$$= \sin^n\theta \sum_{k=0}^{n} \binom{n}{k} i^k \cot^{n-k}\theta,$$

we obtain the trigonometric identity

$$\sin n\theta = \sin^n\theta \left\{ \binom{n}{1} \cot^{n-1}\theta - \binom{n}{3} \cot^{n-3}\theta + \binom{n}{5} \cot^{n-5}\theta - + \cdots \right\}.$$

Take $n = 2m + 1$ and write this in the form

$$\sin(2m+1)\theta = \sin^{2m+1}\theta P_m(\cot^2\theta) \text{ with } 0 < \theta < \frac{\pi}{2}, \tag{3}$$

where $P_m$ is the polynomial of degree $m$ given by

$$P_m(x) = \binom{2m+1}{1} x^m - \binom{2m+1}{3} x^{m-1} + \binom{2m+1}{5} x^{m-2} - + \cdots .$$

Since $\sin\theta \neq 0$ for $0 < \theta < \pi/2$, equation (3) shows that $P_m(\cot^2\theta) = 0$ if and only if $(2m+1)\theta = k\pi$ for some integer $k$. Therefore $P_m(x)$ vanishes at the $m$ distinct points $x_k = \cot^2 \pi k/(2m+1)$ for $k = 1, 2, \cdots, m$. These are all the zeros of $P_m(x)$ and their sum is

$$\sum_{k=1}^{m} \cot^2 \frac{\pi k}{2m+1} = \binom{2m+1}{3} \Big/ \binom{2m+1}{1} = \frac{m(2m-1)}{3},$$

which proves (2).

NOTE. This paper was translated from a Greek manuscript and communicated to the MONTHLY on behalf of the author by Tom M. Apostol, California Institute of Technology. After this paper was written it was learned that the same proof was discovered independently and published in Norwegian by Finn Holme in Nordisk Matematisk Tidskrift, vol. 18 (1970), pp. 91–92. See also A. M. Yaglom and I. M. Yaglom, *Challenging mathematical problems with elementary solutions*, vol. II, Holden-Day, San Francisco, 1967, problem 145.

## ANOTHER ELEMENTARY PROOF OF EULER'S FORMULA FOR $\zeta(2n)$

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**1. Introduction.** The classic formula

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!} \tag{1}$$

which expresses $\zeta(2n)$ as a rational multiple of $\pi^{2n}$ was discovered by Euler [**2**]. The numbers $B_n$ are Bernoulli numbers and can be defined by the recursion formula

$$B_0 = 1, \; B_n = \sum_{s=0}^{n} \binom{n}{s} B_s \text{ for } n \geqq 2,$$

or equivalently, as the coefficients in the power series expansion

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}, \qquad |z| < 2\pi. \tag{2}$$

In this notation we have

$$B_1 = -\frac{1}{2}, \; B_{2n+1} = 0 \text{ for } n \geqq 1, \tag{3}$$

and

$$B_2 = \frac{1}{6}, \; B_4 = -\frac{1}{30}, \; B_6 = \frac{1}{42}, \; B_8 = -\frac{1}{30}, \; B_{10} = \frac{5}{66}.$$

Euler's original proof of (1) was obtained from two distinct representations of $\pi z \cot \pi z$, a power series expansion obtainable from (2),

$$\pi z \cot \pi z = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2\pi z)^{2n} B_{2n}}{(2n)!}, \text{ valid for } |z| < 1,$$

and the partial fraction decomposition

$$\pi z \cot \pi z = 1 - 2 \sum_{k=1}^{\infty} \frac{z^2}{k^2 - z^2}, \text{ valid for } z \neq 0, \pm 1, \pm 2, \cdots.$$

If $|z| < 1$, each term in the last sum can be expanded in a geometric series giving us

$$\pi z \cot \pi z = 1 - 2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{z^2}{k^2}\right)^n = 1 - 2 \sum_{n=1}^{\infty} \zeta(2n) z^{2n}.$$

Equation (1) follows by equating coefficients of $z^{2n}$ in the two power series expansions of $\pi z \cot \pi z$. Details justifying this argument are given in Knopp [**4**], pp. 203–207, 236.

