1. (a) Let's write elements of l_p as $\mathbf{a} = (a_i)_{i\geq 1}$. Let $\mathbf{e}_i$, $i \geq 1$, denote the standard basis for l^p. Observe that evaluation $\varepsilon_i(\mathbf{a}) = a_i$ is multiplicative for each $i \geq 1$. Suppose that φ is a multiplicative linear functional. Since $\operatorname{span}\{\mathbf{e}_i : i \geq 1\}$ is dense in l^p, there is some j so that $\varphi(\mathbf{e}_j) = \lambda \neq 0$. Then if $i \neq j$, $0 = \varphi(\mathbf{e}_i\mathbf{e}_j) = \varphi(\mathbf{e}_i)\lambda$; whence $\varphi(\mathbf{e}_i) = 0$. While $\lambda^2 = \varphi(\mathbf{e}_j^2) = \varphi(\mathbf{e}_j) = \lambda$ shows that $\varphi(\mathbf{e}_j) = 1$. Thus φ agrees with ε_j on $\operatorname{span}\{\mathbf{e}_i : i \geq 1\}$, and hence on all of l_p. So $\mathcal{M}_{l^p} = \{\varepsilon_i : i \geq 1\}$.

   (b) The set $J = \operatorname{span}(l_p)^2$ is an ideal. It is proper because it lies in the subspace $l_{p/2}$; that is, if $\mathbf{a} = (a_i)$ and $\mathbf{b} = (b_i)$ are in l_p, then by Cauchy-Schwartz,
$$\sum |a_ib_i|^{p/2} \leq \Big(\sum |a_i|^p\Big)^{1/2}\Big(\sum |b_i|^p\Big)^{1/2} < \infty.$$
   This extends to the span by standard methods even when $p < 2$. The algebra l_p/J has the trivial product, and thus every subspace is an ideal. Consequently any subspace of l_p containing J is an ideal. Take a subspace M of codimension 1 containing J. This is then a maximal ideal. However it is not closed because J is dense, and is not modular because l_p/M has the trivial multiplication and so it isn't unital.

2. If $f \in \mathcal{I}_0(E)$, let U be open containing E such that $f|_U = 0$. For each $x \in X \setminus U$, there is a function $g_x \in \mathcal{I}$ such that $g_x(x) \neq 0$. So $V_x = g_x^{-1}(\mathbb{C} \setminus \{0\})$ is an open cover of $X \setminus U$. Let $V_{x_1}, \dots, V_{x_n}$ be a finite open cover. Then $g = \sum_{i=1}^n g_{x_i}\overline{g_{x_i}}$ is an element of $\mathcal{I}$ which is strictly positive on $X \setminus U$. Therefore $h = \begin{cases} f(x)/g(x) & \text{for } x \in X \setminus U \\ 0 & \text{for } x \in U \end{cases}$ is continuous on X. Hence $f = hg$ belongs to $\mathcal{I}$. The second inclusion is trivial.

3. (a) If φ is a multiplicative linear functional on $\mathfrak{B}$, then $\varphi\theta$ is multiplicative on $\mathfrak{A}$. Thus if $a \in \operatorname{rad}\mathfrak{A}$, then $0 = \varphi\theta(a)$ for all φ. So $\sigma(\theta(a)) = \{\varphi(\theta(a)\} = \{0\}$. Since $\mathfrak{B}$ is semisimple, $\theta(a) = 0$. Since θ is injective, $a = 0$. So $\mathfrak{A}$ is also semisimple.

   (b) Point evaluation ε_x for $x \in [0,1]$ are clearly multiplicative. These separate points, and so $\operatorname{rad} C^1[0,1] = \{0\}$; i.e., it is semisimple. The closed subspace $\mathcal{J}$ is an ideal because if $f \in \mathcal{J}$ and $g \in C^1[0,1]$, $fg(0) = 0$ and $(fg)'(0) = f'(0)g(0) + f(0)g'(0) = 0$. So $\mathfrak{B} = \mathfrak{A}/\mathcal{J}$ is a Banach algebra. Let φ be a multiplicative functional on $\mathfrak{B}$. Then φq is multiplicative on $\mathfrak{A}$, where q is the quotient map. This must coincide with some ε_x which vanishes on $\mathcal{J}$, and hence it must be ε_0. Thus there is a unique multiplicative functional. Therefore $\operatorname{rad}\mathfrak{B} = \ker\varphi$ is proper.

