POSITIVE MAPS AND ENTANGLEMENT IN REAL HILBERT SPACES

GIULIO CHIRIBELLA, KENNETH R. DAVIDSON, VERN I. PAULSEN, AND MIZANUR RAHAMAN

Abstract. The theory of positive maps plays a central role in operator algebras and functional analysis, and has countless applications in quantum information science. The theory was originally developed for operators acting on complex Hilbert spaces, and little is known about its variant on real Hilbert spaces. In this article we study positive maps acting on a full matrix algebra over the reals, pointing out a number of fundamental differences with the complex case and discussing their implications in quantum information.

We provide a necessary and sufficient condition for a real map to admit a positive complexification, and connect the existence of positive maps with non-positive complexification with the existence of mixed states that are entangled in real Hilbert space quantum mechanics, but separable in the complex version, providing explicit examples both for the maps and for the states. Finally, we discuss entanglement breaking and PPT maps, and we show that a straightforward real version of the PPT-squared conjecture is false even in dimension 2. Nevertheless, we show that the original PPT-squared conjecture implies a different conjecture for real maps, in which the PPT property is replaced by a stronger property of invariance under partial transposition (IPT). When the IPT property is assumed, we prove an asymptotic version of the conjecture.

1. Introduction

Positive and completely positive maps play a central role in the theory of operator algebras and in the related fields of dilation theory, random matrix theory, free probability, semidefinite optimization theory and quantum information [4, 16, 36]. In quantum theory, completely positive maps are ubiquitous as they represent general evolutions of quantum states, while positive maps are a powerful tool for characterizing quantum entanglement [17, 18, 26, 27, 39, 42, 48, 49, 55, 56] (see [29] for a review).

Much of the theory of positive maps was originally developed in the case where the underlying number field is the complex numbers. In contrast, the case of operator algebras over the reals has not been looked at as extensively. Notable exceptions are the works of Ruan [45, 46], and, very recently, Blecher and Tepsan [6]. The real variant quantum theory was considered in an early work by Stückelberg [53], and was later found to differ from standard complex quantum theory in a number of essential features. Araki [2] observed that, while the state space of composite systems in complex quantum theory is equal to the product of the dimension of the components, in real quantum theory it is strictly larger. Dually, this fact implies that product observables are not sufficient to identify general entangled states [23, 59]. Explicitly, Wootters [59] provided an example of two mixed states that have orthogonal support, and yet give rise to the same correlations for all possible local observables. Building on this example, it was later shown that there exist evolutions in real quantum theory that are indistinguishable when acting on a single system, but can be distinguished perfectly when applied locally on an entangled state [10]. These phenomena can be summarized by the statement: *real quantum theory violates the Local Tomography Axiom* (see [12] for a review), often assumed in axiomatic reconstructions of standard finite-dimensional quantum theory (see e.g. [11, 21, 22, 24, 25]). Further differences between real and complex quantum theory were observed in [1, 13, 60, 61]. Very recently, an experimental test that distinguishes between complex quantum theory and real quantum theory was proposed [43].

In this paper we develop the theory of positive maps acting on a full matrix algebra over the reals, and discuss their applications to the study of entanglement in real quantum mechanics. We characterize the subset of real maps with a positive complexification, providing explicit examples of positive real maps outside this subset. See also our paper [14]. Then, we prove that the existence of positive real maps with non-positive complexification is equivalent to the existence of quantum states that are entangled in real quantum theory, but are separable when regarded as elements of the larger state space of complex quantum theory. These states, of which we also provide explicit examples, cannot violate Bell inequalities [5] (see [7] for a review), even if pre-processing operations are allowed [40] and if an arbitrarily large number of identically prepared systems is available [35, 37]. Operationally, the existence of real-entangled, but complex-separable states can be interpreted in a resource-theoretic framework [62], as the ability to prepare a larger class of states by performing a larger set of local operations that are not restricted to have real matrix elements in a given basis.

Finally, we characterize the real maps that break quantum entanglement, extending a classic characterization by Horodecki, Shor and Ruskai [28] to the real domain, and exploring the relation between entanglement breaking maps and another class of maps, known as PPT. In the complex case, Christandl conjectured (see [47]) that $\Phi \circ \Phi$ is entanglement breaking for every PPT map Φ and in [16] (also see [9]) it was shown that this conjecture (now known as the PPT-squared conjecture) holds in dimension 3. In stark contrast, here we show that the real version of the so called PPT-squared conjecture is false in even dimension 2. Nevertheless, we show that the original PPT-squared conjecture (stated in the complex case) implies another conjecture for real maps satisfying a suitable strengthening of the PPT property. When this stronger property is assumed, we prove that the real version of the PPT-squared conjecture holds asymptotically. Moreover, when assumed trace preserving and unital, we prove that finite number of iterations of PPT maps become real entanglement breaking.

The rest of the paper is structured as follows. In Section 2 we discuss the notion of positivity on real matrix algebras, pointing out the need to include a Hermiticity requirement in the definition of a positive map. In Section 3, we consider p-positive maps and characterize their Choi matrices. The results of this section are mostly adaptations of existing results in the complex case. In Section 4 we characterize the set of p-positive real maps that admit a p-positive complexification, and in Section 5 we provide concrete examples of p-positive real maps that violate this condition. We then move to the study of entanglement on real Hilbert spaces, characterizing entanglement witnesses and showing that the existence of positive real maps with non-positive complexification is equivalent to the existence of mixed states that are entangled in real quantum theory but separable in the complex version (Section 6). Examples of real-entangled but complex-separable quantum states are provided in Section 7. In Section 8 we consider the notion of entanglement p-breaking channels, which transform arbitrary entangled states into states with Schmidt rank at most p, and show an explicit example of a map that breaks complex-entanglement but preserves real-entanglement. Finally, we conclude with a discussion of the relation between entanglement breaking channels and PPT channels, proposing a real version of the PPT-squared conjecture and showing that its asymptotic version holds true (Section 9).

2. Positivity and Hermiticity

Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$. A matrix $P \in M_n(\mathbb{K})$ is called positive (denoted $P \geq 0$) if it is of the form $P = A^*A$ for some $A \in M_n(\mathbb{K})$, where A^* is the adjoint of A (for $\mathbb{K} = \mathbb{R}$, the adjoint A^* coincides with the transpose A^t). The set of all positive matrices in $M_n(\mathbb{K})$ will be denoted by $\mathrm{PSD}(\mathbb{K}^n)$.

A matrix is Hermitian if $A = A^*$, and we write $\mathrm{Herm}(\mathbb{K}^n)$ for the set of Hermitian matrices in $M_n(\mathbb{K})$. For a Hermitian matrix $A \in \mathrm{Herm}(\mathbb{K}^n)$, positivity is equivalent to the condition $\langle v|Av\rangle \geq 0$ for every $v \in \mathbb{K}^n$. Here, the product is defined as $\langle v|w\rangle = v^*w$, regarding the vectors $v, w \in \mathbb{K}^n$ as column vectors.

We say that a linear map $\Phi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$ is *positivity preserving* if $\Phi(P) \geq 0$ for all $P \geq 0$, and *Hermitian* if $\Phi(A^*) = \Phi(A)^*$ for all $A \in M_n(\mathbb{K})$. If the map Φ is positivity preserving and Hermitian, we call it *positive*.

In the complex field, preservation of positivity implies Hermiticity. This implication, however, does not hold on the real field. Counterexamples are abundant. For example, the map

$$\Phi : M_2(\mathbb{R}) \to M_1(\mathbb{R}), \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (a + d + b - c)/2$$

is positivity preserving but not Hermitian.

Let $\mathrm{Sym}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A^t = A\}$ be the space of all symmetric matrices, and let $\mathrm{Asym}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A^t = -A\}$. It is immediate to see that a map Φ is Hermitian if and only if

$$\Phi\left(\mathrm{Sym}_n(\mathbb{R})\right) \subseteq \mathrm{Sym}_m(\mathbb{R}) \quad \text{and} \quad \Phi\left(\mathrm{Asym}_n(\mathbb{R})\right) \subseteq \mathrm{Asym}_m(\mathbb{R}).$$

In special cases, Hermiticity follows from positivity preservation, plus additional properties, such as unitality and unit norm 1:

Proposition 2.1. *If $\Phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$ is positivity preserving, unital and norm 1, then Φ is Hermitian.*

Proof. Since Φ is positivity preserving, $\Phi\left(\mathrm{Sym}_n(\mathbb{R})\right) \subseteq \mathrm{Sym}_m(\mathbb{R})$. We now show that unitality and the norm-1 condition imply that $\Phi\left(\mathrm{Asym}_n(\mathbb{R})\right) \subseteq \mathrm{Asym}_m(\mathbb{R})$.

The proof is by contradiction. Let C be an antisymmetric matrix, with $\|C\| = 1$ without loss of generality. Suppose that $\Phi(C) = X$ is not antisymmetric. Then $B = \frac{1}{2}(X + X^t) \neq 0$. Since $B^t = B \neq 0$, the spectrum of B is real and not $\{0\}$. Let λ be an eigenvalue of B with $|\lambda| = \|B\|$.

Since C is antisymmetric and norm 1, we have $-I_n \leq C^2 \leq 0$ and -1 is an eigenvalue of C^2. For $t \in \mathbb{R}$,

$$\|I_n + tC\|^2 = \|I_n - t^2C^2\| = 1 + t^2.$$

Now choose the sign of t so that $|t\lambda|$ belongs to the spectrum of tB; then

$$\|\Phi(I_n + tC)\| = \|I_m + tX\| \geq 1 + |t\lambda|.$$

For sufficiently small t, this yields a contradiction with the norm 1 property of Φ. ∎

One may ask if Hermiticity follows from positivity preservation and from the property $\|\Phi\| = \|\Phi(I_n)\|$, without requiring unitality. The following counterexample answers the question in the negative.

Example 2.2. We provide an example of a positivity preserving map Φ on $M_2(\mathbb{R})$ that satisfies the condition $\|\Phi\| = \|\Phi(I_2)\|$ but is not Hermitian. Define

$$\Phi(A) = \Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a+d}{2} & 0 \\ 0 & \frac{b-c}{2} \end{pmatrix}.$$

The map Φ is positivity preserving, since $\Phi\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right) = \frac{a+d}{2}E_{11}$, for any positive matrix. Moreover, it satisfies

$$\|\Phi(A)\| = \max\left\{\left|\frac{a+d}{2}\right|, \left|\frac{b-c}{2}\right|\right\} \leq \|A\|$$

Hence, $\|\Phi\| = \|\Phi(I_2)\| = 1$. However, Φ maps antisymmetric matrices into symmetric ones, and therefore is not Hermitian.

We conclude with an elementary lemma that will be useful later.

Lemma 2.3. *A Hermitian map $\Phi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$ is positive if and only if $\langle w|\Phi(vv^*)w\rangle \geq 0$ for every pair of vectors $v \in \mathbb{K}^n$ and $w \in \mathbb{K}^m$, where vv^* is the rank-1 matrix with entries $(vv^*)_{ij} := v_i\overline{v}_j$, $\overline{v}_j$ denoting the complex conjugate of v_j.*

Proof. Since every positive matrix P is a sum of matrices of the form vv^*, Φ is positive if and only if $\Phi(vv^*)$ is positive for every $v \in \mathbb{K}^n$. Note that $\Phi(vv^*)$ is Hermitian, because Φ is a Hermitian. Hence, $\Phi(vv^*)$ is positive if and only if $\langle w|\Phi(vv^*)w\rangle$ for all $w \in \mathbb{K}^m$. ■

3. p-positive maps and their Choi representation

A linear map $\Phi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$ is called *p-positive* if the map $\Phi\otimes id_p : M_n(\mathbb{K}) \otimes M_p(\mathbb{K}) \to M_m(\mathbb{K}) \otimes M_p(\mathbb{K})$ is positive, where id_p is the identity map on $M_p(\mathbb{K})$. Positive maps correspond to the special case $p = 1$. A map is *completely positive* if it is p-positive for every $p \in \mathbb{N}$.