Another well-known proof is obtained by putting $s = 2n$ in Riemann's functional equation

$$\zeta(1 - s) = 2(2\pi)^{-s} \Gamma(s) \cos \frac{\pi s}{2} \zeta(s)$$

and using the fact that $\zeta(1 - 2n) = -B_{2n}/(2n)$. These results are deduced by applying residue calculus to a contour integral representation of $\zeta(s)$. (See Titchmarsh [**8**], pp. 18-20.)

Several writers have given more elementary proofs of (1) that do not require concepts from advanced real or complex analysis. For example, Titchmarsh [**7**] obtained a set of complicated recursion formulas which can be used to evaluate $\zeta(4), \zeta(6), \cdots$, successively in terms of $\zeta(2)$. Estermann [**1**] obtained a simpler formula of the same type. These recursion formulas, which show that $\zeta(2n)$ is a rational multiple of $\zeta(2)^n$, were deduced by rearranging absolutely convergent infinite series but did not require any function theory. Estermann also gave an elementary proof of the formula $\zeta(2) = \pi^2/6$ as a consequence of Gregory's series $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \pi/4$.

A recursion formula simpler than those of Titchmarsh and Estermann was proved by G. T. Williams [**11**] who showed by elementary methods that

$$\left(n+\frac{1}{2}\right)\zeta(2n)=\sum_{k=1}^{n-1}\zeta(2k)\zeta(2n-2k). \tag{4}$$

He also obtained the companion result

$$\left(n-\frac{1}{2}\right)(1-2^{-2n})\zeta(2n)=\sum_{k=1}^{n}\xi(2k-1)\xi(2n-2k+1), \tag{5}$$

where

$$\xi(s)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^s}\quad\text{for } s>0.$$

Note that $\xi(1)$ is Gregory's series for $\pi/4$. Taking $n=1$ in (5) we find $\frac{3}{8}\zeta(2)=\xi^2(1)=\pi^2/16$, so $\zeta(2)=\pi^2/6$. This result, in conjunction with (4), gives a completely elementary evaluation of $\zeta(2n)$ as a rational multiple of $\pi^{2n}$. Williams also points out that (4) is equivalent to the following recursion formula for Bernoulli numbers,

$$-(2n+1)B_{2n}=\sum_{k=1}^{n-1}\binom{2n}{2k}B_{2k}B_{2n-2k}.$$

This relation appears in Nielsen's book [**5**] and was also discovered independently by R. S. Underwood [**9**] who used it to evaluate the sums $\sum_{k=1}^{m}k^n$ in terms of Bernoulli polynomials.

The purpose of this note is to show that the elementary method used by Papadimitriou to evaluate $\zeta(2)$ in the foregoing paper [**6**] can be extended to evaluate $\zeta(2n)$ and leads directly to Equation (1) rather than to a recursion formula. The interplay of ideas from elementary algebra and trigonometry makes the proof especially suitable for an elementary calculus course.

**2. Elementary Proof of (1).** The key ingredient in Papadimitriou's proof is the formula

$$\sum_{k=1}^{m}\cot^2\frac{k\pi}{2m+1}=\frac{m(2m-1)}{3},$$

or rather the asymptotic relation

$$\sum_{k=1}^{m}\cot^2\frac{k\pi}{2m+1}=\frac{2}{3}m^2+O(m) \tag{6}$$

which it implies. Our evaluation of $\zeta(2n)$ makes use of the following lemma which provides a generalization of (6).

