PM 810 **Assignment 4** **Due Thursday March 28.**

1. (a) For a von Neumann algebra $\mathcal{N}$, show that the extreme points of $\overline{b_1(\mathcal{N}_+)}$ are precisely the projections.

   (b) Show that for $\mathcal{N} = \mathcal{B}(H)$, the set of extreme points of the unit ball is strictly larger than the set of unitaries.

2. Suppose that ρ is a representation of a C*-algebra A on a Hilbert space H with a unit cyclic vector x. Define a state by $f(a) = \langle \rho(a)x, x\rangle$. Show that ρ is unitarily equivalent to the GNS construct π_f via a unitary operator U such that $Ux_f = x$. This shows that the GNS construction is unique. **Hint:** set $U\dot{a} = \rho(a)x$.

3. Suppose that π and σ are irreducible representations of a C*-algebra $\mathfrak{A}$. Suppose that there is a non-zero operator T such that $\pi(a)T = T\sigma(a)$ for all $a \in \mathfrak{A}$. Prove that π and σ are unitarily equivalent. **Hint:** show that the partial isometry in the polar decomposition of T is the desired unitary.

4. An operator algebra $\mathcal{A} \subset \mathcal{B}(H)$ is *reflexive* if every $T \in \mathcal{B}(H)$ satisfying $TM \subset M$ for every invariant subspace M of $\mathcal{A}$ belongs to $\mathcal{A}$. An operator A is *reflexive* if the WOT-closed (nonself-adjoint, unital) algebra $W(A)$ generated by A is reflexive.

   (a) Prove that every von Neumann algebra is reflexive.

   (b) Prove that normal operators on separable Hilbert space are reflexive.
   Hint: $W(N) \subset W^*(N) \simeq L^\infty(\mu)$. If $h(N) \in W^*(N) \setminus W(N)$, use Hahn–Banach to find a separating functional f in $L^1(\mu)$. Factor $f = g\bar{k}$ where $g, k \in L^2(\mu)$.

5. Use the polar decomposition of a compact operator K to show that it may be written as $K = \sum_{n\geq 1} s_n e_n f_n^*$ where s_n is a positive sequence decreasing to 0, and $\{e_n\}$ and $\{f_n\}$ are orthonormal sequences. Here ef^* is the rank one operator $ef^*(x) = \langle x, f\rangle e$. The sequence $s_n(K)$ are called the **singular values** of K.

6. A compact operator K is trace class if $\|K\|_1 := \sum_{n\geq 1} s_n(K) < \infty$. The collection of all trace class operators on a separable Hilbert space H is denoted by $\mathfrak{S}_1$.

   (a) Show that if x_n and y_n are orthonormal sequences, then $\sum_{n\geq 1} |\langle Kx_n, y_n\rangle| \leq \|K\|_1$.

   (b) Show that $\mathfrak{S}_1$ is complete subspace in the trace norm, and that the ideal of finite rank operators is dense in $\mathfrak{S}_1$.

   (c) Show that $\|AKB\|_1 \leq \|A\|\,\|K\|_1\,\|B\|$. Hence $\mathfrak{S}_1$ is a non-closed 2-sided ideal of $\mathcal{B}(H)$.

   (d) Fix an orthonormal basis $\{e_n\}$ and define the trace by $\operatorname{Tr}(K) = \sum_{n\geq 1}\langle Ke_n, e_n\rangle$. Show that $\operatorname{Tr}(KT) = \operatorname{Tr}(TK)$ for all $K \in \mathfrak{S}_1$ and $T \in \mathcal{B}(H)$. Hence deduce that Tr is independent of the choice of basis.

   (e) Each $T \in \mathcal{B}(H)$ defines a linear functional φ_T on $\mathfrak{S}_1$ by $\varphi_T(K) = \operatorname{Tr}(TK)$. Show that $\|\varphi_T\| = \|T\|$.

   (f) Show that if φ is a linear functional on $\mathfrak{S}_1$, then the sesquilinear form $\langle x, y\rangle := \varphi(xy^*)$ determines a bounded linear operator T such that $\varphi = \varphi_T$.

   (g) Deduce that $\mathcal{B}(H)$ is the dual space of $\mathfrak{S}_1$. Show that this weak-$*$ topology on $\mathcal{B}(H)$ corresponds to the ultraweak topology.