A linear map $\Phi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$ is conveniently represented in terms of its Choi matrix [15], defined as

$$C_\Phi := \sum_{i,j} E_{i,j} \otimes \Phi(E_{i,j})\,,$$

where $E_{i,j}$ are the standard matrix units of $M_n(\mathbb{K})$. A related representation is the Jamiołkowski matrix [31] $J_\Phi := \sum_{i,j} E_{i,j} \otimes \Phi(E_{j,i})$, originally introduced in the study of positive maps.

We will often use the following lemma, the complex version of which is well known.

Lemma 3.1. *A linear map $\Phi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$ is*

1. *Hermitian if and only if its Choi matrix C_Φ is Hermitian,*
2. *p-positive if and only if C_Φ is Hermitian and $\langle V|C_\Phi V\rangle \geq 0$ for every vector $V \in \mathbb{K}^n \otimes \mathbb{K}^m$ with Schmidt rank at most p.*
3. *p-positive if and only if its adjoint map Φ^* is p-positive.*

Proof. The complex version of (1) was proven in [20,42], while the complex version of (2) was proven in [31] (for $p = 1$, see also [51, Proposition 4.1.11]) and [17,42] (general p). The proof of both items uses the relation

$$\mathrm{Tr}(C_\Phi(A^t \otimes B)) = \mathrm{Tr}(\Phi(A)B), \tag{1}$$

valid for arbitrary matrices $A \in M_n(\mathbb{K})$ and $B \in M_m(\mathbb{K})$.

If Φ is Hermitian, then $C_\Phi^* = \sum_{i,j} E_{i,j}^* \otimes \Phi(E_{i,j})^* = \sum_{i,j} E_{j,i} \otimes \Phi(E_{i,j}^*) = C_\Phi$. Conversely, if C_Φ is Hermitian, then one has

$$\begin{aligned}(\Phi(A)^*)_{i,j} &= \mathrm{Tr}[\Phi(A)^*\,E_{j,i}] = \overline{\mathrm{Tr}[\Phi(A)\,E_{j,i}^*]} = \overline{\mathrm{Tr}(C_\Phi(A^t \otimes E_{j,i}^*))}\\ &= \mathrm{Tr}(C_\Phi^*((A^t)^* \otimes E_{j,i})) = \mathrm{Tr}(C_\Phi((A^*)^t \otimes E_{j,i})) = \mathrm{Tr}[\Phi(A^*)\,E_{j,i}]\\ &= \Phi(A^*)_{i,j}\,.\end{aligned}$$

This concludes the proof of (1).

If Φ is $p-$positive, then $\Phi \otimes id_p$ is Hermitian. Then, Lemma 2.3 implies $\langle w|(\Phi \otimes id_p)(vv^*)w\rangle \geq 0$ for every $v \in \mathbb{K}^n \otimes \mathbb{K}^p$ and every $w \in \mathbb{K}^m \otimes \mathbb{K}^p$. Let us expand the vectors as $v = \sum_{i=1}^p a_i \otimes e_i^{(p)}$ and $w = \sum_{j=1}^p b_j \otimes e_j^{(p)}$, where $\{e_i^{(p)}\}$ is the standard orthonormal basis in $\mathbb{K}^p$, and $a_i \in \mathbb{K}^n$, $b_j \in \mathbb{K}^m$ are arbitrary vectors. With this expansion, we have

$$\begin{aligned}\langle w|(\Phi \otimes id_p)(vv^*)w\rangle &= \sum_{i,j=1}^p \langle b_i|\Phi(a_i a_j^*) b_j\rangle = \sum_{i,j=1}^p \mathrm{Tr}(\Phi(a_i a_j^*) b_j b_i^*) \\ &= \sum_{i,j=1}^p \mathrm{Tr}(C_\Phi\left((a_i a_j^*)^t \otimes b_j b_i^*\right)) \\ &= \sum_{i,j=1}^p \langle \overline{a}_i \otimes b_i | C_\Phi(\overline{a}_j \otimes b_j)\rangle = \langle V|C_\Phi V\rangle\,, \end{aligned} \tag{2}$$

with $V := \sum_{i=1}^p \overline{a}_i \otimes b_i$. Since the vectors a_i and b_i are arbitrary, V is an arbitrary vector with Schmidt rank at most p. Summarizing, the condition $\langle w|(\Phi \otimes id_p)(vv^*)w\rangle \geq 0$ implies $\langle V|C_\Phi V\rangle \geq 0$ for every vector V with Schmidt rank at most p.

Conversely, suppose that C_Φ is Hermitian and satisfies $\langle V|C_\Phi V\rangle \geq 0$ for every vector V with Schmidt rank at most p. Since C_Φ and id_p are both Hermitian, item 1 of this proof implies that $\Phi \otimes id_p$ is Hermitian. Lemma 2.3 then implies that $\Phi \otimes id_p$ is positive if $\langle w|(\Phi \otimes id_p)(vv^*)w\rangle \geq 0$ for every v and w. But Eq. (2) guarantees that this condition follows from the condition $\langle V|C_\Phi V\rangle \geq 0$ for every vector V with Schmidt rank at most p. This concludes the proof of (2).

To prove (3), note that, for every pair of matrices A and B, one has

$$\begin{aligned}\mathrm{Tr}(C_{\Phi^*}(B^t \otimes A)) &= \mathrm{Tr}(\Phi^*(B)A) = \mathrm{Tr}(B\Phi(A)) \\ &= \mathrm{Tr}(C_\Phi(A^t \otimes B)) = \mathrm{Tr}(C_\Phi^t\, S(B^t \otimes A)S^*)\,,\end{aligned}$$

where $S : \mathbb{K}^m \otimes \mathbb{K}^n \to \mathbb{K}^n \otimes \mathbb{K}^m$ is the linear operator defined by $S(w \otimes v) = v \otimes w$ for all $v \in \mathbb{K}^n$ and $w \in \mathbb{K}^m$. Since A and B are arbitrary, we conclude

$$C_{\Phi^*} = S^* C_\Phi^t S. \tag{3}$$

Clearly, C_{Φ^*} is Hermitian if and only if C_Φ is, and $\langle W|C_{\Phi^*}W\rangle \geq 0$ for every vector $W \in \mathbb{K}^m \otimes \mathbb{K}^n$ with Schmidt rank $\leq p$ if and only if $\langle V|C_\Phi V\rangle \geq 0$ for every vector $V \in \mathbb{K}^n \otimes \mathbb{K}^m$ with Schmidt rank $\leq p$. Hence, (2) implies that Φ^* is p-positive if and only if Φ is. ■

4. Real maps with p-positive complexification

Given a linear map $\Phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$ its *complexification* is the linear map $\widetilde{\Phi} : M_n(\mathbb{C}) \to M_m(\mathbb{C})$, defined by

$$\widetilde{\Phi}(X + iY) = \Phi(X) + i\Phi(Y) \quad \text{for all} \quad X, Y \in M_n(\mathbb{R}).$$

We now characterize the set of maps with a p-positive complexification. The characterization uses the correspondence between matrices $A = (a_{ij}) \in M_{n,m}(\mathbb{K})$ and vectors in $\mathrm{Vec}(A) \in \mathbb{K}^n \otimes \mathbb{K}^m$ given by

$$\mathrm{Vec}(A) := \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}\, e_i^{(n)} \otimes e_j^{(m)},$$

where, for $r \in \{m, n\}$, $\{e_i^{(r)}\}_{i=1}^{r}$ is the standard orthonormal basis for $\mathbb{K}^r$.

Theorem 4.1. *A map $\Phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$ has a p-positive complexification if and only if its Choi matrix C_Φ is Hermitian and satisfies the condition $\langle \mathrm{Vec}(X) | C_\Phi \, \mathrm{Vec}(X) \rangle + \langle \mathrm{Vec}(Y) | C_\Phi \, \mathrm{Vec}(Y) \rangle \geq 0$ for every pair of matrices $X, Y \in M_{n,m}(\mathbb{R})$ such that $\mathrm{rank}(X + iY) \leq p$.*

Proof. By Lemma 3.1, $\widetilde{\Phi}$ is p-positive if and only if its Choi matrix $C_{\widetilde{\Phi}} = C_\Phi$ is Hermitian and satisfies the condition $\langle V | C_\Phi V \rangle \geq 0$ for every vector $V \in \mathbb{C}^n \otimes \mathbb{C}^m$ with Schmidt rank at most p. In the vector notation, this condition is equivalent to $\langle \mathrm{Vec}(A) | C_\Phi \, \mathrm{Vec}(A) \rangle \geq 0$ for every matrix $A \in M_{n,m}(\mathbb{C})$ with $\mathrm{rank}(A) \leq p$. Writing $A = X + iY$, with $X, Y \in M_{n,m}(\mathbb{R})$, and using the fact that C_Φ is a symmetric matrix, we obtain

$$\langle \mathrm{Vec}(A) | C_\Phi \, \mathrm{Vec}(A) \rangle = \langle \mathrm{Vec}(X) | C_\Phi \, \mathrm{Vec}(X) \rangle + \langle \mathrm{Vec}(Y) | C_\Phi \, \mathrm{Vec}(Y) \rangle. \quad \blacksquare$$

Corollary 4.2. *If a map $\Phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$ is $2p$-positive, then its complexification is p-positive.*

Proof. Immediate from Theorem 4.1, Lemma 3.1, and the fact that, for $A = X + iY$, $\mathrm{rank}(X) = \mathrm{rank}(A + \overline{A}) \leq 2\,\mathrm{rank}(A)$, and similarly for $\mathrm{rank}(Y)$. $\blacksquare$

Note that the corollary above establishes that if Φ is completely positive, then the complexification is also completely positive, a fact proved in Lemma 2.3 in [6]. In particular, this corollary shows that 2-positivity on the real field is sufficient for positivity of the complexification. In general, however, 2-positivity is not necessary: for example, the maps $\Lambda_q : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by

$$\Lambda_q(A) = \mathrm{Tr}(A)\, I_n - qA$$

have p-positive complexification whenever $q \leq 1/p$ [54]. They are not $(p+1)$-positive (and therefore not $2p$ positive) for $q > 1/(p+1)$, as one can verify from the relation $\langle V_l | C_{\Lambda_q} V_l \rangle = 1 - ql$, with $V_l := \sum_{i=1}^{l} e_i^{(n)} \otimes e_i^{(n)} / \sqrt{l}$. Similar counterexamples can be constructed using the spectral methods developed by Chruściński and Kossakowski in [17].

5. Examples of positive real maps with non-positive complexification

In this section, we provide several examples of positive maps whose complexification is not. The first deals with p-positivity for general p. The others deal with $p = 1$.