LEMMA 1. *For any integers $m\geqq 1$, $n\geqq 1$, we have*

$$\sum_{k=1}^{m} \cot^{2n}\frac{k\pi}{2m+1} = (-1)^{n-1}\frac{2^{4n-1}B_{2n}}{(2n)!}m^{2n} + O(m^{2n-1}), \tag{7}$$

*where the constant implied by the O-symbol is independent of m.*

First we show how the lemma implies (1) and then we prove the lemma. The inequality $\sin x < x < \tan x$ for $0 < x < \pi/2$ implies

$$\cot^{2n}x < \frac{1}{x^{2n}} < (1+\cot^2 x)^n$$

for each integer $n \geqq 1$. We take $x = k/(2m+1)$ and sum on $k$ to obtain

$$\sum_{k=1}^{m} \cot^{2n}\frac{k\pi}{2m+1} < \frac{(2m+1)^{2n}}{\pi^{2n}}\sum_{k=1}^{m}\frac{1}{k^{2n}} < \sum_{k=1}^{m}\left(1+\cot^2\frac{k\pi}{2m+1}\right)^n. \tag{8}$$

From (7) and the binomial theorem we see that

$$\sum_{k=1}^{m}\left(1+\cot^2\frac{k\pi}{2m+1}\right)^n = \sum_{k=1}^{m}\cot^{2n}\frac{k\pi}{2m+1} + O(m^{2n-1}).$$

Therefore if we multiply (8) by $\pi^{2n}/(2m)^{2n}$ and let $m \to \infty$ we obtain

$$\lim_{m\to\infty}\sum_{k=1}^{m}\frac{1}{k^{2n}} = (-1)^{n-1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!},$$

which proves (1).

**3. Proof of Lemma 1.** As in Papadimitriou's paper we use the polynomial

$$P_m(x) = \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \binom{2m+1}{5}x^{m-2} - + \cdots$$

whose $m$ zeros are the numbers

$$x_k = \cot^2\frac{k\pi}{2m+1}, \qquad k = 1, 2, \cdots, m.$$

Let $s_n = x_1^n + \cdots + x_m^n$. This sum appears on the left of (7) and we are to prove that

$$s_n = (-1)^{n-1}\frac{2^{4n-1}B_{2n}}{(2n)!}m^{2n} + O(m^{2n-1}). \tag{9}$$

The proof is by induction on $n$. The case $n = 1$ was proved in Papadimitriou's paper. Now we assume that (9) is true for $n = 1, 2, \cdots, r-1$, and prove it for $n = r$, with the help of Newton's formulas (see [**10**], p. 261)

$$-s_r = (-1)^r r\sigma_r + \sum_{k=1}^{r-1}(-1)^{r-k}s_k\sigma_{r-k}, \qquad r = 1, 2, \cdots, m, \tag{10}$$

where $\sigma_1, \sigma_2, \cdots, \sigma_m$ are the elementary symmetric functions of the zeros $x_1, \cdots, x_m$. In this case we have

$$\sigma_r = \binom{2m+1}{2r+1} \Big/ \binom{2m+1}{1} = \frac{2m(2m-1)\cdots(2m-2r+1)}{(2r+1)!} = \frac{2^{2r}}{(2r+1)!} m^{2r} + O(m^{2r-1}), \tag{11}$$

for $r = 1, 2, \cdots, m$. Using this with (9) we find

$$(-1)^{r-k} s_k \sigma_{r-k} = (-1)^{r-1} \frac{2^{2r+2k-1} B_{2k}}{(2k)!(2r+1-2k)!} m^{2r} + O(m^{2r-1}),$$

so (10) becomes

$$\begin{aligned} -s_r &= \frac{2r(-1)^r 2^{2r-1}}{(2r+1)!} m^{2r} + (-1)^{r-1} 2^{2r-1} m^{2r} \sum_{k=1}^{r-1} \frac{2^{2k} B_{2k}}{(2k)!(2r+1-2k)!} + O(m^{2r-1}) \\ &= (-1)^r 2^{2r-1} m^{2r} \left\{ \frac{1}{(2r)!} - \sum_{k=0}^{r-1} \frac{2^{2k} B_{2k}}{(2k)!(2r+1-2k)!} \right\} + O(m^{2r-1}). \end{aligned}$$

Now we use Lemma 2 (stated below) to evaluate the expression in braces and we find

$$-s_r = (-1)^r \frac{2^{4r-1} B_{2r}}{(2r)!} m^{2r} + O(m^{2r-1}),$$

which proves (9) by induction.

