PM 810 **Assignment 3** **Due Thursday March 14.**

1. (a) Show that if $\mathcal{A}$ is a separable C*-algebra, then it has an approximate unit consisting of an increasing sequence.

   (b) Show that if $0 \le a$, $0 \le b$ and $\|a\| < 1$ and $\|b\| < 1$, then
$$c = b + (1-b)^{1/2}[(1-b)^{-1/2}(a-b)(1-b)^{-1/2}]_+(1-b)^{1/2}$$
   satisfies $b \le c$, $a \le c$ and $\|c\| < 1$.

2. Show that if $\mathcal{J}$ is a closed ideal of a C*-algebra $\mathcal{A}$ and $a \in \mathcal{A}$, then there is an element $j \in \mathcal{J}$ such that $\|a - j\| = \operatorname{dist}(a, \mathcal{J})$. **Hint:** write $|a| - \|a + \mathcal{J}\| 1$ as $b_+ - b_-$.

3. Show that if S and T are two normal operators, then there is a $*$-isomorphism between $\mathrm{C}^*(S)$ and $\mathrm{C}^*(T)$ taking S to T if and only if $\sigma(S) = \sigma(T)$.

4. (a) The ultraweak (or σ-weak) topology is the weakest topology on $\mathcal{B}(H)$ such that the functionals $\omega(T) = \sum_{n\ge1}\langle Tx_n, y_n\rangle$ are continuous for all families $\{x_n, y_n : n \ge 1\}$ of vectors such that $\sum_{n\ge1}\|x_n\|\,\|y_n\| < \infty$. Likewise the ultrastrong topology is the topology determined by seminorms of the form $\rho(T) = \big(\sum_{n\ge1}\|Tx_n\|^2\big)^{1/2}$ where $\sum_{n\ge1}\|x_n\|^2 < \infty$. Prove that these two topologies have the same continuous linear functionals.

   (b) Find a linear functional on $\mathcal{B}(H)$ which is ultraweak continuous but not WOT continuous. Hence find a subspace of $\mathcal{B}(H)$ such that its ultraweak closure is properly contained in its WOT-dense closure.

   (c) Rework the proof of von Neumann's Double Commutant Theorem to show that if $\mathcal{A}$ is a non-degenerate C*-subalgebra of $\mathcal{B}(H)$, then $\mathcal{A}''$ equals the ultraweak closure of $\mathcal{A}$.

5. Let $\mathcal{A} \subset \mathcal{B}(H)$ be a concrete C*-algebra. Suppose that $A \in \mathcal{A}$ has a polar decomposition $A = U|A|$. Show that $Uf(|A|)$ belongs to $\mathcal{A}$ provided that $f \in \mathrm{C}(\sigma(|A|))$ and $f(0) = 0$.

6. (a) If X commutes with a normal operator N, show that it also commutes with N^*.
   Hint: The entire function $f(z) = e^{zN^*}Xe^{-zN^*} = e^{zN^*}e^{-\bar{z}N}Xe^{\bar{z}N}e^{-zN^*}$ is bounded.

   (b) Show that if M and N are normal and $MX = XN$, then $M^*X = XN^*$.
   Hint: $\left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$ commutes with $\left[\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right]$.

   (c) Show that two similar normal operators are unitarily equivalent. **Hint:** take the unitary part of the polar decomposition of the similarity and apply part (a).

7. Let $\mathcal{A}$ be a finite dimensional C*-algebra.

   (a) Let π be an irreducible representation of $\mathcal{A}$ on $\mathcal{B}(H)$. Prove that $\dim H < \infty$, that $\pi(\mathcal{A}) = \mathcal{B}(H)$ and that $\ker\pi$ is a maximal ideal.

   (b) Show that $\mathcal{A}$ has a faithful finite dimensional representation.

   (c) Prove that $\mathcal{A}$ is $*$-isomorphic to the direct sum of a finite number of full matrix algebras.