Example 5.1. We provide an example of a $(2p-1)$-positive real map such that its complefixication is not p-positive. Define $\Gamma_q : M_{2p}(\mathbb{R}) \to M_{2p}(\mathbb{R})$ by

$$\Gamma_q(A) = \mathrm{Tr}(A) I_{2p} - q(O_+ A O_+ + O_- A O_-),$$

where $O_+ = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ and $O_- = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}$. The Choi matrix of this map is

$$C_{\Gamma_q} = I_{2p} \otimes I_{2p} - q\big(\mathrm{Vec}(O_+)\,\mathrm{Vec}(O_+)^t + \mathrm{Vec}(O_-)\,\mathrm{Vec}(O_-)^t\big).$$

Note that the vectors $\mathrm{Vec}(O_+)$ and $\mathrm{Vec}(O_-)$ are orthogonal and each has length $\sqrt{2p}$, so that

$$\mathcal{P} = \frac{1}{2p}\big(\mathrm{Vec}(O_+)\,\mathrm{Vec}(O_+)^t + \mathrm{Vec}(O_-)\,\mathrm{Vec}(O_-)^t\big)$$

is a projection. Thus $C_{\Gamma_q} = I_{2p} \otimes I_{2p} - 2pq\mathcal{P}$. Moreover the unit vectors in $\mathrm{ran}\,\mathcal{P} = \mathrm{span}\{O_+, O_-\}$ are $\{\frac{1}{\sqrt{2p}}(\cos\theta\, O_+ + \sin\theta\, O_-) : \theta \in [0, 2\pi]\}$. Now $O_\theta := \cos\theta\, O_+ + \sin\theta\, O_- \in \mathcal{O}_{2n}$ is an orthogonal matrix for all θ.

Given a unit vector $V = \mathrm{Vec}(A)$, so that $\mathrm{Tr}(A^t A) = 1$, one has

$$\begin{aligned} \langle V | C_{\Gamma_q} V\rangle &= \|V\|^2 - 2pq\|\mathcal{P}V\|^2 \\ &= 1 - 2pq \max_{\substack{W=\mathcal{P}W \\ \|W\|=1}} \langle V|W\rangle^2 \\ &= 1 - \sqrt{2p}q \max_{\theta\in[0,2\pi]} \mathrm{Tr}(A^t O_\theta). \end{aligned}$$

If the unit vector V has Schmid number at most $2p-1$, i.e., $\mathrm{rank}(A) \le 2p-1$, we obtain

$$\mathrm{Tr}(A^t O_\theta) \le \mathrm{Tr}\,|A| \le \sqrt{2p-1}.$$

Hence, we obtain the inequality $\langle V|C_{\Gamma_q} V\rangle \ge 1 - q\sqrt{2p(2p-1)}$ for every vector V with Schmidt rank $\le 2p-1$. By Lemma 3.1, we conclude that the map Φ is $(2p-1)$-positive for every $\sqrt{2p(2p-1)}q \le 1$.

Take q with $\frac{1}{2p} < q \le \frac{1}{\sqrt{2p(2p-1)}}$. Then Γ_q is $(2p-1)$-positive, but its complexification $\widetilde{\Gamma_q}$ is not p-positive. To see this, let

$$V = \frac{1}{2\sqrt{p}}\,\mathrm{Vec}(O_+ + iO_-).$$

Then $\mathrm{rank}(O_+ + iO_-) = p\,\mathrm{rank}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} = p$. Compute

$$\langle V|C_{\widetilde{\Gamma_q}}|V\rangle = \langle V|C_{\Gamma_q}|V\rangle = 1 - 2pq < 0.$$

Therefore $\widetilde{\Gamma_q}$ is not p-positive by Lemma 3.1 again.

In the rest of the section we focus on the $p = 1$ case.

Example 5.2. Let us start with a simple example of positivity preserving real map with non-positive complexification. Define $\Phi_s : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by

$$\Phi_s(A) = A + s(A - A^t).$$

Then, Φ_s is a positivity preserving for all $s \in \mathbb{R}$. However it is not Hermitian if $s \neq 0$. Hence $\widetilde{\Phi}_s$ is not Hermitian, and thus is not positive. Explicitly, if we set

$$P = E_{1,1} + E_{2,2} + i(E_{1,2} - E_{2,1}) \geq 0,$$

where $E_{i,j}$ are the standard matrix units, then

$$\widetilde{\Phi}_s(P) = E_{1,1} + E_{2,2} + (1 + 2s)i(E_{1,2} - E_{2,1}),$$

which has eigenvalue $-2s$, and therefore is not positive for $s > 0$.

In the complex field, a result of Russo and Dye (Corollary 2.9 in [38]) states that a positive map Φ on a unital C*-algebra satisfies the condition $\|\Phi\| = \|\Phi(I)\|$. Hence, a necessary condition for positivity of the complexification is that Φ is positivity preserving, Hermitian, and satisfies the Russo-Dye condition $\|\Phi\| = \|\Phi(I)\|$. We now provide counterexamples, showing that the Russo-Dye condition is not sufficient for positive complexification.

Example 5.3. Here we provide an example of a Hermitian, positive, unital, norm one map on $M_2(\mathbb{R})$ whose complexification is not positive. Define

$$\Phi(A) = \Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & \frac{b-c}{2} \\ \frac{c-b}{2} & d \end{pmatrix}.$$

This map is clearly unital and Hermitian. It is positive since $\Phi \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right) = dI_2$. Since the two columns of $\Phi(A)$ are orthogonal, we have

$$\|\Phi(A)\| = \left\| \begin{pmatrix} d & \frac{b-c}{2} \\ \frac{c-b}{2} & d \end{pmatrix} \right\| = \sqrt{d^2 + \frac{(b-c)^2}{4}}.$$

Now,

$$\begin{aligned} \|A\| &\geq \frac{1}{2} \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & -c \\ -b & d \end{pmatrix} \right\| = \left\| \begin{pmatrix} a & \frac{b-c}{2} \\ \frac{c-b}{2} & d \end{pmatrix} \right\| \\ &\geq \sqrt{d^2 + \frac{(b-c)^2}{4}} = \|\Phi(A)\|. \end{aligned}$$

Hence, $\|\Phi\| = 1$. Finally, the non-positivity of $\widetilde{\Phi}$ can be seen from the relation $\widetilde{\Phi} \left(\begin{smallmatrix} \lambda & i \\ -i & 1/\lambda \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1/\lambda & i \\ -i & 1/\lambda \end{smallmatrix}\right)$ for $\lambda > 1$.

Example 5.4. Here we provide an example of a Hermitian, positive, trace-preserving map on $M_2(\mathbb{R})$ that satisfies the Russo-Dye condition but has a non-positive complexification. Define

$$\begin{aligned}\Psi(A) = \Psi\begin{pmatrix} a & b \\ c & d\end{pmatrix} &:= \begin{pmatrix} 0 & \frac{b-c}{2} \\ \frac{c-b}{2} & a+d\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} a & b \\ c & d\end{pmatrix} + \frac{1}{2}\begin{pmatrix} d & -c \\ -b & a\end{pmatrix} + \begin{pmatrix} -\frac{a+d}{2} & 0 \\ 0 & \frac{a+d}{2}\end{pmatrix}.\end{aligned}$$

This map is clearly trace-preserving and Hermitian. It is positive since $\Psi\left(\begin{smallmatrix} a & b \\ b & d\end{smallmatrix}\right) = (a+d)\,E_{22}$. Moreover, one has

$$\|\Psi(A)\| \le \frac{\|A\|}{2} + \frac{\|A\|}{2} + \frac{|\operatorname{Tr}(A)|}{2} \le 2\|A\|.$$

Hence, $\|\Psi\| = \|\Psi(I_2)\| = 2$. On the other hand, $\widetilde{\Psi}$ is not positive, as one can see from the relation

$$\widetilde{\Psi}\begin{pmatrix} 1 & i \\ -i & 1\end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 2\end{pmatrix}.$$

Example 5.5. Finally, we provide an example of Hermitian, positive, trace-preserving and unital map on $M_3(\mathbb{R})$ that satisfies the Russo-Dye condition, but has a non-positive complexification. Define the one-parameter family of maps for $t \in [0,1]$,

$$\rho_t(A) = \rho_t\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} := \begin{pmatrix} \frac{a_{11}+a_{22}+2a_{33}}{4} & \frac{(a_{12}-a_{21})t}{2\sqrt{2}} & 0 \\ \frac{(a_{21}-a_{12})t}{2\sqrt{2}} & \frac{a_{11}+a_{22}+2a_{33}}{4} & 0 \\ 0 & 0 & \frac{a_{11}+a_{22}}{2}\end{pmatrix}.$$

This map is clearly unital, trace-preserving, positive, and Hermitian.

Set $x = \frac{a_{11}+a_{22}}{2}$, $y = \frac{a_{12}-a_{21}}{2}$, $z = a_{33}$, and $r = \max\{\sqrt{x^2+y^2}, |z|\}$. Compute

$$\begin{aligned}\|A\| &\ge \left\|\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33}\end{pmatrix}\right\| \\ &\ge \max\left\{\left\|\frac{1}{2}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} + \frac{1}{2}\begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11}\end{pmatrix}\right\|,\ |z|\right\} \\ &= \max\left\{\left\|\begin{pmatrix} x & y \\ -y & x\end{pmatrix}\right\|,\ |z|\right\} = \max\left\{\sqrt{x^2+y^2}, |z|\right\} = r.\end{aligned}$$

The three columns of $\rho_t(A)$ are orthogonal. Hence

$$\|\rho_t(A)\| = \left\| \begin{pmatrix} \frac{x+z}{2} & \frac{yt}{\sqrt{2}} & 0 \\ \frac{-yt}{\sqrt{2}} & \frac{x+z}{2} & 0 \\ 0 & 0 & x \end{pmatrix} \right\| \leq \max\left\{ \sqrt{\left(\frac{x+z}{2}\right)^2 + \frac{t^2\, y^2}{2}}, |x| \right\}$$
$$\leq \max\left\{ \sqrt{\left(\frac{|x|+|z|}{2}\right)^2 + \frac{y^2}{2}}, r \right\}.$$

Let $s = \frac{|x|}{r}$ and $c = \frac{|y|}{r}$ and note that $s^2 + c^2 \leq 1$. Hence

$$\left(\frac{|x|+|z|}{2}\right)^2 + \frac{y^2}{2} \leq \frac{r^2}{4}\big((s+1)^2 + 2c^2\big) \leq \frac{r^2}{4}\big(s^2 + 2s + 1 + 2 - 2s^2\big)$$
$$= \frac{r^2}{4}\big(4 - (1-s)^2\big) \leq r^2.$$

Therefore

$$\|\rho_t(A)\| \leq r \leq \|A\|.$$

Thus ρ_t has norm 1 for every $t \in [0,1]$.

On the other hand, $\widetilde{\rho_t}$ is not positive for every $t > 1/\sqrt{2}$ because

$$\widetilde{\rho_t} \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & it\sqrt{2} & 0 \\ -it\sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We also observe that $\widetilde{\rho_t}$ is positive for $t \leq \frac{1}{\sqrt{2}}$. Indeed, if $A = \big[a_{ij}\big] \geq 0$,

$$\widetilde{\rho_t}(A) = \widetilde{\rho_t} \begin{pmatrix} a_{11} & a_{12} & 0 \\ \overline{a_{12}} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} \frac{a_{11}+a_{22}+2a_{33}}{4} & \frac{i\mathrm{Im}(a_{12})t}{\sqrt{2}} & 0 \\ \frac{-i\mathrm{Im}(a_{12})t}{\sqrt{2}} & \frac{a_{11}+a_{22}+2a_{33}}{4} & 0 \\ 0 & 0 & \frac{a_{11}+a_{22}}{2} \end{pmatrix}.$$

Since

$$\left|\frac{i\mathrm{Im}(a_{12})\, t}{\sqrt{2}}\right|^2 \leq \frac{|a_{12}|^2}{4} \leq \frac{a_{11}a_{22}}{4} \leq \left(\frac{a_{11}+a_{22}}{4}\right)^2,$$

we obtain $\widetilde{\rho_t}(A) \geq 0$.

This example extends to any dimension $n > 3$ by defining

$$\sigma_{n,t} \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} \rho_t(A) & 0 \\ 0 & D \end{pmatrix}$$

for $A \in M_3(\mathbb{R})$, $B \in M_{2,n-3}(\mathbb{R})$, $C \in M_{n-3,2}(\mathbb{R})$ and $D \in M_{n-3,n-3}(\mathbb{R})$. The map $\sigma_{n,t}$ for $t \in (\frac{1}{\sqrt{2}}, 1]$ and $n > 3$ is an example of a positive, unital, trace-preserving, norm-1 map on M_n that has no positive complexification.

