1. (a) Rewrite in standard form $y' = \frac{2xy}{x^2 - y^2}$. The RHS is homogeneous of order 0, so we can substitute $y = xz$. Then $z + xz' = \frac{2z}{1 - z^2}$.

Thus $z' = \frac{2z}{1 - z^2} - z = \frac{z(1 + z^2)}{1 - z^2}$. Separate variables: \( \frac{(1 - z^2)z'}{z(1 + z^2)} = \frac{1}{x} \).

Partial fractions: \( \frac{1 - z^2}{z(1 + z^2)} = \frac{a}{z} + \frac{bz + c}{1 + z^2} = \frac{a + cz + (a + b)z^2}{z(1 + z^2)} \).

Therefore $a = 1$, $c = 0$ and $b = -2$. So \( \frac{1}{z} - \frac{2z}{1 + z^2} dz = \frac{dx}{x} \).

Integrate: $\log z - \log(1 + z^2) = \log x + c$. Thus $\frac{y}{1 + y^2} = c_1 x$.

Substitute $z = y/x$ to get $c = \frac{y}{x^2(1 + y^2/x^2)} = \frac{y}{x^2 + y^2}$.

Rewrite this as $x^2 + y^2 = 2c_2 y$ or $x^2 + (y - c_2)^2 = c_2^2$.

(b) The curves $x^2 + (y - c)^2 = c^2$ are circles with centres on the $y$-axis that pass through the origin $(0, 0)$. However since $y'$ must be defined, the points on the lines $y = \pm x$ excluding $(0, 0)$ are not on the solution curves as the tangent is vertical at those points. See figure.

(c) $1^2 + (2 - c)^2 = c^2$ implies $c = 5/4$. So $x^2 + (y - \frac{5}{4})^2 = \frac{25}{16}$.

Therefore $y = \frac{5}{4} + \sqrt{x^2 - \frac{25}{16}}$ for $-\frac{5}{4} < x < \frac{5}{4}$.

2. (a) Separate variables: $\frac{y'}{y} = -\frac{2x}{1 + x^2}$. Integrate: $\log y = -\log(1 + x^2) + c$.

Therefore $y = \frac{c}{1 + x^2}$.

(b) In standard form $y' = \frac{-2x}{1 + x^2}y + \frac{\cot x}{1 + x^2}$. This is linear with forcing term.

Thus

$$y = \frac{1}{1 + x^2} \int \frac{\cot x}{1 + x^2} (1 + x^2) \, dx = \frac{1}{1 + x^2} \left( \log |\sin x| + c \right)$$

$$= \frac{1}{1 + x^2} \log |\sin x| + c \frac{1}{1 + x^2}.$$

3. (a) Set $z = y^{1-a}$; so that $z' = (1 - a)y^{-a}y'$. Multiply the DE by $y^{-a}$ to get: $y^{-a}y' + p(x)y^{1-a} = q(x)$.

This becomes: $\frac{1}{1 - a} z' + p(x)z = q(x)$, which is linear.
4. (a) As developed in class, a drop of water starting at height \( h \) and initial velocity 0 dropping with the force of gravity satisfies \( y(0) = h, y'(0) = 0 \) and \( y'' = -g \). Therefore \( y'(t) = -gt \) and \( y = h - \frac{1}{2} gt^2 \).

So \( y(t) = 0 \) at time \( t_0 = \sqrt{\frac{2h}{g}} \). At this point, the velocity is \( y'(t_0) = -\sqrt{2hg} \).

