b v_2) = U_2 v_3. \qquad \blacksquare$$

Theorem 4.3. *Suppose that A_1, A_2 and A_3 are three commuting 3×3 matrix contractions which are all of the form scalar plus nilpotent of order 2. Then there exist commuting isometric coextensions S_1, S_2 and S_3 of A_1, A_2 and A_3.*

Proof. First assume that the A_i are nilpotent of the form $A_i = v_i f^*$ for $i = 1, 2, 3$, where the v_i are *unit vectors* in $(\mathbb{C}f)^\perp \simeq \mathbb{C}^2$. Choose a basis for $\mathbb{C}^2$ so that $v_1 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Let U_i be the commuting 2×2 unitaries given by Lemma 4.2. Then the construction yields the identities $U_i v_1 = v_i$ for $i = 1, 2, 3$. So the second column of each U_i is v_i, say $U_i = \begin{bmatrix} \gamma_i & \alpha_i \\ \delta_i & \beta_i \end{bmatrix}$.

Observe that if U is the bilateral shift on ℓ^2, then $W_i = U \otimes U_i$ are commuting unitaries of the form:

$$W_i = \left[\begin{array}{ccc|cc|ccc} \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ \ddots & O & O & O & O & O & O & \dots \\ \ddots & U_i & O & O & O & O & O & \dots \\ \hline \dots & O & U_i & O & O & O & O & \dots \\ \dots & O & O & U_i & O & O & O & \dots \\ \hline \dots & O & O & O & U_i & O & O & \ddots \\ \dots & O & O & O & O & U_i & O & \ddots \\ & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{array}\right]$$

on $\mathcal{K} = \mathcal{K}_- \oplus \mathbb{C}^4 \oplus \mathcal{K}_+$. Here O is a 2×2 zero matrix. Decompose the central 4×4 block of W_i as $\mathbb{C} \oplus \mathcal{H}$:

$$\begin{bmatrix} O & O \\ U_i & O \end{bmatrix} = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \gamma_i & \alpha_i & 0 & 0 \\ \delta_i & \beta_i & 0 & 0 \end{array}\right] = \begin{bmatrix} 0 & 0_3^* \\ u_i & A_i \end{bmatrix}$$

The spaces $\mathcal{K}_+$ and $\mathcal{H} \oplus \mathcal{K}_+$ are invariant for each W_i. Therefore W_i are commuting unitary (power) dilations of the A_i. Note that W_i^* then form commuting unitary dilations of the adjoints, A_i^*; and $\mathcal{H}$ is the difference of invariant subspaces $\mathcal{K}_- \oplus \mathbb{C}^4$ and $\mathcal{K}_- \oplus \mathbb{C}$.

By Lemma 3.1, if A_i are commuting 3×3 nilpotents of order 2, then either they have the form $A_i = v_i f^*$ or their adjoints do. By Lemma 4.1, it suffices to dilate the normalized matrices $\tilde{A}_i := A_i/\|A_i\|$. Hence we can assume that each $\|v_i\| = 1$. The argument above produces commuting unitary dilations. The restriction of these unitaries to the smallest common invariant subspace containing the original 3-dimensional space yields commuting isometric coextensions of the $\tilde{A}_i$.

We reduce the general case to the nilpotent case as in the previous section. Suppose that $A_i = \lambda_i I + N_i$ where $N_i^2 = 0$. Then

$$|\lambda_i| \le \|A_i\| \le 1.$$

Moreover, if $|\lambda_i| = 1$, then $A_i = \lambda_i I$ is already an isometry, and we coextend it to $\lambda_i I$ on the larger space. When $|\lambda_i| < 1$, define the Möbius map

$$b_i(z) = \frac{z - \lambda_i}{1 - \overline{\lambda_i} z}.$$

Then $b_i(A_i)$ are commuting nilpotents of order 2. Dilate them to commuting unitaries W_i as above. Then define $U_i = b_i^{-1}(W_i)$. These are commuting unitaries dilating A_i. ∎

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