PM 950 **Assignment 3** **Due Thursday November 27**

1. Suppose that $\mathfrak{A}$ is a unital subalgebra of $\mathcal{M}_n$ such that $\operatorname{Lat}\mathfrak{A} = \{0, \mathbb{C}^n\}$ is trivial. (Such an algebra is called *transitive*.) Show that $\mathfrak{A} = \mathcal{M}_n$ as follows:
   (a) Show that if T commutes with $\mathfrak{A}$, then T is scalar. **Hint:** consider $\ker(T - \lambda I)$.
   (b) Show that if x and y are linearly independent vectors, then there is an $A \in \mathfrak{A}$ such that $Ax = 0$ and $Ay \neq 0$.
   Hint: otherwise $T(Ax) = Ay$ defines a well defined operator in $\mathfrak{A}'$.
   (c) Construct all of the matrix units of $\mathcal{M}_n$ in $\mathfrak{A}$.

2. Show that every unital subalgebra of $\mathcal{M}_2$ is similar to one of five possible algebras, and determine which are reflexive.

3. Let P be the $n \times n$ projection matrix with $p_{ij} = 1/n$ for all $1 \leq i, j \leq n$. Compute $\operatorname{dist}(P, \mathcal{T}_n)$, and find an explicit upper triangular matrix T so that $\|P - T\| = \operatorname{dist}(P, \mathcal{T}_n)$.

4. (a) Let $\mathcal{N}$ be a nest of order type $\omega = \mathbb{N} \cup \{\infty\}$. Given $K \in \mathcal{K}(\mathcal{H})$, find $T \in \mathcal{T}(\mathcal{N}) \cap \mathcal{K}$ such that $\|K - T\| = \operatorname{dist}(K, \mathcal{T}(\mathcal{N}))$.
   Hint: find $N_0 < \mathcal{H}$ in $\mathcal{N}$ so that $\|P_{N_0}^\perp K\| \leq \frac{1}{2} \operatorname{dist}(K, \mathcal{T}(\mathcal{N}))$, and use this to reduce the problem to a finite nest.
   (b) Let $\mathcal{N}$ be the Volterra nest on $L^2(0,1)$, and let $C = 11^*$. Compute $\operatorname{dist}(C, \mathcal{T}(\mathcal{N}))$ and show that there is no closest compact operator.
   Hint: suppose that $T \in \mathcal{T}(\mathcal{N})$ satisfies $\|C - T\| = \operatorname{dist}(C, \mathcal{T}(\mathcal{N}))$. Let $x = \sqrt{2}\chi_{(0,1/2)}$. Show that $P_{N_t}^\perp T P_{N_t}^\perp x = \frac{1}{2} P_{N_t}^\perp x$ for all $t \in [0, 1/2)$.
   (c) Show that one closest element of $\mathcal{T}(\mathcal{N})$ to C is $V + D$ where V is the Volterra operator $Vf(t) = \int_t^1 f(x)\,dx$ and $D = M_h$ where $h(x) = \min\{x, 1-x\}$. **Hint:** $C = V + V^*$. Chop the nest into 2^n equal pieces. Use Q3 to find a block upper triangular operator which achieves the distance to the block upper triangulars. Take a limit.

5. Let $\mathcal{M} = \{M_n\}_{n\geq 1} \cup \{\mathcal{H}\}$ and $\mathcal{N} = \{N_n\}_{n\geq 1} \cup \{\mathcal{H}\}$ be two order ω nests such that $\dim M_n = \dim N_n = n$ for $n \geq 1$ and $\lim_{n\to\infty} \|P_{M_n} - P_{N_n}\| = 0$.
   (a) Show explicitly how to construct a sequence of unitaries U_k such that $U_k - I$ are finite rank so that $\lim_{k\to\infty} \sup_{n\geq 1} \|U_k P_{M_n} - P_{N_n} U_k\| = 0$.
   (b) Hence show that there is an invertible operator S with $S - I$ compact such that $SM_n = N_n$ for all $n \geq 1$.
   (c) Also show that there is a unitary operator U such that $UM_n = N_n$ for all $n \geq 1$.
   (d) **Bonus** Are there examples where you cannot take $U - I$ in the compacts?

6. Let $\mathcal{N}$ be the Volterra nest. Let $\mathcal{D}(\mathcal{N}) = \mathcal{T}(\mathcal{N}) \cap \mathcal{T}(\mathcal{N})^*$.
   (a) Show that there are operators A and B in $\mathcal{T}(\mathcal{N})$ such that $(AB - BA)^2 = I$.
   Hint: find A and B in $\mathcal{T}(\mathcal{N} \oplus \mathcal{N})$ first, where $\mathcal{N} \oplus \mathcal{N} = \{N_t \oplus N_t : 0 \leq t \leq 1\}$.
   (b) Show that there is no ideal $\mathcal{I}$ of $\mathcal{T}(\mathcal{N})$ such that $\mathcal{T}(\mathcal{N}) = \mathcal{D}(\mathcal{N}) + \mathcal{I}$.