Math 148 Assignment 7
Due 2:00 p.m. Wednesday, April 1 in the Math 148 dropbox.

1. (a) Solve the DE \((x^2 - y^2)y' - 2xy = 0\).

(b) Sketch the set of solution curves.

(c) If \((1, 2)\) is a point on the solution curve, find the solution.

2. (a) Solve the DE \((1 + x^2)y' + 2xy = 0\).

(b) Solve the DE \((1 + x^2)y' + 2xy = \cot x\).

3. (a) Consider the DE \(y' + p(x)y = q(x)y^a\) where \(a \not\in \{0, 1\}\) is a real number.

    Set \(z = y^{1-a}\). Turn this DE into a linear DE in \(z\).

(b) Use this to solve \(xy^2y' + y^3 = x \cos x\).

4. Torricelli’s Law for fluid flowing out the bottom of a tank states that the velocity is calculated as if the fluid dropped from the surface of the water. A hemispherical container of radius \(R\) is completely full of water. A small round hole of radius \(r\) at the bottom is unplugged. (You can ignore the small difference in the height caused by the hole.)

(a) Use Newtonian mechanics to deduce how long it takes a drop of water to fall distance \(h\) from a position of rest, and use this to compute the velocity at that point.

(b) Compute the volume \(V(h)\) in the bowl when the water depth is \(h\).

(c) How long does it take the bowl to empty?

    **Hint:** compute \(\frac{dV}{dt}\) in two ways. Get a DE in \(h\).

5. Consider the homogeneous linear DE \(y'' + p(x)y' + q(x)y = 0\). Suppose that \(y_1\) and \(y_2\) are two solutions on \([a, b]\). Define \(W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)\).

(a) Find a first order DE satisfied by \(W\) and solve it.

(b) Prove that if the vectors \((y_1(c), y_1'(c))\) and \((y_2(c), y_2'(c))\) are linearly independent for some \(c \in [a, b]\), then \(W(x)\) never vanishes. Hence show that \((y_1(x), y_1'(x))\) and \((y_2(x), y_2'(x))\) are linearly independent vectors for every \(x \in [a, b]\).

6. Consider a curve given in polar coordinates by \(r(\theta) = \frac{1}{1 + e \cos \theta}\), where \(e \geq 0\).

(a) Show that the distance of each point on this curve to the line \(x = \frac{1}{e}\) is a constant multiple of \(r(\theta)\).

(b) When \(e > 1\), show that the curve approaches two asymptotes, find them and sketch the curve. **Hint:** If the critical angles are \(\pm \theta_0\), compute the vertical distance of the point of the curve at angle \(\theta = \theta_0 + h\) to the line \(\theta = \theta_0\), and take a limit using Taylor approximations.

(c) Observe that the curve is bounded if and only if \(e < 1\). Show that the curve is an ellipse as follows: Let \(a\) be the midpoint between the two points intersecting the \(x\)-axis. Show that \((1 - e^2)(x - a)^2 + y^2\) is constant.

(d) What happens when \(e = 1\)?