Math 148 Assignment 7

Due 2:00 p.m. Wednesday, April 1 in the Math 148 dropbox.

1. (a) Solve the DE $(x^2 - y^2)y' - 2xy = 0$.
   (b) Sketch the set of solution curves.
   (c) If $(1, 2)$ is a point on the solution curve, find the solution.

2. (a) Solve the DE $(1 + x^2)y' + 2xy = 0$.
   (b) Solve the DE $(1 + x^2)y' + 2xy = \cot x$.

3. (a) Consider the DE $y' + p(x)y = q(x)y^a$ where $a \notin \{0, 1\}$ is a real number. Set $z = y^{1-a}$. Turn this DE into a linear DE in z.
   (b) Use this to solve $xy^2y' + y^3 = x \cos x$.

4. Torricelli's Law for fluid flowing out the bottom of a tank states that the velocity is calculated as if the fluid dropped from the surface of the water. A hemispherical container of radius R is completely full of water. A small round hole of radius r at the bottom is unplugged. (You can ignore the small difference in the height caused by the hole.)
   (a) Use Newtonian mechanics to deduce how long it takes a drop of water to fall distance h from a position of rest, and use this to compute the velocity at that point.
   (b) Compute the volume $V(h)$ in the bowl when the water depth is h.
   (c) How long does it take the bowl to empty?
   Hint: compute $\frac{dV}{dt}$ in two ways. Get a DE in h.

5. Consider the homogeneous linear DE $y'' + p(x)y' + q(x)y = 0$. Suppose that y_1 and y_2 are two solutions on $[a, b]$. Define $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$.
   (a) Find a first order DE satisfied by W and solve it.
   (b) Prove that if the vectors $(y_1(c), y_1'(c))$ and $(y_2(c), y_2'(c))$ are linearly independent for some $c \in [a, b]$, then $W(x)$ never vanishes. Hence show that $(y_1(x), y_1'(x))$ and $(y_2(x), y_2'(x))$ are linearly independent vectors for every $x \in [a, b]$.

6. Consider a curve given in polar coordinates by $r(\theta) = \dfrac{1}{1 + e\cos\theta}$, where $e \geq 0$.
   (a) Show that the distance of each point on this curve to the line $x = \frac{1}{e}$ is a constant multiple of $r(\theta)$.
   (b) When $e > 1$, show that the curve approaches two asymptotes, find them and sketch the curve. **Hint:** If the critical angles are $\pm\theta_0$, compute the vertical distance of the point of the curve at angle $\theta = \theta_0 + h$ to the line $\theta = \theta_0$, and take a limit using Taylor approximations.
   (c) Observe that the curve is bounded if and only if $e < 1$. Show that the curve is an ellipse as follows: Let a be the midpoint between the two points intersecting the x-axis. Show that $(1 - e^2)(x - a)^2 + y^2$ is constant.
   (d) What happens when $e = 1$?