Math 148 Assignment 4 Solutions

1. Let $b_i = a_{2^n}$ if $2^n \leq i < 2^{n+1}$. Then each $a_{2^n}$ is repeated $2^n$ times. Therefore

$$
\sum_{i=1}^{\infty} b_i = \sum_{n=0}^{\infty} 2^n a_{2^n}.
$$

So $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges if and only if $\sum_{i=1}^{\infty} b_i$ converges.

If $2^n \leq i < 2^{n+1}$, then $a_i = a_{2^n} = b_i$.

Thus by the Comparison Test, if $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{n=1}^{\infty} a_i$ converges.

On the other hand, if $2^n - 1 \leq i < 2^n$, then $a_{2i} = a_{2^n} \leq a_i$.

So by the Comparison Test, if $\sum_{n=1}^{\infty} a_i$ converges, then $\sum_{i=1}^{\infty} b_{2i}$ converges.

But $\sum_{i=1}^{\infty} b_{2i} = \sum_{n=1}^{\infty} 2^{n-1} a_{2^n}$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

2. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ converges absolutely by the Integral Test since

$$
\int_{1}^{\infty} \frac{1}{x^4} \, dx = \frac{-1}{3x^3} \bigg|_{1}^{\infty} = \frac{1}{3} < \infty.
$$

So all rearrangements converge to the same sum. Therefore

$$
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^{4 \text{ even}}} - \sum_{n=1}^{\infty} \frac{1}{n^{4 \text{ odd}}} = 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^4} - \sum_{n=1}^{\infty} \frac{1}{n^4} = -\frac{7}{8} \sum_{n=1}^{\infty} \frac{1}{n^4} = -\frac{7\pi^4}{720}.
$$

3. Claim: If $a_n \leq b_n \leq c_n$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} c_n$ converge, then $\sum_{n=1}^{\infty} b_n$ converges.

Use the Cauchy Criterion. Given $\varepsilon > 0$, the convergence of the two series means that there is an $N$ so that

$$
\left| \sum_{i=n+1}^{m} a_i \right| < \varepsilon \quad \text{and} \quad \left| \sum_{i=n+1}^{m} c_i \right| < \varepsilon \quad \text{for all} \quad N \leq n < m.
$$

Therefore

$$
-\varepsilon < \sum_{i=n+1}^{m} a_i \leq \sum_{i=n+1}^{m} b_i \leq \sum_{i=n+1}^{m} c_i < \varepsilon.
$$

Hence by the Cauchy Criterion, $\sum_{n=1}^{\infty} b_n$ converges.
4. This is called Abel’s Test. Let \( c_n = b_n - B \). This converges monotonely to 0.

The series \( \sum_{n=1}^{\infty} a_n \) converges, and hence the partial sums are bounded.

Therefore by Dirichlet’s Test, \( \sum_{n=1}^{\infty} a_n c_n \) converges.

It is also true that \( \sum_{n=1}^{\infty} a_n B \) converges to \( AB \). Therefore

\[
\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n c_n + \sum_{n=1}^{\infty} a_n B \quad \text{converges.}
\]

Here we are using the elementary fact that if \( \sum_{n=1}^{\infty} x_n = X \) and \( \sum_{n=1}^{\infty} y_n = Y \),

then \( \sum_{n=1}^{\infty} x_n + y_n \) converges to \( X + Y \). This is called Abel’s Test.

5. (a) \( \sqrt{n+1} - \sqrt{n} = \frac{\sqrt{n+1} - \sqrt{n}}{n^a} \cdot \frac{n^a}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{n^a(\sqrt{n+1} + \sqrt{n})} \).

Thus

\[
\frac{1}{2n^{a+\frac{1}{2}}} \leq \frac{\sqrt{n+1} - \sqrt{n}}{n^a} \leq \frac{1}{n^{a+\frac{1}{2}}},
\]

It follows from the Comparison Test that our series converges if and only if \( \sum_{n=1}^{\infty} \frac{1}{n^{a+\frac{1}{2}}} \) converges. By the Integral Test (as done in class), this converges absolutely if and only if \( a > 1/2 \), and otherwise diverges.

(b) \( \sqrt{n} = e^{\log n} - 1 \). But \( f(x) = \frac{\log x}{x} \) attains its maximum at \( x = e \), namely \( 1/e \).

So \( \sqrt{n} - 1 < e^{1/e} - 1 < \frac{1}{2} \). Therefore \( (\sqrt{n} - 1)^n < 2^{-n} \).

Hence this series converges absolutely by the Comparison Test.

(c) This series does not converge absolutely because \( \frac{\arctan(n)}{n} > \frac{1}{n} \) for \( n \geq 2 \).

But \( f(x) = \frac{\arctan x}{x} \) is monotone decreasing on \([1, \infty)\).

So this series converges conditionally by the Alternating Series Test.

(d) Since \( \frac{(-1)^{n+1} (n+1)^{42}}{n^{32} (n+1)!} = (1 + \frac{1}{n})^{42} \left( -\frac{1}{n+2} \right) \) converges to 0, this series converges absolutely by the Ratio Test.

(e) As noted in (b), \( n^{1/n} < 3/2 \). So \( \frac{1}{n^{1+1/n}} > \frac{2}{3n} \). Therefore this series diverges by comparison with the harmonic series.

