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A Short Path to the Shortest Path

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steps of 0.1. This picture inspires one to look for a single equation satisfied by  $\sigma_1$  and  $\sigma_2$ . In fact, it is not too hard to show that

$$(1 - \sigma_1)\sigma_2 = \frac{1}{3}. \tag{11}$$

To see this, translate  $p$  so that  $r_2 = 0$ . Then  $p'(x) = 3x^2 - 2(r_1 + r_3)x + r_1r_3$ . From this, we see that the product of the two roots of  $p'$  is  $r_1r_3/3$ . However, the roots of  $p'$  are  $(1 - \sigma_1)r_1$  and  $\sigma_2r_3$ .<sup>1</sup>

Figure 2 was produced in a similar manner but using quartic polynomials of the form  $p(x) = x(x - r_2)(x - r_3)(x - r_4)$ . The critical points were approximated by numerically solving the cubic equation  $p'(x) = 0$  using Maple's **fsolve** procedure. This time  $Y_4$  clearly appears to be a smooth surface in  $X_4$ .

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## A Short Path to the Shortest Path

**Peter D. Lax**

This note contains a demonstration of the isoperimetric inequality. Our proof is somewhat simpler and more straightforward than the usual ones; it is eminently suitable for presentation in an honors calculus course.

1. *The Isoperimetric Inequality* says that a closed plane curve of length  $2\pi$  encloses an area  $\leq \pi$ . Equality holds only for a circle.

Let  $x(s), y(s)$  be the parametric presentation of the curve,  $s$  arclength,  $0 \leq s \leq 2\pi$ . Suppose that we have so positioned the curve that the points  $x(0), y(0)$  and  $x(\pi), y(\pi)$  lie on the  $x$ -axis, i.e.

$$y(0) = 0 = y(\pi). \tag{1}$$

The area enclosed by the curve is given by the formula

$$A = \int_0^{2\pi} y\dot{x} ds, \tag{2}$$

where the dot denotes differentiation with respect to  $s$ . We write this integral as the sum  $A_1 + A_2$  of an integral from 0 to  $\pi$  and from  $\pi$  to  $2\pi$ , and show that each is  $\leq \frac{\pi}{2}$ .

<sup>1</sup>The author thanks the referee for this particularly nice derivation of (11).

According to a basic inequality,

$$ab \leq \frac{a^2 + b^2}{2};$$

equality holds only when  $a = b$ . Applying this to  $y = a$ ,  $\dot{x} = b$ , we get

$$A_1 = \int_0^\pi y \dot{x} ds \leq \frac{1}{2} \int_0^\pi (y^2 + \dot{x}^2) ds. \quad (3)$$

Since  $s$  is arclength,  $\dot{x}^2 + \dot{y}^2 = 1$ ; so we can rewrite (3) as

$$A_1 \leq \frac{1}{2} \int_0^\pi (y^2 + 1 - \dot{y}^2) ds. \quad (3')$$

Since  $y = 0$  at  $s = 0$  and  $\pi$ , we can factor  $y$  as

$$y(s) = u(s) \sin s, \quad (4)$$

$u$  bounded and differentiable. Differentiate (4):

$$\dot{y} = \dot{u} \sin s + u \cos s.$$

Setting this into (3') gives

$$A_1 \leq \frac{1}{2} \int_0^\pi [u^2(\sin^2 s - \cos^2 s) - 2u\dot{u} \sin s \cos s - \dot{u}^2 \sin^2 s + 1] ds. \quad (5)$$

The product  $2u\dot{u}$  is the derivative of  $u^2$ ; integrating by parts changes (5) into

$$A_1 \leq \frac{1}{2} \int_0^\pi (1 - \dot{u}^2 \sin^2 s) ds,$$

clearly  $\leq \pi/2$ . Equality holds only if  $\dot{u} \equiv 0$ , which makes  $y(s) \equiv \text{constant} \sin s$ . Since equality in (3) holds only if  $y = \dot{x} = \sqrt{1 - \dot{y}^2}$ ,  $y(s) \equiv \pm \sin s$ ,  $x(s) \equiv \mp \cos s + \text{constant}$ . This is a semicircle. Q.e.d.

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## A Note on Entire Solutions of the Eiconal Equation

Dmitry Khavinson

The eiconal equation  $\sum_{i=1}^n (\partial u / \partial x_i)^2 = 1$ ,  $u: \mathbf{R}^n \rightarrow \mathbf{R}$  is one of the main equations of geometrical optics. Its characteristics represent the light rays, while the level surfaces of solution  $u$  can be thought of as wave fronts (cf., e.g., [3]). Here,