6. Witnesses of real entanglement

We now show that the existence of positive real maps with non-positive complexification is equivalent to the existence of quantum states that are entangled in real Hilbert space quantum mechanics, and yet are separable when regarded as states on complex Hilbert spaces. The equivalence is established by proving a duality between the cone of real positive maps and the cone of separable states in real Hilbert space quantum mechanics.

For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the cone of separable matrices is

$$\mathrm{SEP}(\mathbb{K}^n \otimes \mathbb{K}^m) = \Big\{ \sum_i A_i \otimes B_i : A_i \in \mathrm{PSD}(\mathbb{K}^n),\ B_i \in \mathrm{PSD}(\mathbb{K}^m) \Big\}.$$

The elements of $\mathrm{SEP}(\mathbb{K}^n \otimes \mathbb{K}^m)$ will be called $\mathbb{K}$-*separable*, and the elements of $\mathrm{PSD}(\mathbb{K}^n \otimes \mathbb{K}^m) \setminus \mathrm{SEP}(\mathbb{K}^n \otimes \mathbb{K}^m)$ will be called $\mathbb{K}$-*entangled*.

More generally, we consider the cone of positive matrices with Schmidt rank at most p [55]. We call these matrices $\mathbb{K}$-p-*separable* and denote them by

$$\mathrm{SEP}_p(\mathbb{K}^n \otimes \mathbb{K}^m) = \Big\{ \sum_j \mathrm{Vec}(A_j)\,\mathrm{Vec}(A_j)^* : A_j \in M_{n,m}(\mathbb{K})\,, \mathrm{rank}(A_j) \leq p \Big\}.$$

Note that $\mathrm{SEP}_1(\mathbb{K}^n \otimes \mathbb{K}^m) = \mathrm{SEP}(\mathbb{K}^n \otimes \mathbb{K}^m)$. The elements of $\mathrm{PSD}(\mathbb{K}^n \otimes \mathbb{K}^m) \setminus \mathrm{SEP}_p(\mathbb{K}^n \otimes \mathbb{K}^m)$ will be called $\mathbb{K}$-p-*entangled*.

We also define the cone consisting of all real positive matrices that are $\mathbb{C}$-p-separable, denoted as

$$\begin{aligned}\mathrm{CSEP}_p(\mathbb{R}^n \otimes \mathbb{R}^m) &:= \mathrm{SEP}_p(\mathbb{C}^n \otimes \mathbb{C}^m) \cap \mathrm{PSD}(\mathbb{R}^n \otimes \mathbb{R}^m) \\ &= \Big\{ \sum_j \mathrm{Vec}(A_j)\,\mathrm{Vec}(A_j)^* + \mathrm{Vec}(\overline{A}_j)\,\mathrm{Vec}(\overline{A}_j)^* : \\ &\qquad A_j \in M_{n,m}(\mathbb{C}) \text{ and } \mathrm{rank}(A_j) \leq p \text{ for all } j \Big\}.\end{aligned}$$

For $p = 1$, we use the notation $\mathrm{CSEP}(\mathbb{R}^n \otimes \mathbb{R}^m) := \mathrm{CSEP}_1(\mathbb{R}^n \otimes \mathbb{R}^m)$.

We now provide an equivalent characterization of $\mathrm{CSEP}_p(\mathbb{R}^n \otimes \mathbb{R}^m)$, which shows that every $\mathbb{C}$-p-separable real matrix is also $\mathbb{R}$-$2p$-separable:

Proposition 6.1. *One has*

$$\begin{aligned}\mathrm{CSEP}_p(\mathbb{R}^n \otimes \mathbb{R}^m) = \Big\{ \sum_j \mathrm{Vec}(X_j)\,\mathrm{Vec}(X_j)^t + \mathrm{Vec}(Y_j)\,\mathrm{Vec}(Y_j)^t : \\ X_j, Y_j \in M_{n,m}(\mathbb{R}),\ \mathrm{rank}(X_j + iY_j) \leq p\,, \forall j \Big\}\end{aligned}$$

and $\mathrm{CSEP}_p(\mathbb{R}^n \otimes \mathbb{R}^m) \subset \mathrm{SEP}_{2p}(\mathbb{R}^n \otimes \mathbb{R}^m)$.

Proof. The characterization follows from decomposing A_j as $A_j = X_j + iY_j$, with $X_j, Y_j \in M_{m,n}(\mathbb{R})$, and from the relation

$$\mathrm{Vec}(A_j)\,\mathrm{Vec}(A_j)^* + \mathrm{Vec}(\overline{A}_j)\,\mathrm{Vec}(\overline{A}_j)^* = \mathrm{Vec}(X_j)\,\mathrm{Vec}(X_j)^t + \mathrm{Vec}(Y_j)\,\mathrm{Vec}(Y_j)^t.$$

The inclusion equality follows from the first and from the fact that $\mathrm{rank}(A_j) \leq p$ implies $\mathrm{rank}(X) \leq 2p$ and $\mathrm{rank}(Y) \leq 2p$. ∎

In the next section we will provide an example showing that the inclusion $\mathrm{CSEP}_p(\mathbb{R}^n \otimes \mathbb{R}^m) \subset \mathrm{SEP}_{2p}(\mathbb{R}^n \otimes \mathbb{R}^m)$ is strict.

Remark 6.2. Note that the cone $\mathrm{SEP}_p(\mathbb{K}^n \otimes \mathbb{K}^m)$ is closed. To see this it is enough to show that the convex set consisting of the elements in these cones of trace 1 is a closed convex set. But the elements in $\mathrm{SEP}_p(\mathbb{K}^n \otimes \mathbb{K}^m)$ of trace one is the convex hull of the set of the matrices of the form $A = VV^*$ where V is a unit vector with Schmidt rank $\leq p$. The set of such matrices is compact and, in finite dimensions, the convex hull of a compact set is again a compact set by Caratheodory's Theorem. The cone $\mathrm{CSEP}_p(\mathbb{R}^n \otimes \mathbb{R}^m)$ is also closed since it is the intersection of two closed sets.

The complex version of part (1) of the next theorem was proven in [49].

Theorem 6.3. *Let $\Phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$ be a Hermitian map and let C_Φ be its Choi matrix. Then,*

(1) *Φ is p-positive if and only if*
$$\mathrm{Tr}(C_\Phi\, P) \geq 0 \quad \textit{for all} \quad P \in \mathrm{SEP}_p(\mathbb{K}^n \otimes \mathbb{K}^m),$$
(2) *The complexification $\widetilde{\Phi} : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is p-positive if and only if*
$$\mathrm{Tr}(C_\Phi\, P) \geq 0 \quad \textit{for all} \quad P \in \mathrm{CSEP}_p(\mathbb{R}^n \otimes \mathbb{R}^m).$$

Proof. The set $\mathrm{SEP}_p(\mathbb{K}^n \otimes \mathbb{K}^m)$ is the convex hull of its rank-one elements, of the form $\mathrm{Vec}(A)\,\mathrm{Vec}(A)^*$ with $\mathrm{rank}(A) \leq p$. The condition $\mathrm{Tr}(C_\Phi P) \geq 0 \quad$ for all $\quad P \in \mathrm{SEP}_p(\mathbb{K}^n \otimes \mathbb{K}^m)$ is equivalent to
$$0 \leq \mathrm{Tr}(C_\Phi\ \mathrm{Vec}(A)\,\mathrm{Vec}(A)^*) = \langle \mathrm{Vec}(A) | C_\Phi\,\mathrm{Vec}(A)\rangle$$
for all matrices with rank at most p. In turn, this condition is equivalent to $\langle V | C_\Phi V\rangle \geq 0$ for every vector V with Schmidt number at most p. By Lemma 3.1, this condition is equivalent to p-positivity of Φ. This proves (1).

We now prove (2). By Proposition 6.1, the condition $\mathrm{Tr}(C_\Phi P) \geq 0$ for all $P \in \mathrm{CSEP}_p(\mathbb{R}^n \otimes \mathbb{R}^m)$ is equivalent to
$$\begin{aligned} 0 &\leq \mathrm{Tr}(C_\Phi\,(\mathrm{Vec}(X)\,\mathrm{Vec}(X)^t + \mathrm{Vec}(Y)\,\mathrm{Vec}(Y)^t)) \\ &= \langle \mathrm{Vec}(X) | C_\Phi\,\mathrm{Vec}(X)\rangle + \langle \mathrm{Vec}(Y) | C_\Phi\,\mathrm{Vec}(Y)\rangle\,, \end{aligned}$$
for all real matrices X, Y such that $\mathrm{rank}(X + iY) \leq p$. By Theorem 4.1, this condition is equivalent to p-positivity of $\widetilde{\Phi}$. ∎

Corollary 6.4. *Let $P \in \mathrm{PSD}(\mathbb{K}^n \otimes \mathbb{K}^m)$. Then:*

(1) *P is $\mathbb{K}$-p-separable if and only if* $\mathrm{Tr}(C_\Phi P) \geq 0$*, for all p-positive* $\Phi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$,
(2) *for $\mathbb{K} = \mathbb{R}$, P is $\mathbb{C}$-p-separable if and only if* $\mathrm{Tr}(C_\Phi P) \geq 0$ *for all* $\Phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$ *such that $\widetilde{\Phi}$ is p-positive.*

Proof. Use the fact that these cones are closed and apply the standard Hahn-Banach results for separating a point from a closed cone. ∎

This corollary implies that, if a real positive matrix P is $\mathbb{R}$-p-entangled, then there must exist a p-positive map Φ such that $\mathrm{Tr}(C_\Phi P) < 0$. We say that Φ is a *witness of $\mathbb{R}$-p-entanglement* for P. Similarly, if a real positive matrix P is $\mathbb{C}$-p-entangled, then there must exist a positive map Ψ with positive complexification such that $\mathrm{Tr}(C_\Psi P) < 0$. We say that Ψ is a *witness of $\mathbb{C}$-p-entanglement* for P. In the literature, the term entanglement witness was typically used for $p = 1$, while witnesses for $p > 1$ were sometimes called Schmidt number witnesses [48, 56].

Before proceeding to the next result we need a lemma for which we refer to [16]. The assertion holds for any field.

Lemma 6.5. *Every vector $V \in \mathbb{K}^n \otimes \mathbb{K}^m$ with Schmidt rank $\leq p$ can be written as*

$$V = (I_n \otimes S)W = (T \otimes I_m)Z,$$

where W is a vector in $\mathbb{K}^n \otimes \mathbb{K}^p$, Z is a vector in $\mathbb{K}^p \otimes \mathbb{K}^m$ with $S : \mathbb{K}^p \to \mathbb{K}^m$ and $T : \mathbb{K}^p \to \mathbb{K}^n$ are partial isometries.

The fact that p-positive maps are witnesses of p-entanglement is equivalently expressed by the following corollary, the complex version of which was established in [26] for $p = 1$ and in [55] for general p.