(f) Since \( \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^{n} = \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^{n} = \frac{1}{e} < 1 \), this series converges absolutely by the Root Test.
(g) If \( \theta \in 2\pi \mathbb{Z} \), then this series is just \( \sum_{n=2}^{\infty} \frac{1}{\log n} \), which diverges by comparison with the harmonic series. Otherwise, \( \sum_{n=2}^{\infty} \cos n\theta \) has bounded partial sums (which can be checked the way we summed sines in class) and \( b_n = \frac{1}{\log n} \) for \( n \geq 2 \) converges monotonely to 0. So this series converges by the Dirichlet Test.

Convergence is conditional. Indeed if \( |\theta| \leq \pi/2 \), at least one of the terms \( |\cos(2n-1)\theta| \) or \( |\cos(2n)\theta| \) must be as large as \( \sin |\theta|/2 \). So comparison with \( \sin |\theta|/2 \sum_{n=2}^{\infty} \frac{1}{\log 2n} \) shows divergence of \( \sum_{n=2}^{\infty} \frac{1}{n} \). For \( \pi/2 < |\theta| \leq \pi \), we have \( |\cos n(\pi - \theta)| = |\cos n\theta| \). So the same estimates hold, and convergence is conditional.

(h) Simple calculus shows that \( f(x) = \frac{\log x}{x^{1/4}} \leq \frac{4}{e} < 2 \). So \( \frac{1}{\sqrt{n(\log n)^2}} > \frac{1}{4n} \).

Hence this series is not absolutely convergent.

However \( \frac{1}{\sqrt{n(\log n)^2}} \) is monotone decreasing to 0, and thus this series converges conditionally by the Alternating Series Test.

(i) This series is \(-1 + \frac{1}{3(2)} - \frac{1}{3} - \frac{1}{3(4)} - \frac{1}{5} + \frac{1}{3(6)} + \ldots\). Observe that

\[
s_{2k} = \sum_{n=1}^{2k} \frac{(-1)^n}{(2 + (-1)^n)n} = -\frac{2}{3} \sum_{n=1}^{k} \frac{1}{2n-1} + \frac{1}{3} \sum_{n=1}^{2k} \frac{(-1)^n}{n}.
\]

The second series converges by the Alternating Series Test to a finite limit, while the first diverges by comparison with a multiple of the harmonic series. So this series diverges.

6. It is clear that a sequence of positive numbers \((x_n)\) has a positive limit if and only if the sequence \((\log x_n)\) has a finite limit. This is used in both parts.

(a) When \( a_i \geq 0 \),

\[
\log \prod_{i=1}^{n} 1 + a_i = \sum_{i=1}^{n} \log(1 + a_i).
\]

By the Mean Value Theorem for \( f(x) = \log(1 + x) \), we have a point \( c \in (1, 1 + x) \) so that

\[
\frac{\log(1 + x)}{x} = \frac{\log(1 + x) - \log 1}{(1 + x) - 1} = f'(c) = \frac{1}{1 + c}.
\]

Therefore

\[
\frac{x}{1 + x} < \log(1 + x) = \frac{x}{1 + c} < x.
\]

Thus \( \log(1 + a_i) < a_i \). So by the Comparison Test, if \( \sum_{i=1}^{\infty} a_i \) converges, then

\[
\sum_{i=1}^{\infty} \log(1 + a_i) \text{ converges; and thus } \prod_{i=1}^{\infty} 1 + a_i \text{ converges.}
\]

Conversely, if \( \sum_{i=1}^{\infty} \log(1 + a_i) \) converges, then \( \lim_{i \to \infty} \log(1 + a_i) = 0 \) and hence \( \lim_{i \to \infty} a_i = 0.\)
So there is a bound \( a_i \leq A \) for all \( i \geq 1 \). Thus
\[
\frac{a_i}{1 + A} < \log(1 + a_i) \quad \text{or} \quad a_i < (A + 1) \log(1 + a_i).
\]

Therefore if \( \sum_{i=1}^{\infty} \log(1 + a_i) \) converges, then \( \sum_{i=1}^{\infty} a_i \) converges by the Comparison Test.

(b) When \( 0 \leq a_i < 1 \), the argument is similar. In this case, the sequence of partial products is decreasing and bounded below by 0, so it always has a limit. The question is whether it is positive. Taking logs, one sees as in (a) that
\[
\lim_{i \to \infty} \prod_{i=1}^{\infty} (1 - a_i) > 0 \quad \text{if and only if} \quad \lim_{i \to \infty} \log(1 - a_i) \quad \text{converges}.
\]

Applying Mean Value Theorem to \( f(x) = \log(1 - x) \) for \( 0 \leq x < 1 \) yields some \( c \in (0, x) \) so that
\[
\frac{\log(1 - x)}{x} = \frac{\log(1 - x) - \log 1}{x - 0} = f'(c) = \frac{-1}{1 - c}.
\]

Therefore
\[
x < -\log(1 - x) = \frac{x}{1 - c} < \frac{x}{1 - x}.
\]

Hence if \( \lim_{i \to \infty} \log(1 - a_i) \) converges, the Comparison Test shows that \( \lim_{i \to \infty} a_i \) converges.

Conversely, if \( \lim_{i \to \infty} a_i \) converges, then \( \alpha = \sup a_i < 1 \), so that \(- \log(1 - a_i) < \frac{a_i}{1 - \alpha} \).

So by the Comparison Test, \( -\sum_{i=1}^{\infty} \log(1 - a_i) \) converges.