Math 148 Assignment 5 Solutions

1. Clearly \( f_n(0) = 0 \) converges to 0. For \( x > 0 \), the ratio test yields
\[
\lim_{n \to \infty} \frac{f_{n+1}(x)}{f_n(x)} = \lim_{n \to \infty} \frac{n+1}{ne^x} = e^{-x} < 1.
\]
Therefore \( \lim_{n \to \infty} f_n(x) = 0 \) pointwise by the ratio test.

To compute \( \|f_n\|_\infty \), note that \( 0 \leq f_n(x) \leq \lim_{x \to \infty} f_n(x) \).
Differentiate to get \( f'_n(x) = ne^{-nx}(1-nx) \).
This has a critical point at \( x = 1/n \), and \( f_n(1/n) = 1/e \). This must be a maximum.
Therefore \( \|f_n\|_\infty = 1/e \). Hence \( f_n \) does not converge to 0 uniformly.

2. Clearly \( f_n(0) = 0 \) converges to 0.
Since \( |f_n(x)| \leq 1/n|x| \) for \( x \neq 0 \), we obtain \( \lim_{n \to \infty} f_n(x) = 0 \) pointwise.
This estimate also shows that also have \( \lim_{|x| \to \infty} |f_n(x)| = 0 \).
As \( f_n(0) = 0 \) for all \( n \), we compute \( \|f_n\|_\infty \) by computing the extrema.
Differentiate to obtain \( f'_n(x) = \frac{1-nx^2}{(1+nx^2)^2} \). The critical points are \( \pm 1/\sqrt{n} \).
Hence \( \|f_n\|_\infty = |f_n(\pm 1/\sqrt{n})| = \frac{1}{2\sqrt{n}} \).
Since \( \lim_{n \to \infty} \frac{1}{2\sqrt{n}} = 0 \), the sequence \( f_n \) converges uniformly to 0 uniformly on \( \mathbb{R} \).

3. By the Extreme Value Theorem for continuous functions on \( [a, b] \), \( \|f\|_\infty \) and \( \|g\|_\infty \) are finite. Because \( \{f_n\} \) converge uniformly to \( f \), there is an \( n \) so that
\[
\|f_n - f\|_\infty < 1 \quad \text{for all} \quad n \geq N.
\]
Hence \( \|f_n\|_\infty < \|f\|_\infty + 1 \) if \( n \geq N \). Therefore, for \( n \geq N \), we get:
\[
\|fg - f_ng_n\|_\infty \leq \|fg - f_ng\|_\infty + \|f_ng - f_ng_n\|_\infty \\
\leq \|f - f_n\|_\infty \|g\|_\infty + (\|f\|_\infty + 1)\|g - g_n\|_\infty.
\]
Therefore
\[
\lim_{n \to \infty} \|fg - f_ng_n\|_\infty \leq \lim_{n \to \infty} \|f - f_n\|_\infty \|g\|_\infty + (\|f\|_\infty + 1)\|g - g_n\|_\infty = 0.
\]
This implies that \( f_ng_n \) converges uniformly to \( fg \).
4. (a) \[ \sum_{n \geq 0} \frac{x^2}{(1 + x^2)^n} = \begin{cases} 0 & \text{if } x = 0 \\ \frac{x^2}{1 - \frac{1}{1 + x^2}} = 1 + x^2 & \text{if } x \neq 0. \end{cases} \]

(b) The limit function is discontinuous at \( x = 0 \). Uniform limits of continuous functions are continuous, and therefore if \( 0 \in [a, b] \), convergence is not uniform. Suppose that \( 0 < a < b \leq \infty \). Differentiate

\[ f'_n(x) = \frac{(2x)(1 - (n - 1)x^2)}{(1 + x^2)^{n+1}}. \]

This has critical points at \( x = \frac{\pm 1}{\sqrt{n-1}} \) and \( x = 0 \).

When \( n = 1 \), \( f_n \) is increasing on \([0, \infty)\) and \( \lim_{x \to \infty} f_1(x) = 1 \). So \( \|f_1\|_\infty = 1 \).

When \( n \geq 2 \), \( f_n \) is decreasing for \( x \geq \frac{1}{\sqrt{n-1}} \). Hence \( \|f_n\|_\infty = f(\frac{1}{\sqrt{n-1}}) < \infty \).

Moreover we have \( \|f_n\|_{[a,b]} = f_n(a) \) for \( n \geq 1 + \frac{1}{a^2} \). Consequently

\[
\sum_{n \geq 1} \|f_n\|_{[a,b]} = \|f_1\|_{[a,b]} + \sum_{2 \leq n < 1 + a^{-2}} \|f_n\|_{[a,b]} + \sum_{n \geq 1 + a^{-2}} \|f_n\|_{[a,b]} \\
\leq 1 + \sum_{2 \leq n < 1 + a^{-2}} \|f_n\|_{[a,b]} + \sum_{n \geq 1 + a^{-2}} f_n(a) \\
\leq 2 + a^2 + \sum_{2 \leq n < 1 + a^{-2}} \|f_n\|_{[a,b]} < \infty.
\]

Therefore the series converges uniformly on \([a, b]\) by the Weierstrass M-test. Since \( f_n \) are even functions, the series converges uniformly on \([a, b]\) if \( -\infty \leq a < b < 0 \).

