PM 810 **Assignment 2** **Due Thursday February 14.**

1. Let ℓ_p, for $1 \le p < \infty$, be a Banach algebra with pointwise multiplication.
   (a) Find its maximal ideal space.
   (b) Prove that it has maximal ideals which are not modular. **Hint:** $(l_p)^2$ is a proper ideal.

2. Let X be a compact Hausdorff space, and let $\mathcal{I}$ be an ideal of $\mathrm{C}(X)$, not necessarily closed. Let $E = \ker \mathcal{I} = \{x \in X : f(x) = 0 \text{ for all } f \in \mathcal{I}\}$. Let $I_0(E)$ be the ideal of functions that vanish on an open neighbourhood of E, and let $I(E)$ denote the ideal of functions which vanish on E. Prove that $I_0(E) \subset \mathcal{I} \subset I(E)$.

3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital commutative Banach algebras.
   (a) Suppose that $\theta : \mathfrak{A} \to \mathfrak{B}$ is an injective homomorphism, and $\mathfrak{B}$ is semisimple. Prove that $\mathfrak{A}$ is semisimple. **Hint:** if $a \in \mathrm{rad}(\mathfrak{A})$, what can you say about $\sigma(\theta(a))$?
   (b) Let $\mathfrak{A} = C^1[0,1]$ with $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Let $\mathcal{J} = \{f \in \mathfrak{A} : f(0) = f'(0) = 0\}$. Show that $\mathfrak{A}$ is semisimple but $\mathfrak{A}/\mathcal{J}$ is not.

4. Let $\mathfrak{A}$ be a unital commutative Banach algebra. If $a_1, \dots, a_n \in \mathfrak{A}$, define the *joint spectrum* to be $\sigma(a_1, \dots, a_n) = \{\lambda = (\varphi(a_1), \dots, \varphi(a_n)) \in \mathbb{C}^n : \varphi \in \mathcal{M}(\mathfrak{A})\}$.
   (a) Show that $\lambda = (\lambda_1, \dots, \lambda_n) \notin \sigma(a_1, \dots, a_n)$ if and only if there are $b_1, \dots, b_n \in \mathfrak{A}$ such that $\sum_{i=1}^n (a_i - \lambda_i) b_i = 1$.
   (b) We say $a_1, \dots, a_n$ generate $\mathfrak{A}$ if the set of polynomials in $a_1, \dots, a_n$ is dense in $\mathfrak{A}$. If this occurs, show that $\Phi : \mathcal{M}(\mathfrak{A}) \to \sigma(a_1, \dots, a_n)$ given by $\Phi(\varphi) = (\varphi(a_1), \dots, \varphi(a_n))$ is a homeomorphism.
   (c) If $a_1, \dots, a_n$ generate $\mathfrak{A}$, show that $\sigma(a_1, \dots, a_n)$ is *polynomially convex*; i.e. if $\lambda \in \mathbb{C}^n$ and $|p(\lambda)| \le \sup\{|p(z)| : z \in \sigma(a_1, \dots, a_n)\}$, then $\lambda \in \sigma(a_1, \dots, a_n)$. **Hint:** if $\lambda \notin \sigma(a_1, \dots, a_n)$, use part (a) and approximate $1 - \sum_{i=1}^n (a_i - \lambda_i) b_i = 0$ by a polynomial.

5. Let $\mathfrak{A}$ be a (non-commutative) Banach algebra.
   (a) Prove that $\mathfrak{A}/\mathrm{rad}(\mathfrak{A})$ is semisimple.
   (b) If $\mathfrak{J}$ is a closed ideal of $\mathfrak{A}$, show that $\mathrm{rad}(\mathfrak{J}) = \mathfrak{J} \cap \mathrm{rad}(\mathfrak{A})$.

6. Let $\mathfrak{A}, \mathfrak{B}$ be (non-commutative) Banach algebras, and let $\theta : \mathfrak{A} \to \mathfrak{B}$ be a surjective homomorphism.
   (a) Show that $\theta(\mathrm{rad}(\mathfrak{A})) \subset \mathrm{rad}(\mathfrak{B})$.
   (b) If $\mathfrak{B}$ is semisimple, prove that $\ker \theta$ is closed.