Corollary 6.6. *For a matrix $P \in \mathrm{PSD}(\mathbb{K}^n \otimes \mathbb{K}^m)$, the following are equivalent*

(1) *P is $\mathbb{K}$-p-separable,*
(2) *$\Phi \otimes id_r(P) \geq 0$ for every $r \in \mathbb{N}$ and for every p-positive map $\Phi : M_n(\mathbb{K}) \to M_r(\mathbb{K})$,*
(3) *$\Phi \otimes id_m(P) \geq 0$ for every p-positive map $\Phi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$.*
(4) *$id_n \otimes \Phi(P) \geq 0$ for every p-positive map $\Phi : M_m(\mathbb{K}) \to M_l(\mathbb{K})$*

Proof. (1) $\Rightarrow$ (2). If P is $\mathbb{K}$-p-separable, it can be written as $P = \sum_i V_i V_i^*$, where each V_i is a vector in $\mathbb{K}^n \otimes \mathbb{K}^m$ with Schmidt rank at most p. Using lemma 6.5 we know that every vector $V_i \in \mathbb{K}^n \otimes \mathbb{K}^m$ with Schmidt rank $\leq p$ can be written as $V_i = (I_n \otimes S_i)W_i$, where W_i is a vector in $\mathbb{K}^n \otimes \mathbb{K}^p$ and $S_i : \mathbb{K}^p \to \mathbb{K}^m$ is a partial isometry. Then, for every map $\Phi : M_n(\mathbb{K}) \to M_r(\mathbb{K})$, we have

$$\Phi \otimes id_m(P) = \sum_i (I_r \otimes S_i)\ (\Phi \otimes id_p(W_i W_i^*))\ (I_r \otimes S_i^*) \geq 0.$$

If Φ is p-positive, then each term in this sum is positive; so $\Phi \otimes id_m(P) \geq 0$.

(2) $\Rightarrow$ (3). Take $r = m$.

(3) $\Rightarrow$ (1). Given a map Φ, the condition $\Phi \otimes id_m(P) \geq 0$ implies

$$\begin{aligned} 0 &\leq \sum_{i,j} \operatorname{Tr}((E_{i,j} \otimes E_{i,j})\Phi \otimes id_m(P)) = \sum_{i,j} \operatorname{Tr}((\Phi^*(E_{i,j}) \otimes E_{i,j})\, P) \\ &= \operatorname{Tr}(SC_{\Phi^*}S^*P) = \operatorname{Tr}(C_{\Phi^*}S^*PS), \end{aligned}$$

where $S : \mathbb{K}^m \otimes \mathbb{K}^n \to \mathbb{K}^m \otimes \mathbb{K}^n$ is the linear operator defined by $S(w \otimes v) = v \otimes w$ for all $v \in \mathbb{K}^n$ for all $w \in \mathbb{K}^m$. If $\Phi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$ is an arbitrary p-positive map, then $\Phi^* : M_m(\mathbb{K}) \to M_n(\mathbb{K})$ is an arbitrary p-positive map by Lemma 3.1. For $\mathbb{K} = \mathbb{R}$, Corollary 6.4 implies that S^*PS, and therefore P, is $\mathbb{R}$-p-separable.

The equivalence of (1) and (4) follows exactly like the equivalence of (1) and (3) using the second equality of Lemma 6.5. ■

Corollary 6.7. *For every $p \in \mathbb{N}$, the following are equivalent*

1. *there exists a p-positive map $\Phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$ with non-p-positive complexification,*
2. *there exists an $\mathbb{C}$-p-separable matrix $P \in \operatorname{PSD}(\mathbb{R}^n \otimes \mathbb{R}^m)$ that is not $\mathbb{R}$-p-separable.*

Proof. Immediate from Corollary 6.4. ■

7. Examples of $\mathbb{R}$-entangled states that are $\mathbb{C}$-separable

In this section, we use the distinctions between the various cones of positive maps to provide explicit examples. The first example is an $\mathbb{R}$-$(2p-1)$-entangled but $\mathbb{C}$-p-separable matrix. The remaining examples focus on the $p = 1$ case, providing examples of quantum states that are entangled in real Hilbert space quantum mechanics, but separable in the complex version. The key observation is that real separable states must be invariant under partial transposition (IPT).

Example 7.1. Here we provide an example of a matrix $P \in \operatorname{PSD}(\mathbb{R}^{2p} \otimes \mathbb{R}^{2p})$ that is $\mathbb{R}$-$(2p-1)$-entangled, but $\mathbb{C}$-p-separable. The matrix is

$$P = \frac{VV^* + \overline{V}V^t}{2},$$

where $V \in \mathbb{C}^{2p} \otimes \mathbb{C}^{2p}$ is the complex vector $V = \sum_{j=1}^{p} \alpha_j \otimes \alpha_j / \sqrt{p}$, with $\alpha_j = (e_j^{(2p)} + i\, e_{j+p}^{(2p)})/\sqrt{2}$. It is evident from the definition that P is $\mathbb{C}$-p-separable. On the other hand, P can be equivalently rewritten as

$$P = \frac{\operatorname{Vec}(O_+)\operatorname{Vec}(O_+)^t + \operatorname{Vec}(O_-)\operatorname{Vec}(O_-)^t}{4p},$$

with $O_+ = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ and $O_- = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}$. If P were $\mathbb{R}$-$(2p-1)$-separable, there should exist at least one real vector with Schmidt rank $\leq 2p-1$ in the linear span of $\mathrm{Vec}(O_+)$ and $\mathrm{Vec}(O_-)$. But this is not possible, because the real span of O_+ and O_- contains only full-rank matrices.

Lemma 7.2. *Every $P \in \mathrm{SEP}(\mathbb{R}^n \otimes \mathbb{R}^m)$ must satisfy the condition $P = \tau_n \otimes id_m(P) = id_n \otimes \tau_m(P)$, where $\tau_r : M_r(\mathbb{R}) \to M_r(\mathbb{R})$, $A \mapsto A^t$ is the transpose map on $M_r(\mathbb{R})$.*

Proof. By definition every $P \in \mathrm{SEP}(\mathbb{R}^n \otimes \mathbb{R}^m)$ can be decomposed as $P = \sum_i A_i \otimes B_i$ for some $A_i \in \mathrm{PSD}(\mathbb{R}^n)$ and $B_i \in \mathrm{PSD}(\mathbb{R}^m)$. Applying the transpose map to the first factor, one gets

$$\tau_n \otimes id_m(P) = \sum_i A_i^t \otimes B_i = \sum A_i \otimes B_i = X,$$

and similarly when the transpose is applied to the second factor. ∎

Invariance under partial transpose is a necessary condition for separability of real quantum states, similarly to the positive partial transpose (PPT) criterion for separability in the complex case [26, 27, 39]. Based on this analogy, we call the invariance under partial transpose the IPT criterion for separability of real quantum states.

Proposition 7.3. *For $n, m \geq 2$, the inclusions*

$$\mathrm{SEP}(\mathbb{R}^n \otimes \mathbb{R}^m) \subsetneq \mathrm{CSEP}(\mathbb{R}^n \otimes \mathbb{R}^m) \subsetneq \mathrm{PSD}(\mathbb{R}^n \otimes \mathbb{R}^m),$$

are both strict.

Proof. Strictness of the second inclusion is trivial: for example, the matrix $\sum_{i,j=1}^{min(n,m)} E_{i,j} \otimes E_{i,j}$ is in $\mathrm{PSD}(\mathbb{R}^n \otimes \mathbb{R}^m)$, but it is not $\mathbb{C}$-separable, and therefore is not in $\mathrm{CSEP}(\mathbb{R}^n \otimes \mathbb{R}^m)$.

To prove strictness of the first inclusion, we show that there exist elements of $\mathrm{CSEP}(\mathbb{R}^n \otimes \mathbb{R}^m)$ that are not invariant under partial transpose. For example, take the matrix $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and consider the Hermitian operator

$$P = I_2 \otimes I_2 + A \otimes A = \left(\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right) \in \mathrm{PSD}(\mathbb{R}^2 \otimes \mathbb{R}^2)$$

Since $I_2 \pm A \geq 0$ and

$$2P = (I_2 + A) \otimes (I_2 + A) + (I_2 - A) \otimes (I_2 - A),$$

P belongs to $\mathrm{CSEP}(\mathbb{R}^n \otimes \mathbb{R}^m)$. However, $id \otimes T(P) \neq P$ and hence $P \notin \mathrm{SEP}(\mathbb{R}^n \otimes \mathbb{R}^m)$.

This takes care of the case $n = m = 2$. For the general case, one can form matrices $A_n = A \oplus 0_{n-2} \in M_n(\mathbb{C})$ and $A_m = A \oplus 0_{m-2} \in M_m(\mathbb{C})$ and observe that the same argument shows that $I_n \otimes I_m + A_n \otimes A_m$ is $\mathbb{R}\mathbb{C}$-separable but not $\mathbb{R}$-separable. ■

Other examples of $\mathbb{C}$-separable but are not equal to their partial transpose (and therefore are $\mathbb{R}$-entangled) are provided in the following.

Example 7.4. Consider the real positive matrix

$$\begin{aligned} P_s &= \frac{s}{n(n+1)}(I_n \otimes I_n + W) + \frac{1-s}{n(n-1)}(I_n \otimes 1_n - W) \\ &= \frac{n+1-2s}{n^3-n}\Big(I_n \otimes I_n + \big(2ns - (n+1)\big)W\Big) \quad \text{for} \quad s \in [0,1]\,, \end{aligned}$$

where $W = \sum_{i,j} E_{i,j} \otimes E_{j,i} \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. It is known [58] (see also chapter 6 of [57]) that P_s is $\mathbb{C}$-separable for $s \in [\frac{1}{2}, 1]$. On the other hand, P_s is not equal to its partial transpose unless $s = \frac{n+1}{2n}$, and therefore P_s is $\mathbb{R}$-entangled for s in $\left[\frac{1}{2}, 1\right] \setminus \left\{\frac{n+1}{2n}\right\}$.

The above example can be generalized as follows:

Proposition 7.5. *For every symmetric matrix $A \in M_n(\mathbb{R}) \otimes M_m(R)$ with $id \otimes T(A) \neq A$, there exists an $s_* > 0$ such that the matrix*

$$P_s = (1-s) I_n \otimes I_m + s\,A$$

is $\mathbb{R}$-entangled and $\mathbb{C}$-separable for every $s \in (0, s_*]$.

Proof. A result of Gurvits and Barnum [30] states that, for every Hermitian matrix $H \in \mathrm{Herm}(\mathbb{C}^n \otimes \mathbb{C}^m)$ with $\|H\|_2 \leq 1$, the matrix $I_n \otimes I_m + H$ is $\mathbb{C}$-separable. If A is a real symmetric matrix, then this result implies that the matrix $I_n \otimes I_m + \frac{s}{1-s} A$ is $\mathbb{C}$-separable whenever $s/(1-s) \leq \|A\|_2$. Therefore the matrix P_s is $\mathbb{C}$-separable whenever $s \leq s_* := \|A\|_2 / (\|A\|_2 + 1)$. On the other hand, $id \otimes T(P_s) \neq P_s$ for every $s > 0$, and therefore P_s is $\mathbb{R}$-entangled for every $s > 0$. ■

8. Entanglement p-breaking maps

In [28], Horodecki, Shor, and Ruskai introduced the notion of *entanglement breaking* maps, that is, maps that transform every entangled state into a separable state when acting on one of the components. In general, we say that a map $\Phi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$ is $\mathbb{K}$-*entanglement p-breaking* if

$$\Phi \otimes id_r(P) \in \mathrm{SEP}_p(\mathbb{K}^m \otimes \mathbb{K}^r) \quad \text{for all} \quad P \in \mathrm{PSD}(\mathbb{K}^n \otimes \mathbb{K}^r) \text{ and } r \geq 1.$$

For $p = 1$, we just call the map Φ $\mathbb{K}$-entanglement breaking. In the complex case, entanglement p-breaking maps had been considered in [49], where they were called p-superpositive maps.