   Remark: The map taking f to $\begin{bmatrix} f(0) & f'(0) \\ 0 & f(0) \end{bmatrix}$ is a homomorphism with kernel $\mathcal{J}$. So $\mathfrak{B}$ is isomorphic to the matrices of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ and the radical has the form $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$.

4. (a) Consider the ideal generated by $\{a_i - \lambda_i : 1 \leq i \leq n\}$. If this is a proper ideal, then it is contained in a maximal ideal (by Zorn's Lemma), say $\ker\varphi$ for $\varphi \in \mathcal{M}(A)$. Thus $0 = \varphi(a_i - \lambda_i) = \varphi(a_i) - \lambda_i$; whence $\lambda \notin \sigma(a_1, \dots, a_n)$. In this case, the ideal does not contain 1, and so there are no b_i's satisfying the identity.

   On the other hand, if it is not a proper ideal, then it does contain 1. So there are $b_1, \dots, b_n \in A$ such that $\sum_{i=1}^n (a_i - \lambda_i)b_i = 1$. In this case, there is no maximal ideal containing all $\{a_i - \lambda_i : 1 \leq i \leq n\}$; so $\lambda \notin \sigma(a_1, \dots, a_n)$.

(b) The map Φ is continuous by definition of the weak-$*$ topology, and the range is compact. Because $a_1, \dots, a_n$ generate A, each $\varphi \in \mathcal{M}(A)$ is determined uniquely by its values on $a_1, \dots, a_n$. Therefore Φ is one-to-one. A continuous map of one compact Hausdorff space onto another is a homeomorphism.

(c) If $\lambda \notin \sigma(a_1, \dots, a_n)$, use part (a) to find $b_1, \dots, b_n \in A$ so that $\sum_{i=1}^n (a_i - \lambda_i) b_i = 1$. As $a_1, \dots, a_n$ generate A, there are polynomials $p_i \in \mathbb{C}[z_1, \dots, z_n]$ so that $\|b_i - p_i(a_1, \dots, a_n)\|$ are sufficiently small so that

$$\left\|1 - \sum_{i=1}^n (a_i - \lambda_i) p_i(a_1, \dots, a_n)\right\| < \frac{1}{2}.$$

Let $p(z_1, \dots, z_n) = 1 - \sum_{i=1}^n (z_i - \lambda_i) p_i(z_1, \dots, z_n)$. Then for any $z \in \sigma(a_1, \dots, a_n)$, we have $|p(z)| = |\hat{z}(p(a_1, \dots, a_n))| \le \|p(a_1, \dots, a_n)\| < 1/2$ while $p(\lambda) = 1$. So λ is not in the polynomial convex hull of $\sigma(a_1, \dots, a_n)$. Hence the spectrum is polynomially convex.

Alternatively: suppose that $\lambda \in \mathbb{C}^n$ is in the polynomially convex hull of $\sigma(a)$. Then

$$|p(\lambda)| \le \sup_{z \in \sigma(a)} |p(z)| = \sup_{\varphi \in \mathcal{M}(A)} |\varphi(p(a))| \le \|p(a)\| \quad \text{for all} \quad p \in \mathbb{C}[z_1, \dots, z_n].$$

Thus evaluation at λ is a continuous multiplicative functional on the polynomials in $a_1, \dots, a_n$. So it extends to a character ψ on A by continuity. Thus $\lambda = \psi(a_1, \dots, a_n)$ belongs to $\sigma(a)$.

