1. If $\overline{b_r(z_0)} \subset \Omega$ and $\varphi \in X^*$, $\{\varphi\big(\frac{1}{h}(f(z_0+h) - f(z_0))\big) : |h| \le r\}$ is bounded because $\varphi \circ f$ is analytic. So $\{\frac{1}{h}(f(z_0+h) - f(z_0)) : |h| \le r\}$ is bounded by the Uniform Boundedness Principle (Banach-Steinhaus Theorem). In particular, f is continuous at z_0 for each $z_0 \in \Omega$. Define $x_n = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})(re^{it})^{-n}\,dt$. Complex analysis shows that $\varphi \circ f(z_0+z) = \sum_{n\ge0}\varphi(x_n)z^n$ for all $|z| \le r$. Thus the Uniform Boundedness Principle implies that $\sup_{n\ge0}\|x_n\|r^n < \infty$, so that the series converges in X for $|z| < r$; and the Hahn-Banach Theorem implies that $f(z_0+z) = \sum_{n\ge0}x_nz^n$. Hence f is strongly analytic.
   Some people failed to observe that the series actually converges on a disc.

2. Let f be analytic on $U \supset \sigma(a)$ and g analytic on $V \supset f(\sigma(a)) = \sigma(f(a))$. By the spectral mapping theorem, $\sigma(f(a)) = f(\sigma(a))$, so $g(f(a))$ is defined. Use $U_0 = U \cap f^{-1}(V)$. Choose a contour C_1 is $U_0 \setminus \sigma(a)$ so that $\operatorname{ind}_{C_1}(z) = 1$ for $z \in \sigma(a)$ and equals 0 on U_0^c. Let C_2 be a contour in V such that $\operatorname{ind}_{C_2}(z) = 1$ for $z \in f(\sigma(a)) \cup f(C_1)$ and equals 0 on V^c. Then using interchange of variables and Cauchy's Theorem,

$$\begin{aligned} g(f(a)) &= \frac{1}{2\pi i}\int_{C_2} g(z)(z - f(a))^{-1}\,dz \\ &= \frac{1}{2\pi i}\int_{C_2} g(z)\frac{1}{2\pi i}\int_{C_1}\frac{R(a,w)}{z - f(w)}\,dw\,dz \\ &= \frac{1}{2\pi i}\int_{C_1} R(a,w)\frac{1}{2\pi i}\int_{C_2}\frac{g(z)}{z - f(w)}\,dz\,dw \\ &= \frac{1}{2\pi i}\int_{C_1} R(a,w)g(f(w))\,dw = g(f(a)). \end{aligned}$$

3. (a) This is a routine calculation, but you must establish absolute convergence so that rearrangement of the sums is valid.

   (b) A standard argument from complex analysis shows that there is a branch f of $\log(z)$ which is analytic in a neighbourhood of $\sigma(a)$. Let $b = f(a)$ and use Q2 to show that $e^b = (\exp\circ\log)(a) = \operatorname{id}(a) = a$.

4. (a) A^{2^n} shifts to the right 2^n places, and its weights are the product of 2^n consecutive weights of A. All but one term is the same for each weight except for the multiple of 2^n, which is often 2^{-n} but can be smaller. The norm of A^{2^n} is the largest, the product of the first 2^n weights. This is a power of 2^{-1} where there are 2^{n-1} even terms, $\dots$, 2^{n-k} multiples of 2^k, up to $k = n$. Thus the logarithm base 2 of the product is $-2^{n-1} - 2^{n-2} - \dots - 2^1 - 2^0 = 1 - 2^n$. Divide by 2^n and find the limit is -1. Therefore $\operatorname{spr}(A) = \lim_{n\to\infty}\|A^{2^n}\|^{1/2^n} = 2^{-1}$.

   (b) Let $U = \operatorname{diag}(\lambda^n)_{n\ge0}$. Compute UAU^*. The spectrum is invariant under rotation about 0.

   (c) Every 2^{k+1}st weight of A_k is zero, so $A_k^{2^{k+1}} = 0$ is a nilpotent operator. Therefore $\sigma(A_k) = \{0\}$. Also $\|A - A_k\| = 2^{-k-1}$.

   (d) If $\lambda - A$ is invertible, solve $e_0 = (\lambda - A)\sum c_ne_n$ to get $c_0 = \lambda^{-1}$ and $c_n = b_n\lambda^{-n-1}$ for $n \ge 1$. Since $\lim_{n\to\infty} c_n = 0$, this means that $|\lambda| \ge \lim b_n^{1/n} = \operatorname{spr}(A)$. Since $\sigma(A)$ has circular symmetry, this means that $\sigma(A)$ is the disc of radius $1/2$.

(e) $A_k \to A$, but the spectral radius has limit 0; and thus is discontinuous. You can also say that the spectrum is discontinuous, say with respect to the Hausdorff metric between compact sets of $\mathbb{C}$

$$d_H(A,B) = \max\big\{ \sup_{a\in A} \inf_{b\in B} |a-b|,\ \sup_{b\in B} \inf_{a\in A} |a-b| \big\}.$$

5. (a) On $\mathbb{C} \setminus \sigma(a)_\varepsilon$, $R(a,z)$ is analytic and tends to 0 as $|z| \to \infty$. In particular, it is continuous and vanishes at infinity; so it is bounded, say by M. If $\|a-b\| < 1/M$, then for $\lambda \in \mathbb{C}\setminus\sigma(a)_\varepsilon$, $\|(a-\lambda)-(b-\lambda)\| < 1/M \le 1/\|R(a,\lambda)\|$. Therefore $b-\lambda$ is invertible and thus $\sigma(b) \subset \sigma(a)_\varepsilon$.