The cone of $\mathbb{K}$-entanglement breaking ($\mathbb{K}$-entanglement p-breaking) maps $\Phi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$ will be denoted as $\mathbb{K}\text{-}\,\mathrm{EB}(n,m)$ ($\mathbb{K}\text{-}\,\mathrm{EB}_p(m,n)$). We now provide a characterization of the $\mathbb{K}$-entanglement p-breaking channels. Note that since the cone $\mathrm{SEP}_p(\mathbb{K}^m \otimes \mathbb{K}^r)$ is larger for larger p, meaning that $\mathbb{K}$-entanglement p-breaking is a weaker condition than $\mathbb{K}$-entanglement breaking when $p \geq 2$.

In the following result, the complex versions of (1)-(3) and (4) were provided for $p = 1$ in [28] and [32], respectively. The complex version of (5) and (6) was provided in [49] for general p.

Theorem 8.1. *Let $\Phi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$ be a linear map. Then the following are equivalent.*

1. *Φ is $\mathbb{K}$-entanglement p-breaking.*
2. *$C_\Phi \in \mathrm{SEP}_p(\mathbb{K}^n \otimes \mathbb{K}^m)$,*
3. *there exist matrices $(C_i)_{i=1}^k \subset M_{m,n}(\mathbb{K})$ such that $\mathrm{rank}(C_i) \leq p$ for every i and*
$$\Phi(X) = \sum_{i=1}^k C_i X C_i^*,$$
4. *$\Phi = \Delta \circ \Gamma$ where $\Gamma : M_n(\mathbb{K}) \to l_k^\infty(\mathbb{K}) \otimes M_p(\mathbb{K})$ and $\Delta : l_k^\infty(\mathbb{K}) \otimes M_p(\mathbb{K}) \to M_m(\mathbb{K})$ are completely positive maps for some $k \geq 1$.*
5. *for every r and every p-positive map $\Psi : M_m(\mathbb{K}) \to M_r(\mathbb{K})$ the map $\Psi \circ \Phi$ is completely positive,*
6. *for every r and every p-positive map $\Psi : M_r(\mathbb{R}) \to M_n(\mathbb{R})$ the map $\Phi \circ \Psi$ is completely positive.*

Proof. (1) $\Rightarrow$ (2): Applying $id_n \otimes \Phi$ to $X = \sum_{i,j} E_{ij} \otimes E_{ij}$, we obtain that $C_\Phi = (id_n \otimes \Phi)\big(\sum_{ij} E_{ij} \otimes E_{ij}\big)$ belongs to $\mathrm{SEP}_p(\mathbb{K}^n \otimes \mathbb{K}^m)$.

(2) $\Rightarrow$ (3): If $C_\Phi \in \mathrm{SEP}_p(\mathbb{K}^n \otimes \mathbb{K}^m)$, then it can be written as $C_\Phi = \sum_{i=1}^k \mathrm{Vec}(A_i)\,\mathrm{Vec}(A_i)^*$, with $\mathrm{rank}(A_i) \leq p$ for every i. Using Eq. (1), we obtain
$$\begin{aligned}\mathrm{Tr}[\Phi(X)\,Y] &= \mathrm{Tr}(C_\Phi(X^t \otimes Y)) \\ &= \sum_{i=1}^k \langle \mathrm{Vec}(A_i), (X^t \otimes Y)\,\mathrm{Vec}(A_i)\rangle \\ &= \sum_{i=1}^k \mathrm{Tr}(A_i^t X \overline{A}_i Y)\,,\end{aligned}$$
for arbitrary matrices X and Y. Since Y is arbitrary, we conclude $\Phi(X) = \sum_{i=1}^k C_i X C_i^*$ with $C_i := A_i^t$.

(3) $\Rightarrow$ (1): If $\Phi(X) = \sum_{i=1}^k C_i X C_i^*$ with $\mathrm{rank}(C_i) \leq p$, then for every $A \in M_{n,m}(\mathbb{K})$, one has
$$(\Phi \otimes id_n)(\mathrm{Vec}(A)\,\mathrm{Vec}(A)^*) = \sum_i \mathrm{Vec}(C_i A)\,\mathrm{Vec}(C_i A)^*.$$

Since $\operatorname{rank}(C_iA) \le \operatorname{rank}(C_i) \le p$, we deduce that $(\Phi\otimes id_n)(\operatorname{Vec}(A)\operatorname{Vec}(A)^*)$ is $\mathbb{K}$-p-separable. Since A is arbitrary, we conclude that $(\Phi\otimes id_n)VV^*$ is $\mathbb{K}$-p-separable for every vector V, and therefore $(\Phi\otimes id_n)(X)$ is $\mathbb{K}$-p-separable for every positive X. This proves the equivalence of (1), (2), and (3).

(3) $\Rightarrow$ (4): Since $\operatorname{rank}(C_i) \le p$, there exists an isometry $S_i : \operatorname{ran}(C_i) \to \mathbb{K}^p$. Let $\delta_1, \dots, \delta_k$ be the standard basis for $l_k^\infty(\mathbb{K})$. Define

$$\Gamma(X) = \sum_{i=1}^{k} \delta_i \otimes S_iC_iXC_i^*S_i^* \quad \text{and} \quad \Delta\big(\sum_{i=1}^{k}\delta_i\otimes Y_i\big) = \sum_{i=1}^{k} S_i^*Y_iS_i.$$

The form of Γ and Δ guarantees that they are completely positive. By construction, $\Phi = \Delta\circ\Gamma$.

(4) $\Rightarrow$ (5): The action of Δ on a generic element of $l_k^\infty(\mathbb{K})\otimes M_p$ can be expressed as $\Delta(\sum_i \delta_i\otimes Y_i) = \sum_i \Delta_i(Y_i)$, with $\Delta_i : M_p(\mathbb{K}) \to M_m(\mathbb{K})$ completely positive. Since the domain of Δ_i is $M_p(\mathbb{K})$, Δ_i is $\mathbb{K}$-entanglement p-breaking. Hence Δ is also $\mathbb{K}$-entanglement p-breaking. Using the implication (1) $\Rightarrow$ (2), we conclude that each C_{Δ_i} is $\mathbb{K}$-p-separable. For an arbitrary p-positive map $\Psi : M_m(\mathbb{K}) \to M_r(\mathbb{K})$, $(\Psi\circ\Delta_i)$ is completely positive. Indeed, $C_{\Psi\circ\Delta_i} = (id_p\otimes\Psi)C_{\Delta_i}$, and Corollary 6.6 implies that $(id_p\otimes\Psi)C_{\Delta_i}$ is positive whenever C_{Δ_i} is $\mathbb{K}$-p-separable. We conclude that $\Psi\circ\Phi$ is completely positive, as it factors as $\Psi\circ\Phi = (\Psi\circ\Delta)\circ\Gamma$, which is the composition of two completely positive maps.

(5) $\Rightarrow$ (6): If $\Psi\circ\Phi$ is completely positive, then $C_{\Psi\circ\Phi} = (id_m\otimes\Psi)(C_\Phi) \ge 0$. If Ψ is an arbitrary p-positive map, this implies that C_Φ is $\mathbb{K}$-p-separable, by Corollary 6.6. Hence, also C_{Φ^*} is $\mathbb{K}$-p-separable. Using the implication (2) $\Rightarrow$ (1), we obtain that Φ^* is $\mathbb{K}$-entanglement p-breaking. Using the implication (1) $\Rightarrow$ (5), we conclude that $\Psi\circ\Phi^*$ is completely positive for every p-positive Ψ. Now, a generic map Γ is completely positive if and only if its adjoint Γ^* is completely positive (by Lemma 3.1). Hence, the map $\Phi\circ\Psi^*$ is completely positive for every p-positive Ψ. But for an arbitrary p-positive map Ψ, Ψ^* is an arbitrary p-positive map (by Lemma 3.1)). Hence, $\Phi\circ\Psi$ is completely positive for every p-positive Ψ.

(6) $\Rightarrow$ (2): For every map Ψ, the map $\Phi\circ\Psi$ is completely positive if and only if its adjoint $\Psi^*\circ\Phi^*$ is completely positive. If Ψ is an arbitrary p-positive map, then Ψ^* is an arbitrary p-positive map (by Lemma 3.1). Therefore C_{Φ^*} is $\mathbb{K}$-p-separable, by Corollary 6.6. Recalling Eq. (3), we conclude that C_Φ is $\mathbb{K}$-p-separable. ∎

Remark 8.2. If $\Phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$, then the complexification $\widetilde{\Phi}$ is $\mathbb{C}$-entanglement p-breaking if and only if $C_\Phi \in \mathrm{CSEP}_p(\mathbb{R}^n\otimes\mathbb{R}^m)$. Since we have the strict inclusion $\mathrm{SEP}_p(\mathbb{R}^n\otimes\mathbb{R}^m) \subsetneq \mathrm{CSEP}_p(\mathbb{R}^n\otimes\mathbb{R}^m)$, the complexification can be $\mathbb{C}$-entanglement p-breaking even if the original map Φ is not $\mathbb{R}$-entanglement p-breaking. In other words, there exist maps that break $\mathbb{C}$-entanglement without breaking $\mathbb{R}$-entanglement.

Example 8.3. Here we provide an example of a map $\Phi : M_{2p}(\mathbb{R}) \to M_{2p}(\mathbb{R})$ that has a $\mathbb{C}$-entanglement p-breaking complexification, but is not $\mathbb{R}$-entanglement $(2p-1)$-breaking (and therefore is also not entanglement p-breaking). The map is

$$\Phi(A) = O_+ A O_+ + O_- A O_- \,,$$

where $O_\pm$ are the orthogonal matrices $O_+ = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ and $O_- = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}$. Its Choi matrix C_Φ was studied in example 7.1, where we showed that C_Φ is $\mathbb{C}$-p-separable, but $\mathbb{R}$-$(2p-1)$-entangled. By Theorem 8.1, this implies that $\widetilde{\Phi}$ is $\mathbb{C}$-entanglement p-breaking, while Φ is not $\mathbb{R}$-entanglement $(2p-1)$-breaking.

For $p = 1$, the above example shows that a completely positive map can break $\mathbb{C}$-entanglement without breaking $\mathbb{R}$-entanglement.

9. Real version of the PPT-squared conjecture

A positive matrix $P \in \mathrm{PSD}(\mathbb{K}^n \otimes \mathbb{K}^m)$ is called

(1) *PPT* (*positive partial transpose*) if $(\tau_n \otimes id_m)(P) \geq 0$,
(2) *IPT* (*invariant under partial transpose*) if $(\tau_n \otimes id_m)(P) = P$.

A completely positive map $\Phi : M_n(\mathbb{K}) \to M_m(\mathbb{K})$ is called PPT (IPT) if its Choi matrix is PPT (IPT).

Clearly, every entanglement breaking map is PPT: if the Choi matrix is separable, then it is also PPT by the Peres-Horodecki criterion [26, 27, 39]. In the real case, entanglement breaking maps and their complexifications are IPT.

Proposition 9.1. *If $\Phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$ is $\mathbb{R}$-entanglement breaking, then its complexification $\widetilde{\Phi}$ is IPT, and $\tau_m \circ \widetilde{\Phi} = \widetilde{\Phi} \circ \tau_n = \widetilde{\Phi}$.*

Proof. Since $C_\Phi \in \mathrm{SEP}(\mathbb{R}^n \otimes \mathbb{R}^m)$, we can write $C_\Phi = \sum_k A_k \otimes B_k$ for $A_k \in \mathrm{PSD}(\mathbb{R}^n)$ and $B_k \in \mathrm{PSD}(\mathbb{R}^m)$. Recalling that $C_{\widetilde{\Phi}} = C_\Phi$, we obtain

$$\tau_n \otimes id_m(C_{\widetilde{\Phi}}) = \sum_k A_k^t \otimes B_k = \sum_k A_k \otimes B_k = C_{\widetilde{\Phi}} \,,$$

that is $\widetilde{\Phi}$ is IPT. Moreover, one has

$$C_{\tau_m \circ \widetilde{\Phi}} = id_m \otimes \tau_m(C_{\widetilde{\Phi}}) = \sum_k A_k \otimes B_k^t = \sum_k A_k \otimes B_k = C_{\widetilde{\Phi}} \,,$$

which implies $\tau_m \circ \widetilde{\Phi} = \Phi$, and and

$$C_{\widetilde{\Phi} \circ \tau_n} = C_{\tau_m \circ \widetilde{\Phi} \circ \tau_n} = \sum_{i,j} E_{i,j} \otimes (\widetilde{\Phi}(E_{i,j}^t))^t = \sum_{i,j} E_{i,j} \otimes (\Phi(E_{i,j}^t))^t = C_\Phi \,,$$

where the last equality follows from the Hermiticity of Φ. ∎

In a presentation at the workshop "Operator structures in quantum information theory" (Banff Research Station, February 26-March 2 2012) [47], Christandl raised what has later become known as the PPT-squared conjecture.

