PM 950 Assignment 1 Due Tuesday Sept. 29.

1. Let A be a function algebra. Find all extreme points of the unit ball of A^*.

2. Prove that if X is metrizable, and A is a function algebra on X, then $\mathrm{Ch}(A)$ is the unique minimal boundary set for A.

3. Let Λ be an uncountable set, let $X = \prod_{\lambda\in\Lambda}[0,1]$, and let $A = \mathrm{C}(X)$. A point in X may be written $x = (x_\lambda)_{\lambda\in\Lambda}$.
   (a) Show that every $f \in A$ depends on only countably many coordinates.
   (b) Show that every point of X is a p-point.
   (c) Show that $E = \{x \in X : x_\lambda = 0 \text{ except countably often}\}$ is a boundary for A.
   (d) Find another boundary which is disjoint from E, showing that there is no minimal boundary.

4. Let $0 < \alpha < 1$ be an irrational number. Consider $\mathbb{T}^2$ as a subset of $\mathbb{C}^2$. Let A_α be the subalgebra of $\mathrm{C}(\mathbb{T}^2)$ spanned by $\{z^m w^n : m, n \in \mathbb{Z} \text{ and } m + n\alpha \geq 0\}$.
   (a) Show that A is a Dirichlet algebra, i.e., $\{f + \bar{g} : f, g \in A\}$ is dense in $\mathrm{C}(\mathbb{T}^2)$.
   (b) Let $\varphi \in \mathcal{M}_{A_\alpha}$ and let $r = |\varphi(z)|$. Prove that $|\varphi(w)| = r^\alpha$.
   (c) Prove that if $f(z)$ is a bounded analytic function on the right half plane H_+ and continuous on $\overline{H_+}$, then $\sup_{z\in H_+} |f(z)| = \sup_{y\in\mathbb{R}} |f(iy)|$. **Hint:** Fix a branch $L(z)$ of $\log(z + 1)$ on H_+. For $\varepsilon > 0$, let $g_\varepsilon(z) = f(z)e^{-\varepsilon L(z)}$. This is continuous at ∞.
   (d) If $0 < r \leq 1$ and $\theta, \eta \in [0, 2\pi)$, show that there is a $\varphi \in \mathcal{M}_{A_\alpha}$ such that $\varphi(z) = re^{i\theta}$ and $\varphi(w) = r^\alpha e^{i\eta}$. **Hint:** if $p(z, w) = \sum a_{m,n} z^m w^n$ is a finite sum of monomials in A_α, define an analytic function at $u = x + iy \in H_+$ by $f(u) = \sum a_{m,n} e^{i(m\theta+n\eta)} e^{-(m+n\alpha)u} = \sum a_{m,n} e^{-(m+n\alpha)x} e^{i(\theta-y)m} e^{i(\eta/\alpha-y)n\alpha}$. Note that $\varphi(p) = f(-\log r)$.
   (e) Decribe $\mathcal{M}_{A_\alpha}$.

5. Let μ be a finite measure on $\mathbb{C}$ with compact support K. Define $\hat{\mu}(w) = \int_K \frac{d\mu(z)}{z-w}$.
   (a) Show that $\hat{\mu}$ is analytic on $\mathbb{C} \setminus K$ and $\lim_{|w|\to\infty} \hat{\mu}(w) = 0$.
   (b) Show that $\hat{\mu}(w)$ is defined and finite m_2 a. e., where m_2 is planar Lebesgue measure. **Hint:** estimate $\int_K |\hat{\mu}(w)|\, dm_2(w)$.
   (c) Show that if $X \subset \mathbb{C}$ is a compact set and $\mu \in M(X)$, then $\mu \perp R(X)$ if and only if $\hat{\mu} = 0$ on $\mathbb{C} \setminus X$.

6. Let $X \subset \mathbb{C}$ be a compact set. Show that $R(X)$ is doubly generated as a function algebra. **Hint:** Fix a sequence a_n, $n \geq 1$, consisting of one point in each bounded component of $\mathbb{C} \setminus X$. Choose $\varepsilon_n > 0$ decreasing sufficiently rapidly to 0, and define $f_1(z) = z$ and $f_2(z) = \sum_{n\geq 1} \frac{\varepsilon_n}{z-a_n}$. Let B be the function algebra generated by f_1 and f_2. If $z - a_1$ is not invertible in B, take $\varphi \in \mathcal{M}_B$ with $\varphi(z) = a_1$; and estimate $|\varphi((z - a_1)f_2)|$. If you have defined (ε_n) properly, you will get a contradiction.