Math 148 Assignment 6

Due 2:00 p.m. Friday, March 20 in the Math 148 dropbox.

1. Find the radius of convergence for the following series, and evaluate the function.

   (a) $f(x) = \sum_{n=0}^{\infty} (n^2 + n)x^n$

   (b) $g(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

   (c) $h(x) = \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n^2 - n}$

   (d) Evaluate $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - n}$. Justify!

2. Suppose that $f(x) = \sum_{j=0}^{\infty} a_j x^j$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$ have positive radii of convergence R_1 and R_2 respectively. Let $c_n = \sum_{j=0}^{n} a_j b_{n-j}$ for $n \geq 0$; and let $R = \min\{R_1, R_2\}$.

   (a) Define $h(x) = \sum_{n=0}^{\infty} c_n x^n$. Prove that $h(x) = f(x)g(x)$ on $(-R, R)$.

   (b) Give an example where h has radius of convergence strictly greater than R.

3. A function defined on $\mathbb{R}$ satisfies the DE $f'(x) = 2xf(x) + 4x$ and $f(0) = 1$.
   Find a power series for the solution, and evaluate it.

4. (a) Verify that $4\tan^{-1}(\frac{1}{5}) - \tan^{-1}(\frac{1}{239}) = \frac{\pi}{4}$.

   (b) Use (a) and Taylor polynomials to calculate π to 6 decimals of accuracy with error estimates.

5. Use Taylor polynomials and *big O* arguments to compute these limits (no L'Hôpital rule).

   (a) $\lim_{x \to 0} \frac{x\cos x - \sin x}{x \sin^2 x}$

   (b) $\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2}$

   (c) $\lim_{x \to 0} \cot^2 x - \frac{1}{x^2}$

   (d) $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$ **Hint:** take the log.

6. (a) Let $X_n = \sum_{k=1}^{n} \frac{1}{k} - \ln(n+1)$ for $n \geq 1$. Express X_n as an area. Hence show that $X_n < X_{n+1} < 1$ for all $n \geq 1$. Deduce that this sequence has a limit, usually called γ.

   (b) Use (a) to obtain a useful expression for $\sum_{k=1}^{p_n} \frac{1}{2k-1} - \sum_{k=1}^{q_n} \frac{1}{2k}$.
   Hint: $\sum_{k=1}^{p} \frac{1}{2k-1} = \sum_{k=1}^{2p} \frac{1}{k} - \sum_{k=1}^{p} \frac{1}{2k}$.

   (c) Consider a *simple rearrangement* of the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ that contains the odd terms in order and the even terms in order, but the first n term of the series consists of p_n odd terms and q_n even terms. Prove that the series converges if and only if the limit $\lim_{n \to \infty} \frac{p_n}{q_n} = \alpha$ exists. Express the limit as a function of α.