Conjecture 9.2 (PPT-squared). *For every PPT map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, $\Phi \circ \Phi$ is $\mathbb{C}$-entanglement breaking.*

In [16], Christandl, Muller-Hermes, and Wolf show that if this conjecture is true, then the composition $\Phi \circ \Psi$ is entanglement breaking for any two PPT maps with compatible domain and range. They also show that the conjecture is true for $n = 3$ [16, Corollary 3.1].

The example below shows that a direct real analogue of Conjecture 9.2 is false. Namely, there exist real PPT maps whose square is not $\mathbb{R}$-entanglement breaking, even for $n = 2$.

In [33] it is shown that if $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is idempotent, i.e., $\Phi \circ \Phi = \Phi$, unital, i.e, $\Phi(I_n) = I_n$ and PPT, then $\Phi \circ \Phi$ is $\mathbb{C}$-entanglement breaking. In [33] it is also shown that if Φ is only unital and PPT, then every limit point of the sequence of iterates $\Phi^k = \Phi \circ \cdots \circ \Phi$ is a $\mathbb{C}$-entanglement breaking map. In [41] it is shown that if $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is unital, trace-preserving and PPT, then Φ^k is $\mathbb{C}$-entanglement breaking for some power of Φ.

The example below also shows that the direct real analogues of these results are false, even for $n = 2$.

Example 9.3. Define $\Phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ with

$$C_\Phi = \frac{1}{2}\begin{bmatrix} I_2 & \gamma \\ -\gamma & I_2 \end{bmatrix} \quad \text{where} \quad \gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then Φ is a unital, completely positive, trace preserving map which is idempotent, $\Phi^2 = \Phi$. Moreover, $\operatorname{ran}\Phi = \operatorname{span}\{I_2, \gamma\}$ is a real abelian C*-algebra with the usual product. Since C_Φ and its partial transpose are positive, Φ is PPT. Since C_Φ is not equal to its partial transpose, Corollary 9.1 shows that Φ is not $\mathbb{R}$-entanglement breaking.

However, C_Φ is in $\mathrm{CSEP}(\mathbb{R}^2 \otimes \mathbb{R}^2)$: let $A_\pm = \frac{1}{2}(I_2 \pm i\gamma) \geq 0$. Then

$$A_+ \otimes A_- + A_- \otimes A_+ = \frac{1}{2}(I_2 \otimes I_2 + \gamma \otimes \gamma) = C_\Phi$$

and

$$A_+ \otimes A_+ + A_- \otimes A_- = \frac{1}{2}(I_2 \otimes I_2 - \gamma \otimes \gamma) = C_{\Phi \circ T}$$

Therefore $\widetilde{\Phi}$ is $\mathbb{C}$-entanglement breaking.

With Corollary 9.1 and Example 9.3 in mind, we formulate a real version of the PPT-squared conjecture, which we call the IPT-squared conjecture. This conjecture also makes sense for $\mathbb{C}$. We formulate both versions.

Conjecture 9.4 (IPT-squared for $\mathbb{K}$). *For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and every IPT map $\Phi : M_n(\mathbb{K}) \to M_n(\mathbb{K})$, $\Phi \circ \Phi$ is $\mathbb{K}$-entanglement breaking.*

The following gives a further indication that the IPT-squared conjecture is the correct real counterpart of the original PPT-conjecture.

Proposition 9.5. *If the* PPT^2 *conjecture is true, then* IPT^2 *Conjecture is true for $\mathbb{C}$. If the* IPT^2 *Conjecture is true for $\mathbb{C}$, then it is true for $\mathbb{R}$.*

In particular, in dimension three, the above conjectures are true.

Proof. The first statement is obvious, since Conjecture 9.4 for $\mathbb{C}$ is a special case of Conjecture 9.2.

Assume that Conjecture 9.4 is true for $\mathbb{C}$. Let $\Phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be IPT. Then $\widetilde{\Phi} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is IPT and hence $\widetilde{\Phi} \circ \widetilde{\Phi} = \widetilde{\Phi \circ \Phi}$ is $\mathbb{C}$-entanglement breaking. This means that we can write

$$C_{\Phi\circ\Phi} = C_{\widetilde{\Phi}\circ\widetilde{\Phi}} = \sum_k P_k \otimes Q_k,$$

with $P_k, Q_k \geq 0$ complex matrices. Write $P_k = A_k + iB_k$, $Q_k = C_k + iD_k$, where $A_k, B_k, C_k, D_k \in M_n(\mathbb{R})$ with A_k, C_k symmetric and B_k, D_k skew-symmetric. Also note that $A_k, C_k \geq 0$.

Since $\Phi \circ \Phi$ is IPT,

$$C_{\Phi\circ\Phi} = C_{T\circ\Phi\circ\Phi} = \sum_k P_k \otimes Q_k^t = \sum_k P_k \otimes (C_k - iD_k).$$

Averaging these two expressions yields

$$C_{\Phi\circ\Phi} = \sum_k P_k \otimes C_k = \sum_k A_k \otimes C_k + i \sum_k B_k \otimes C_k.$$

Since $C_{\Phi\circ\Phi}$ is a real matrix, we have that

$$C_{\Phi\circ\Phi} = \sum_k A_k \otimes C_k.$$

Hence, $\Phi \circ \Phi$ is $\mathbb{R}$-entanglement breaking.

Note that, it is known that in demension three (complex case), the PPT-sqaured conjecture is true (see [16] and also [9]). Hence using the earlier proof method, one can show that the IPT^2 Conjecture is true for $\mathbb{R}$ in demension 3. ■

We do not know if these conjectures are equivalent.

We can however establish the three results from [33] and [41] cited above for IPT maps.

Proposition 9.6. *Let $\Phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be unital, trace-preserving and IPT. Then there is a power k such that Φ^k is $\mathbb{R}$-entanglement breaking.*

PROOF. Since $\widetilde{\Phi}$ is unital, trace-preserving and PPT, there is some k so that $(\widetilde{\Phi})^k = \widetilde{\Phi^k}$ is $\mathbb{C}$-entanglement breaking. Using the fact that $\tau_n \circ \Phi = \Phi$ implies $\tau_n \circ \Phi^k = \Phi^k$ and arguing as in the previous proof, we obtain that C_{Φ^k} is $\mathbb{R}$-separable. ∎

Before proving the analogues of the results in [33], an analogue of [33, Lemma 3.1] is required. However it is much more difficult in the real case.

When $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is idempotent and unital, i.e., $\Phi \circ \Phi = \Phi$ and $\Phi(I_n) = I_n$, then the range of Φ is an operator system which becomes a C*-algebra when endowed with the Choi-Effros product $A \star B = \Phi(AB)$. In [33], it is shown that this C*-algebra is abelian. They then use the fact that every finite dimensional complex abelian C*-algebra is of the form $\ell_k^\infty(\mathbb{C})$ and the complex analogue of 8.1(5), to deduce that Φ is $\mathbb{C}$-entanglement breaking.

However, not every finite dimensional real abelian C*-algebra is of the form $\ell_k^\infty(\mathbb{R})$. In particular, in Example 9.3, the range of Φ is $*$-isomorphic to the real abelian C*-algebra

$$\mathfrak{A} = \{f \in C(\{a,b\}) : f(b) = \overline{f(a)}\}.$$

This is a 2-dimensional abelian real C*-algebra which is not isomorphic to $l_2^\infty(\mathbb{R})$ or $l_1^\infty(\mathbb{C})$. Note that Φ is PPT but not IPT.

Theorem 9.7. *Let $\Psi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be a unital IPT idempotent map. Then the operator system $\mathcal{R} := \Psi(M_n(\mathbb{R}))$ with the Choi-Effros product is order isomorphic to $l_k^\infty(\mathbb{R})$ for some $k \geq 1$, and Ψ is $\mathbb{R}$-entanglement breaking.*

PROOF. The Choi-Effros product on $\mathcal{R}$ is given by $A \star B := \Psi(AB)$, and this makes $\mathcal{R}$ into a real C*-algebra $\mathfrak{A}$. The complexification $\widetilde{\Psi}$ is PPT, unital and idempotent. By [33, Lemma 3.1], the Choi-Effros product on $\widetilde{\Psi}(M_n(\mathbb{C}))$ is abelian; and hence $\mathfrak{A}$ is abelian. Let $\Gamma : \mathcal{R} \to \mathfrak{A}$ be this identification, which is a complete isometry; so Γ^{-1} is also completely isometric.

An old result of Arens and Kaplansky shows that if $\mathfrak{A}$ is an abelian real C*-algebra, then there is a locally compact Hausdorff space Z and a homeomorphism $\tau : Z \to Z$ with $\tau^2 = id_Z$ such that

$$\mathfrak{A} \simeq \{f \in C_0(Z) : f(\tau(z)) = \overline{f(z)} \text{ for all } x \in Z\}.$$

See [44, Theorem 1.9] for an easy proof. In our case, since $\mathfrak{A}$ is finite dimensional, X is a finite set with the discrete topology. Hence

$$Z = \{x_j, y_k, z_k : 1 \leq j \leq J, 1 \leq k \leq K\}$$

such that $\tau(x_j) = x_j$, $\tau(y_k) = z_k$ and $\tau(z_k) = y_k$. Thus

$$\mathfrak{A} = \Big\{ \sum_j a_j \delta_{x_j} + \sum_k b_k \delta_{y_k} + \overline{b_k} \delta_{z_k} : a_j \in \mathbb{R},\, b_k \in \mathbb{C} \Big\}.$$

A basis for $\mathfrak{A}$ as a real vector space is given by $\{\delta_{x_j}, \delta_{y_k} + \delta_{z_k}, i(\delta_{y_k} - \delta_{z_k})\}$. Write these vectors as $\{\alpha_j, \beta_k, \gamma_k\}$. Then $\alpha_j^* = \alpha_j$, $\beta_k^* = \beta_k$ and $\gamma_k^* = -\gamma_k$.

The map $\Phi = \Gamma\Psi : M_n(\mathbb{R}) \to \mathfrak{A}$ can be written as

$$\Phi(X) = \sum_j \mathrm{Tr}(XA_j)\alpha_j + \sum_k \mathrm{Tr}(XB_k)\beta_k + \mathrm{Tr}(XG_k)\gamma_k$$

for matrices $A_j, B_k, G_k \in M_n(\mathbb{R})$. Since Φ is CP,

$$X \geq 0 \implies \mathrm{Tr}(XA_j) \geq 0,\ \mathrm{Tr}(XB_k) \geq 0 \text{ and } \mathrm{Tr}(XG_k) = 0.$$

Therefore $A_j, B_k \in \mathrm{PSD}(\mathbb{R}^n)$ and $G_k^t = -G_k$, since G_k is orthogonal to all positive matrices, and hence to all symmetric matrices.