(b) The volume can be computed by the slice method with circular slices. At height \( y \) from the bottom of the tank, the radius of the circle is \( \sqrt{R^2 - (R - y)^2} = \sqrt{2Rh - y^2} \). Therefore the volume when the water height is \( h \) is

\[
V(h) = \int_0^h \pi(2Ry - y^2) dy = \pi(R^2 - \frac{1}{3}y^3)\bigg|_0^h = \pi(R^2 - \frac{1}{3}h^3).
\]

(c) Using Torcelli’s Law and part (a), we get

\[
-\pi r^2 \sqrt{2gh} = \frac{dV}{dt} = \pi(2Rh - h^2)h'.
\]

Therefore

\[
-r^2 \sqrt{2g} = (2R^{1/2} - h^{3/2})h'.
\]

Integrate to get

\[
-r^2 \sqrt{2g} t + c = \frac{4}{3} Rh^{3/2} - \frac{2}{5} h^{5/2}.
\]

At \( t = 0 \), we have \( h = R \). So

\[
c = (\frac{4}{3} - \frac{2}{5}) r^{5/2} = \frac{14}{15} R^{5/2}.
\]

Therefore when \( h = 0 \), we have

\[
t = \frac{c}{r^2 \sqrt{2g}} = \frac{14R^{5/2}}{15r^2 \sqrt{2g}}.
\]
5. (a) 
\[ W' = (y_1'y_2' - y_1'y_2') = y_1'y_2' + y_1'y_2' - y_1'y_2 - y_1'y_2 \\
= y_1y_2' - y_1'y_2 = y_1(-p'y_2 - qy_2) - (-p'y_1 - qy_1)y_2 \\
= -p(y_1'y_2 - y_1'y_2) = -pW. \]

Separate variables: \( \frac{W'}{W} = -p. \) Therefore

\[ \log W = -\int p(x) \, dx = -P(x) + c \]

where \( P \) is any antiderivative of \( p. \) Hence \( W(x) = Ce^{-P(x)}. \)

(b) If for some \( c \in [a,b], (y_1(c), y_1'(c)) \) and \( (y_2(c), y_2'(c)) \) are linearly independent, then

\[ Ce^{-P(c)} = W(c) = \det \begin{vmatrix} y_1(c) & y_2(c) \\ y_1'(c) & y_2'(c) \end{vmatrix} \neq 0. \]

Therefore \( C \neq 0, \) and hence \( W(x) \) is never 0. So

\[ 0 \neq W(x) = \det \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}. \]

Thus \((y_1(x), y_1'(x))\) and \((y_2(x), y_2'(x))\) are linearly independent for every \( x \in [a,b]. \)

6. (a) The rectangular coordinates at angle \( \theta \) are \( \left( \frac{\cos \theta}{1 + e \cos \theta}, \frac{\sin \theta}{1 + e \cos \theta} \right). \)

Hence the distance to \( x = \frac{1}{e} \) is

\[ \frac{1}{e} - \frac{\cos \theta}{1 + e \cos \theta} = \frac{1 + e \cos \theta - e \cos \theta}{e(1 + e \cos \theta)} = \frac{1}{e(1 + e \cos \theta)} = \frac{r(\theta)}{e}. \]

Those who studied Euclidean geometry may recognize that this describes a conic section, corresponding to an ellipse when \( e < 1, \) a parabola when \( e = 1 \) and a hyperbola when \( e > 1. \)

(b) When \( e > 1, \) \( r(\theta) \) tends to \( \pm \infty \) when \( \theta \) approaches \( \pm \cos^{-1}(-1/e). \)

Let \( \theta_0 = \cos^{-1}(-1/e) \) in \( (\frac{\pi}{2}, \pi). \) For \( \theta = \theta_0 + h \) for small \( h, \) compare the point \( \left( \frac{\cos \theta}{1 + e \cos \theta}, \frac{\sin \theta}{1 + e \cos \theta} \right) \) on the curve to the point \( \left( \frac{\cos \theta_0}{1 + e \cos \theta_0}, \frac{\sin \theta_0}{1 + e \cos \theta_0} \right) \) on the line.

