PM 810 **Assignment 1** **Due Thursday January 31.**

1. Let Ω be an open subset of $\mathbb{C}$. Show that a function $f : \Omega \to X$ into a Banach space X is weakly analytic if and only if it is strongly analytic.
   Hint: (i) If $\overline{b_r(z_0)} \subset \Omega$, show that $\left\{\frac{1}{h}(f(z_0+h) - f(z_0)) : |h| \le r\right\}$ is bounded.
   (ii) Show that f is continuous.
   (iii) Define $x_n = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})(re^{it})^{-n}\,dt$. Show that $f(z_0+z) = \sum_{n\ge0} x_n z^n$.

2. Let a be an element of a unital Banach algebra $\mathfrak{A}$. Suppose that $f(z)$ is analytic in a neighbourhood of $\sigma(a)$ and that $g(w)$ is analytic in a neighbourhood of $f(\sigma(a))$. Show that $(g \circ f)(a) = g(f(a))$.

3. Define $e^a = \sum_{n\ge0} \frac{1}{n!} a^n$.
   (a) If $ab = ba$, show that $e^{a+b} = e^a e^b$.
   (b) Let $a \in \mathfrak{A}^{-1}$ such that 0 is in the unbounded component of the resolvent. (This includes special case $\|1 - a\| < 1$.) Prove that there is an element $b \in \mathfrak{A}$ such that $a = e^b$.

4. *Kakutani's shift.* Let $\{e_n : n \ge 0\}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Let $a_n = \gcd(n, 2^n)^{-1}$ for $n \ge 1$. Define $A \in \mathcal{B}(\mathcal{H})$ by $Ae_n = a_{n+1}e_{n+1}$ for $n \ge 0$.
   (a) Compute $\operatorname{spr}(A)$. **Hint:** Compute $\|A^{2^n}\|$.
   (b) Show A is unitarily equivalent to λA for $|\lambda| = 1$. What does this say about $\sigma(A)$?
   (c) Define $A_k \in \mathcal{B}(\mathcal{H})$ by $A_k e_n = a_{n,k}e_{n+1}$, where $a_{n,k} = a_{n+1}$ if $a_{n+1} \ge 2^{-k}$ and $a_{n,k} = 0$ otherwise. Find $\sigma(A_k)$. Show that $A = \lim\limits_{k\to\infty} A_k$.
   (d) Show that $\sigma(A)$ is the disk of radius $\operatorname{spr}(A)$. **Hint:** if $(\lambda - A)^{-1}e_0 = \sum c_n e_n$, solve for c_n in terms of λ and $b_n = \|A^n e_0\|$. Use info about b_n from 3(a).
   (e) Conclude that the spectrum and spectral radius are not continuous functions.

5. *Upper semicontinuity of the spectrum.* Let $\mathfrak{A}$ be a unital Banach algebra; $a \in \mathfrak{A}$.
   (a) Given $\varepsilon > 0$, find $\delta > 0$ so that $\|a - b\| < \delta$ implies that $\sigma(b)$ is contained in $\sigma(a)_\varepsilon := \{z : \operatorname{dist}(z, \sigma(a)) < \varepsilon\}$. **Hint:** Enclose $\sigma(a)$ inside a curve $\mathcal{C}$ contained in $\sigma(a)_\varepsilon$. Then $\|R(a,z)\|$ is bounded on $\mathcal{C}$.
   (b) Suppose $\sigma(a) = \sigma_1 \cup \sigma_2$, where σ_i are disjoint non-empty clopen subsets of $\sigma(a)$. Enclose σ_1 inside curves $\mathcal{C}_i$ so that $\overline{\Omega_i} = \mathcal{C}_i \cup \{z : \operatorname{ind}_{\mathcal{C}_i}(z) \ne 0\}$ are disjoint (how?). Find $\delta > 0$ so that $\|a - b\| < \delta$ implies that $\sigma(b) \cap \Omega_i$ is not empty.
   Hint: Use part (a) to get δ. Let $b_t = (1-t)a + tb$ for $0 \le t \le 1$. Consider the Riesz projection e_t for the operator b_t and its spectrum in Ω_1. This is a continuous path.

6. (a) Let $\mathfrak{A}_0^{-1}$ denote the set of all invertible elements of a Banach algebra $\mathfrak{A}$ which may be written as a finite product of exponentials e^{b_i}. Show that this is a normal subgroup of the group $\mathfrak{A}^{-1}$ of invertible elements of $\mathfrak{A}$.
   (b) Show that $\mathfrak{A}_0^{-1}$ consists of all invertible elements which can be connected to the identity by a continuous path of invertible elements. Hence conclude that it is an open subgroup (and thus also closed). **Hint:** problem 3(b).
   (c) Let $\mathfrak{A} = \mathrm{C}(\mathbb{T})$. Describe $\mathrm{C}(\mathbb{T})_0^{-1}$ and $\mathrm{C}(\mathbb{T})^{-1}/\mathrm{C}(\mathbb{T})_0^{-1}$.