5. (a) Once \( n \geq x \), we have \( f_n(x) = e^{-x} \). Thus \( \lim_{n \to \infty} f_n(x) = e^{-x} \) pointwise. Compute

\[
e^{-x} - f_n(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq n \\ e^{-x} - f_n(x) \in [-e^{-n}, e^{-n}] & \text{for } n \leq x \leq n + e^n \\ e^{-x} & \text{for } x \geq n + e^n. \end{cases}
\]

Thus \( \|e^{-x} - f_n\|_{[0,\infty)} \leq e^{-n} \). Hence \( (f_n) \) converges uniformly to \( e^{-x} \).

(b) \[ \int_0^\infty e^{-x} \, dx = -e^{-x}\big|_0^\infty = 1, \]

\[ \int_0^\infty f_n(x) \, dx = \int_0^n e^{-x} \, dx + \int_n^{n+e^n} f_n(x) \, dx = -e^{-x}\big|_0^n + \frac{1}{2} e^{-n} e^n = \frac{3}{2} - e^{-n}. \]

Thus \( \lim_{n \to \infty} \int_0^\infty f_n(x) \, dx = \frac{3}{2} \neq 1 = \int_0^\infty e^{-x} \, dx. \)

(c) This does not contradict Integral Convergence Theorem because the result only applies on a closed \textit{bounded} interval \([a, b]\), not on \([0, \infty)\).
6. (a) Since $f$ is uniformly continuous, for $\varepsilon > 0$, there is a $\delta > 0$ so that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$. Choose $N > \delta^{-1}$. If $n \geq N$, then since $\frac{1}{n} < \delta$,

$$\|f - f_n\|_{\infty} = \sup_{x \in \mathbb{R}} |f(x) - f(x + \frac{1}{n})| \leq \varepsilon.$$ 

Therefore $f_n$ converges uniformly to $f$ on $\mathbb{R}$.

(b) This does not remain true if $f$ is just continuous. Let $f(x) = x^2$. Then

$$\|f - f_n\|_{\infty} = \sup_{x \in \mathbb{R}} |x^2 - (x + \frac{1}{n})^2| = \sup_{x \in \mathbb{R}} \left| \frac{2x}{n} + \frac{1}{n^2} \right| = +\infty.$$ 

7. (a) Proceed by induction. First observe that since $f_1(x) = x \geq 1$, we have

$$f_2(x) = x^2 \geq x^1 = f_1(x).$$

Now if $f_n(x) \geq f_{n-1}(x)$, it follows that

$$f_{n+1}(x) = x^{f_n(x)} \geq x^{f_{n-1}(x)} = f_n(x).$$

So by induction, this is valid for all $n \geq 1$.

(b) Since $f_n(x)$ is monotone increasing, it has a limit if and only if it is bounded above. Suppose $L(x) = \lim_{n \to \infty} f_n(x)$ exists. Then,

$$x^{L(x)} = \lim_{n \to \infty} x^{f_n(x)} = \lim_{n \to \infty} f_{n+1}(x) = L(x).$$

Therefore, $L(x) \log x = \log L(x)$, which implies that $\log x = \frac{\log L(x)}{L(x)}$. Now define $g(t) = \frac{\log t}{t}$. Differentiating gives $g'(t) = \frac{1 - \log t}{t^2}$, so there is one critical point at $(e, \frac{1}{e})$. This is the global maximum for $g$ because

$$\lim_{t \to 0^+} g(t) = 0 = \lim_{t \to \infty} g(t).$$

Therefore, $\log x \leq \frac{1}{e}$, implying that $x \leq e^{1/e}$.

(c) Suppose $x \leq e^{1/e}$, then we claim that $f_n(x) \leq e$ for all $n \geq 1$. To prove this we use induction on $n$. Clearly $f_1(x) = x \leq e^{1/e} \leq e$. Now suppose the proposition is true for $k = n$. Then we have: $f_{n+1}(x) = x^{f_n(x)} \leq (e^{1/e})^e = e$. Therefore, $\lim_{n \to \infty} f_{n+1}(x) = L(x)$ exists if $x \leq e^{1/e}$.

(In fact, $g$ is monotone increasing from $[1, e]$ onto $[0, \frac{1}{e}]$. Hence it has an inverse function $g^{-1} : [0, \frac{1}{e}] \to [1, e]$. Thus $L(x) = g^{-1}(\log x)$ for $x \in [1, e^{1/e}]$.)

(d) For larger $x$, this limit must be $+\infty$ as it cannot be bounded above because of part (b).