The vertical distance is

\[ \frac{\sin(\theta_0 + h)}{1 + e \cos(\theta_0 + h)} = \frac{\sin \theta_0 \cos(\theta_0 + h)}{\cos \theta_0 (1 + e \cos(\theta_0 + h))} = \frac{\sin(\theta_0 + h) \cos \theta_0 - \sin \theta_0 \cos(\theta_0 + h)}{\cos \theta_0 (1 + e \cos(\theta_0 + h))} = \frac{\sin h}{\cos \theta_0 (1 + e(\cos \theta_0 \cos h - \sin \theta_0 \sin h))} = \frac{h + O(h^3)}{(-1/e) (1 + e((-1/e) (1 + O(h^2))) - \sin \theta_0 (h + O(h^3)))} = \frac{-eh + O(h^3)}{\sin \theta_0 + O(h^2)} = \frac{1 + O(h^2)}{-eh \sin \theta_0 + O(h^2)} = \frac{1 + O(h^2)}{\sin \theta_0 + O(h)}. \]
Therefore this converges to \( \csc \theta_0 = \frac{e}{\sqrt{e^2 - 1}} \). It follows that there are two asymptotes

\[
y = \tan \theta_0 x + \csc \theta_0 = -\sqrt{e^2 - 1}x + \frac{e}{\sqrt{e^2 - 1}}
\]

and its reflection in the \( x \)-axis

\[
y = -\tan \theta_0 x - \csc \theta_0 = \sqrt{e^2 - 1}x - \frac{e}{\sqrt{e^2 - 1}}.
\]

(c) Clearly if \( e \geq 1 \), the denominator \( 1 + e \cos \theta \) has a zero, and hence the curve is unbounded; while if \( e < 1 \), it is bounded below by \( 1 - e \), and thus the curve is bounded by \( \frac{1}{1-e} \).

The two points on the \( x \)-axis correspond to \( \theta = 0 \) and \( \theta = \pi \), namely \((\frac{1}{1+e}, 0)\) and \((-\frac{1}{1-e}, 0)\). The midpoint is \((a, 0)\) where \( a = \frac{1}{2} \left( \frac{1}{1+e} - \frac{1}{1-e} \right) = \frac{-e}{1-e^2} \).

Compute for \((x, y)\) on the curve at angle \( \theta \):

\[
(1 - e^2)(x - a)^2 + y^2 = (1 - e^2) \left( \frac{\cos \theta}{1 + e \cos \theta} + \frac{e}{1 - e^2} \right)^2 + \left( \frac{\sin \theta}{1 + e \cos \theta} \right)^2
\]

\[
= (1 - e^2) \left( \frac{(1 - e^2) \cos \theta + e(1 + e \cos \theta)}{(1 + e \cos \theta)(1 - e^2)} \right)^2 + \frac{\sin^2 \theta}{(1 + e \cos \theta)^2}
\]

\[
= \frac{(\cos \theta + e)^2}{(1 + e \cos \theta)^2(1 - e^2)} + \frac{\sin^2 \theta}{(1 + e \cos \theta)^2}
\]

\[
= \cos^2 \theta + 2e \cos \theta + e^2 + (1 - e^2) \sin^2 \theta
\]

\[
= \frac{1 + 2e \cos \theta + e^2 - e^2(1 - \cos^2 \theta)}{(1 + e \cos \theta)^2(1 - e^2)}
\]

\[
= \frac{1 + 2e \cos \theta + e^2 \cos^2 \theta}{(1 + e \cos \theta)^2(1 - e^2)}
\]

\[
= \frac{1}{1 - e^2}.
\]

Therefore this is an ellipse.

(d) When \( e = 1 \), \( r(\theta) = \frac{1}{1 + \cos \theta} \). Then

\[
y^2 = \left( \frac{\sin \theta}{1 + \cos \theta} \right)^2 = \frac{\sin^2 \theta (1 - \cos \theta)}{(1 - \cos^2 \theta)(1 + \cos \theta)} = \frac{1 - \cos \theta}{1 + \cos \theta}.
\]

Hence

\[
y^2 + 2x = \frac{1 - \cos \theta}{1 + \cos \theta} + \frac{2 \cos \theta}{1 + \cos \theta} = 1.
\]

This is a parabola (on its side) with vertex \((\frac{1}{2}, 0)\).