## 4. A lemma on Bernoulli numbers.

LEMMA 2. *If $r \geqq 1$ we have*

$$\sum_{k=0}^{r} \frac{2^{2k} B_{2k}}{(2k)!(2r+1-2k)!} = \frac{1}{(2r)!}. \tag{12}$$

*Proof.* Let $B_n(x)$ denote the Bernoulli polynomial defined by

$$B_n(x) = \sum_{s=0}^{n} \binom{n}{s} B_s x^{n-s} \tag{13}$$

or, equivalently, by the power series expansion

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n, \qquad |z| < 2\pi.$$

A well-known property of $B_n(x)$ is the functional equation

$$B_n(1-x) = (-1)^n B_n(x) \tag{14}$$

which follows at once from the identity

$$\frac{ze^{(1-x)z}}{e^z-1} = \frac{-ze^{-xz}}{e^{-z}-1}.$$

Equation (14) implies

$$B_{2r+1}\left(\frac{1}{2}\right) = 0. \tag{15}$$

Formula (12) is a disguised form of (15). We use (15) along with (13) and multiply by $2^{2r+1}$ to obtain

$$\sum_{s=0}^{2r+1} \binom{2r+1}{s} 2^s B_s = 0.$$

In view of (3) this becomes

$$\binom{2r+1}{1} 2B_1 + \sum_{k=0}^{r} \binom{2r+1}{2k} 2^{2k} B_{2k} = 0,$$

which is the same as (12).

**5. Concluding remarks.** When the method of section 2 is applied to evaluate $\zeta(2n+1)$ we obtain the formula

$$\zeta(2n+1) = \left(\frac{\pi}{2}\right)^{2n+1} \lim_{m\to\infty} \frac{1}{m^{2n+1}} \sum_{k=1}^{m} \cot^{2n+1}\frac{k\pi}{2m+1}, \tag{16}$$

or its equivalent,

$$\sum_{k=1}^{m} \cot^{2n+1}\frac{k\pi}{2m+1} = \left(\frac{2}{\pi}\right)^{2n+1} \zeta(2n+1)m^{2n+1} + o(m^{2n+1}) \text{ as } m\to\infty. \tag{17}$$

Although (16) expresses $\zeta(2n+1)$ as a multiple of $\pi^{2n+1}$ it is not known if this multiple is rational or not. The author has been unable to extend the proof of Lemma 1 to obtain an alternate formula for the asymptotic value (for large $m$) of the sum in (17). All attempts to estimate this sum lead back to (17).

NOTE. After this paper was submitted for publication, a paper appeared by Kenneth S. Williams [12] on the same subject. Williams also uses the cotangent sum of Lemma 1 in his evaluation of $\zeta(2n)$, but his proof, like Euler's, uses complex function theory and cannot be considered elementary. See also I. Skau and E. S. Selmer, Nordisk Mat. Tidskr., 19 (1971) 120–124.

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# MATHEMATICAL EDUCATION

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## AN INTEGRATED SEQUENCE IN THE MATHEMATICAL SCIENCES FOR UNDERGRADUATE BUSINESS STUDENTS

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The courses in mathematical science (mathematics, statistics, and computer programming) which are required for every business student vary widely among colleges and universities. In a recent sample survey of midwestern universities, Rodger Collons [1] found that among the 30 schools surveyed on the semester system, the required hours fell between the extremes of 0 and 21. The median of the required hours among those 30 schools was 9. A typical program might therefore consist of one 3 hour course each in mathematics, statistics, and computer programming. It is the purpose of this article to describe a sequence of two 4 semester hour courses developed at the University of Iowa in which topics from the three areas of mathematics, statistics and computer programming are blended together in an effort to increase the motivation of each of these subject areas. It is hoped that in so doing, the student will acquire more of an overview of the mathematical sciences and how techniques from all three disciplines lend themselves (possibly in conjunction with one another) to the solution of business problems. This article contains some of the details of this sequence and some suggestions for integrating topics from the mathematical sciences.

**1. Course structure.** The number of students entering this sequence each year is