Let $P_j = \Gamma^{-1}(\alpha_j)$, $Q_k = \Gamma^{-1}(\beta_k)$ and $R_k = \Gamma^{-1}(\gamma_k)$. Then P_j, Q_k are positive, and $R_k^t = -R_k$ because Γ and Γ^{-1} are positive. This is a real basis for $\mathcal{R}$. Now we have

$$\Psi(X) = \Gamma^{-1}\Phi(X) = \sum_j \mathrm{Tr}(XA_j)P_j + \sum_k \mathrm{Tr}(XB_k)Q_k + \mathrm{Tr}(XG_k)R_k.$$

The Choi matrix is

$$C_\Psi = \sum_j A_j \otimes P_j + \sum_k B_k \otimes Q_k + \sum_k G_k \otimes R_k.$$

The partial transpose of this map is

$$C_{T\circ\Psi} = \sum_j A_j \otimes P_j + \sum_k B_k \otimes Q_k - \sum_k G_k \otimes R_k.$$

However, $C_{T\circ\Psi} = C_\Psi$ since Ψ is IPT. So $G_k = 0$. Therefore C_Ψ is separable, and hence Ψ is $\mathbb{R}$-entanglement breaking by Theorem 8.1.

Now Γ is an isomorphism, and in particular it is surjective. Therefore $K = 0$ and $Z = \{x_j : 1 \leq j \leq J\}$. Hence $\mathfrak{A} \simeq C_{\mathbb{R}}(Z) \simeq l^\infty_J(\mathbb{R})$. ∎

Now the asymptotic result follows as in [33] by applying Ellis's Theorem that every compact semigroup contains idempotents (and its proof).

Theorem 9.8. *Let $\Phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be a unital IPT map. Then*

$$\lim_{k\to\infty} \mathrm{dist}(\Phi^k, \mathbb{R}\text{-}\,\mathrm{EB}(n,n)) = 0.$$

Acknowledgments

GC was supported by the Hong Kong Research Grant Council through grant 17300918 and through the Senior Research Fellowship Scheme via SRFS2021-7S02, and by the Croucher foundation. KRD and VIP were partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC). MR is supported by the European Research Council (ERC Grant Agreement No. 851716).

References

[1] A. Aleksandrova, V. Borish, and W.K. Wootters, *Real-vector-space quantum theory with a universal quantum bit*, Phys. Rev. A **87** (2013), 052106.

[2] H. Araki, *On a characterization of the state space of quantum mechanics*, Comm. Math. Phys. **75** (1980), no. 1, 1–24.

[3] W.B. Arveson, *Subalgebras of C^*-algebras*, Vol. 123, 1969.

[4] K. Audenaert and B. de Moor, *Optimizing completely positive maps using semidefinite programming*, Physical Review A **65** (2002).

[5] J.S Bell, *On the Einstein Podolsky Rosen paradox*, Phys. Phys. Fiz. **1** (1964), 195.

[6] D.P. Blecher and W. Tepsan, *Real Operator Algebras and Real Positive Maps*, Integral Equations and Operator Theory **93:49** (2021).

[7] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, *Bell nonlocality*, Rev. Modern Phys. **86** (2014), 419.

[8] L. Chen and D. Đoković, *Length filtration of the separable states*, Proc. A. **472** (2016), 20160350, 13.

[9] L. Chen, Y. Yang, and W. Tang, *The positive partial transpose conjecture for n=3*, Phys. Rev. A **99** (2019).

[10] G. Chiribella, G.M. D'Ariano, and P. Perinotti, *Probabilistic theories with purification*, Phys. Rev. A **81** (2010), 062348.

[11] Giulio Chiribella, Giacomo M. D'Ariano, and Paolo Perinotti, *Informational derivation of quantum theory*, Phys. Rev. A (2011), 012311.

[12] G. Chiribella, *Process tomography in general physical theories*, Symmetry **13** (2021), 1985.

[13] G. Chiribella, Y. Yang, and A.C.-C. Yao, *Quantum replication at the Heisenberg limit*, Nat. Commun. **4** (2013).

[14] G. Chiribella, K.R. Davidson, V.I. Paulsen, and M. Rahaman, *Counterexamples to the extendibility of positive unital norm-one maps*, arXiv e-prints (2022), available at `arXiv:2204.08819`.

[15] M.-D. Choi, *Completely positive linear maps on complex matrices*, Linear algebra Appl. **10** (1975), 285–290.

[16] M. Christandl, A. Muller-Hermes, and M. Wolf, *When do composed maps become Entanglement Breaking*, Annales Henri Poincaré **20** (2019), 2295–2322.

[17] D. Chruściński and A. Kossakowski, *Spectral conditions for positive maps*, Comm. Math. Phys. **290** (2009), 1051–1064.

[18] L. Clarisse, *Characterization of distillability of entanglement in terms of positive maps*, Phys. Rev. A **71** (2005), 032332.

[19] B Collins, P. Hayden, and I. Nechita, *Random and free positive maps with application to entanglement detection*, Int. Math. Res. Not. IMRN (2017), 869–894.

[20] J. de Pillis, *Linear transformations which preserve hermitian and positive semidefinite operators*, Pacific J. Math. **23** (1967), 129–137.

[21] L. Hardy, *Quantum theory from five reasonable axioms*, https://arxiv.org/abs/quant-ph/0101012.

[22] ———, *Reformulating and reconstructing quantum theory*, https://arxiv.org/abs/1104.2066.

[23] L. Hardy and W.K. Wootters, *Limited holism and real-vector-space quantum theory*, Found. Phys. **42** (2012), 454–473.

[24] B. Dakić and Č. Brukner, *Quantum theory and beyond: Is entanglement special?*, Deep beauty, 2011, pp. 365–391.

[25] L. Masanes and M.P. Müller, *A derivation of quantum theory from physical requirements*, New J. Phys. **13** (2011).

[26] M. Horodecki, P. Horodecki, and R. Horodecki, *Separability of Mixed States: Necessary and Sufficient Conditions*, Physics Letters A **223** (1996), 1–8.

[27] Pawel Horodecki, *Separability criterion and inseparable mixed states with positive partial transposition*, Phys. Lett. A **232** (1997), 333–339.
[28] M. Horodecki, P. Shor, and M.B. Ruskai, *Entanglement Breaking Channels*, Reviews in Mathematical Physics **15** (2003), 629–641.
[29] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Quantum entanglement*, Rev. Modern Phys. **81** (2009), 865.
[30] L. Gurvits and H. Barnum, *Largest separable balls around the maximally mixed bipartite quantum state*, Phys. Rev. A **66** (2002), 062311.
[31] A. Jamiołkowski, *Linear transformations which preserve trace and positive semidefiniteness of operators*, Reports on Mathematical Physics **3** (1972), 275–278.
[32] N. Johnston, D.W. Kribs, V.I. Paulsen, and R. Pereira, *Minimal and maximal operator spaces and operator systems in entanglement theory*, J. Funct. Anal. **260** (2011), 2407–2423.
[33] M. Kennedy, N.A. Manor, and V.I. Paulsen, *Composition of PPT maps*, Quantum Inf. Comput. **18** (2018), 472–480.
[34] L. Masanes, M.P. Müller, R. Augusiak, and D. Pérez-García, *Existence of an information unit as a postulate of quantum theory*, Proc. Natl. Acad. Sci. USA **110** (2013), 16373–16377.
[35] M. Navascués and T. Vértesi, *Activation of nonlocal quantum resources*, Phys. Rev. Lett. **106** (2011), 060403.
[36] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2011.
[37] C. Palazuelos, *Superactivation of quantum nonlocality*, Phys. RevL̇ett. **109** (2012), 190401.
[38] V.I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002.
[39] A. Peres, *Separability criterion for density matrices*, Phys. Rev. Lett. **77** (1996), 1413.
[40] S. Popescu, *Bell's inequalities and density matrices: revealing "hidden" nonlocality*, Phys. Rev. Lett. **74** (1995), 2619.
[41] M. Rahaman, S. Jaques, and V.I. Paulsen, *Eventually Entanglement Breaking Maps*, J. Math. Phys. (2018), 062201, 11.
[42] K. Ranade and M. Ali, *The Jamiołkowski isomorphism and a simplified proof for the correspondence between vectors having Schmidt number k and k-positive maps*, Open Syst. Inf. Dyn. **14** (2007), 371–378.
[43] M. Renou et al., *Quantum theory based on real numbers can be experimentally falsified*, Nature **600** (2021), 625–629.
[44] J. Rosenberg, *Structure and applications of real C^*-algebras*, Operator algebras and their applications, 2016, pp. 235–258.
[45] Z.-J. Ruan, *On Real Operator Spaces*, Acta Mathematica Sinica, English Series **19** (2003), 485–496.
[46] ———, *Complexifications of real operator spaces*, Illinois Journal of Mathematics **47** (2003), 1047–1062.
[47] M.B. Ruskai, M. Junge, D. Kribs, P. Hayden, and A. Winter, *Operator structures in quantum information theory,* BIRS Report 12w5084, Banff International Research Station, 2012.
[48] A. Sanpera, D. Bruß, and M. Lewenstein, *Schmidt-number witnesses and bound entanglement*, Phys. Rev. A **63** (2001), 050301.
[49] Ł. Skowronek, E. Størmer, and K.l Życzkowski, *Cones of positive maps and their duality relations*, J.f Math.łPhys. **50** (2009), 062106.
[50] E. Størmer, *Extension of positive maps*, Math. Scand. **126** (2020), 256–258.
[51] ———, *Positive linear maps of operator algebras*, Springer Monographs in Mathematics, Springer, Heidelberg, 2013.
[52] ———, *Extension of positive maps* (2018), available at `arXiv:1805.05148`.

[53] E.C.G. Stueckelberg, *Quantum theory in real Hilbert space*, Helv. Phys. Acta **33.727** (1960), 458.
[54] T. Takasaki and J. Tomiyama, *On the geometry of positive maps in matrix algebras*, Math. Z. **184** (1983), 101–108.
[55] B.M. Terhal and P. Horodecki, *Schmidt number for density matrices*, Phys. Rev. A **61** (2000), 040301.
[56] B.M. Terhal, *Bell inequalities and the separability criterion*, Phys. Lett. A **271** (2000), 319–326.
[57] J. Watrous, *The Theory of Quantum Information*, Cambridge University Press, 2018.
[58] R.F. Werner, *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model*, Phys. Rev. A **40** (1989), 4277–4281.
[59] W.K. Wootters, *Local accessibility of quantum states*, Complexity, entropy and the physics of information **8** (1990), 39–46.
[60] ———, *Entanglement sharing in real-vector-space quantum theory*, Foundations of Physics **42** (2012), 19–28.
[61] ———, *The rebit three-tangle and its relation to two-qubit entanglement*, J. Phys. A **47** (2014), 424037.
[62] K.-D. Wu, T.V. Kondra, S. Rana, C.M. Scandolo, G.-Y. Xiang, C.-F. Li, G.-C. Guo, and A. Streltsov, *Operational resource theory of imaginarity*, Phys. Rev. Lett. **126** (2021), 090401.

Department of Computer Science, The University of Hong Kong, Pok Fu Lam Road, Hong Kong
Email address: `giulio.chiribella@cs.ox.ac.uk`

Department of Pure Mathematics, University of Waterloo, Waterloo, ON, Canada N2L 3G1
Email address: `krdavidson@uwaterloo.ca`

Institute for Quantum Computing and Department of Pure Mathematics, University of Waterloo, Waterloo, Waterloo, ON, Canada N2L 3G1
Email address: `vpaulsen@uwaterloo.ca`

Univ Lyon, ENS Lyon, UCBL, CNRS, Inria, LIP, F-69342, Lyon Cedex 07, France
Email address: `mizanur.rahaman@ens-lyon.fr`