5. (a) Let π be an irreducible representation of $\mathfrak{A}/\operatorname{rad}\mathfrak{A}$ and let $q : \mathfrak{A} \to \mathfrak{A}/\operatorname{rad}\mathfrak{A}$ be the quotient map. Then πq is an irreducible representation of $\mathfrak{A}$. Conversely every irreducible representation σ of $\mathfrak{A}$ has kernel containing $\operatorname{rad}\mathfrak{A}$. Thus there is a representation π of $\mathfrak{A}/\operatorname{rad}\mathfrak{A}$ given by $\pi(q(a)) = \sigma(a)$. This is irreducible and $\sigma = \pi q$. Thus

$$\operatorname{rad}(\mathfrak{A}/\operatorname{rad}\mathfrak{A}) = \bigcap \ker \pi = q(\bigcap \ker \pi q) = q(\operatorname{rad}\mathfrak{A}) = \{0\}.$$

So $\mathfrak{A}/\operatorname{rad}(\mathfrak{A})$ is semisimple.

(b) Let (X, π) be an irreducible $\mathfrak{A}$ module. Then $\mathfrak{J}X$ is a submodule. So either it is $\{0\}$ or X. In the second case, X is irreducible for $\mathfrak{J}$ as well. To see this, let $x_0 \ne 0$ and let $x \in X$. There is some $b \in \mathfrak{J}$ and some $x_1 \in X$ so that $bx_1 = x$. Choose $a \in \mathfrak{A}$ so that $ax_0 = x_1$. Then $bax_0 = x$. Since $ba \in \mathfrak{J}$, we see that $\mathfrak{J}$ acts transitively and thus X is irreducible as a $\mathfrak{J}$-module. Conversely, if X is an irreducible $\mathfrak{J}$-module, make it into an $\mathfrak{A}$ module as follows: fix $x_0 \ne 0$ and identify X with $\mathfrak{J}/J_{x_0}$ where here $J_{x_0} = \{a \in \mathfrak{J} : ax_0 = 0\}$. We can make X into an $\mathfrak{A}$ module by defining $a\dot{j} := (\dot{aj})$. This is clearly still irreducible.

Now if $a \in \operatorname{rad}\mathfrak{J}$ and π is an irreducible representation of $\mathfrak{A}$, then it restricts to zero or an irreducible representation of $\mathfrak{J}$ and thus $\pi(a) = 0$. So $\operatorname{rad}\mathfrak{J} \subset \mathfrak{J} \cap \operatorname{rad}\mathfrak{A}$. Conversely, if $a \in \mathfrak{J} \cap \operatorname{rad}\mathfrak{A}$ and π is an irreducible representation of $\mathfrak{J}$, then by the previous paragraph, it extends to an irreducible representation of $\mathfrak{A}$. Thus $\pi(a) = 0$ and hence $a \in rad\mathfrak{J}$.

Remark: One can work instead with maximal modular left ideals. If I is a maximal modular left ideal of $\mathfrak{A}$, then $I \cap \mathfrak{J}$ is either $\mathfrak{J}$ or maximal modular. Conversely if J is a maximal modular left ideal of $\mathfrak{J}$ with right modular unit u, then $I = \{a \in \mathfrak{A} : au \in J\}$ is a maximal modular left ideal of $\mathfrak{A}$. The details are left to you.

6. (a) If (X, π) is an irreducible representation of $\mathfrak{B}$, then $\mathfrak{B}$ acts transitively on X. Since θ is surjective, $\pi\theta(\mathfrak{A}) = \pi(\mathfrak{B})$ acts transitively on X, and hence $\pi\theta$ is irreducible. Thus $\operatorname{rad}\mathfrak{A} \subset \bigcap_{\pi \in \operatorname{irred}(\mathfrak{B})} \ker \pi\theta$. Thus $\theta(\operatorname{rad}\mathfrak{A}) \subset \bigcap_{\pi \in \operatorname{irred}(\mathfrak{B})} \ker \pi = \operatorname{rad}\mathfrak{B}$.

(b) Since $\mathfrak{B}$ is semisimple, $\bigcap_{\pi \in \operatorname{irred}(\mathfrak{B})} \ker \pi = \{0\}$. Each $\pi\theta$ is irreducible, and thus its kernel is primitive and thus closed. So we have $\ker \theta = \bigcap_{\pi \in \operatorname{irred}(\mathfrak{B})} \ker \pi\theta$ is closed.