(b) If $e=e^2$ and $e \ne 0$, then $\operatorname{spr}(e) = \lim \|e\|^{1/n} = 1$; so $\|e\| \ge 1$. Likewise if $e \ne 1$, then $\|1-e\| \ge 1$. Thus among idempotents, $\{0\}$ and $\{1\}$ are isolated points.

If σ_1 is a clopen subset of $\sigma(a)$, choose a contour C in $\rho(a)$ such that $\operatorname{ind}_C(z) = 1$ for $z \in \sigma_1$ and equals 0 on $\mathbb{C}\setminus(\sigma_1)_\varepsilon$. Let $M = \sup\{\|R(a,z)\| : z \in C\}$ and $\delta = 1/M$. If $\|b-a\| < \delta$, and $b_t = (1-t)a + tb$ for $0 \le t \le 1$, then $R(b_t,z)$ is defined for $z \in C$ and $0 \le t \le 1$. Let $e_t = \frac{1}{2\pi i}\int_C R(b_t,z)\,dz$ be the Riesz spectral projection of b_t for σ_1. Since $b_0 = a$, we have $e_0 \ne 0$. Moreover $t \to e_t$ is easily verified to be continuous. So it is always non-zero. Therefore $\emptyset \ne \sigma(b|_{\operatorname{Ran} e_1}) \subset \operatorname{int} C \subset \sigma(a)_\varepsilon$; whence it has non-empty spectrum in a neighbourhood of $\sigma(a)_\varepsilon$.

In the special case that $\sigma_1 = \sigma(a)$, we have $e_0 = 1$. As this is an isolated component of the idempotents, $e_t = 1$ for $0 \le t \le 1$. Whence $e_1 = 1$ and so $\sigma(b) \subset \operatorname{int} C \subset \sigma(a)_\varepsilon$. So this is an alternate proof of 5(a).

6. (a) Clearly $\mathfrak{A}_0^{-1}$ is closed under products and inverses. So it is a subgroup. Also $ae^ba^{-1} = a\sum_{n\ge0}\frac{1}{n!}b^na^{-1} = \sum_{n\ge0}\frac{1}{n!}(aba^{-1})^n = e^{aba^{-1}}$. So this is a normal subgroup of $\mathfrak{A}^{-1}$.

(b) By 3(b), elements in $b_1(1)$ are exponentials. So if $b \in \mathfrak{A}_0^{-1}$ and $\|b-c\| < \|b^{-1}\|^{-1}$, then $c = b(b^{-1}c)$ where $\|b^{-1}c - 1\| \le \|b^{-1}\|\,\|c-b\| < 1$. Thus $b^{-1}c$ is an exponential, and hence c is a finite product of exponentials. So $\mathfrak{A}_0$ is open. As it is a subgroup of $\mathfrak{A}^{-1}$, the complement $\mathfrak{A}^{-1}\setminus\mathfrak{A}_0^{-1}$ is the union of cosets, and hence is also open. Therefore $\mathfrak{A}_0^{-1}$ is (relatively) closed in $\mathfrak{A}^{-1}$; and thus is the union of connected components. Any element $b = e^{a_1}\dots e^{a_n}$ can be connected in $\mathfrak{A}_0$ to 1 by the path $b_t = e^{ta_1}\dots e^{ta_n}$ for $0 \le t \le 1$. Therefore $\mathfrak{A}_0^{-1}$ is the connected component of 1.

(c) $\mathrm{C}(\mathbb{T})^{-1}$ consists of those functions f whose directed image is a closed curve in $\mathbb{C}\setminus\{0\}$. It is a standard fact that every such curve is homotopic to the curve z^n where $n = \operatorname{ind}_{f(\mathbb{T})}(0)$. In particular, if $\operatorname{ind}_{f(\mathbb{T})}(0) = 0$, then f is homotopic in $\mathrm{C}(\mathbb{T})^{-1}$ to 1, and thus lies in $\mathrm{C}(\mathbb{T})_0^{-1}$. Conversely, since winding number is continuous, and hence locally constant, every $f \in \mathrm{C}(\mathbb{T})_0^{-1}$ has winding number 0 around 0. Finally, $\mathrm{C}(\mathbb{T})^{-1}/\mathrm{C}(\mathbb{T})_0^{-1} \simeq \mathbb{Z}$ via the map sending f to $\operatorname{ind}_{f(\mathbb{T})}(0)$. To check that this is a group homomorphism, you need the fact that $\operatorname{ind}_{fg(\mathbb{T})} = \operatorname{ind}_{f(\mathbb{T})} + \operatorname{ind}_{g(\mathbb{T})}$ for $f,g \in \mathrm{C}(\mathbb{T})^{-1}$.