1. Find the radius of convergence for the following series, and evaluate the function.

   (a) \( f(x) = \sum_{n=0}^{\infty} (n^2 + n)x^n \)

   (b) \( g(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \)

   (c) \( h(x) = \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n^2 - n} \)

   (d) Evaluate \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - n} \). Justify!

2. Suppose that \( f(x) = \sum_{j=0}^{\infty} a_j x^j \) and \( g(x) = \sum_{k=0}^{\infty} b_k x^k \) have positive radii of convergence \( R_1 \) and \( R_2 \) respectively. Let \( c_n = \sum_{j=0}^{n} a_j b_{n-j} \) for \( n \geq 0 \); and let \( R = \min\{R_1, R_2\} \).

   (a) Define \( h(x) = \sum_{n=0}^{\infty} c_n x^n \). Prove that \( h(x) = f(x)g(x) \) on \((-R, R)\).

   (b) Give an example where \( h \) has radius of convergence strictly greater than \( R \).

3. A function defined on \( \mathbb{R} \) satisfies the DE \( f'(x) = 2xf(x) + 4x \) and \( f(0) = 1 \). Find a power series for the solution, and evaluate it.

4. (a) Verify that \( 4 \tan^{-1}\left(\frac{1}{3}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \frac{\pi}{4} \).

   (b) Use (a) and Taylor polynomials to calculate \( \pi \) to 6 decimals of accuracy with error estimates.

5. Use Taylor polynomials and big \( O \) arguments to calculate these limits (no L’Hôpital rule).

   (a) \( \lim_{x \to 0} \frac{x \cos x - \sin x}{x \sin^2 x} \)

   (b) \( \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2} \)

   (c) \( \lim_{x \to 0} \cot^2 x - \frac{1}{x^2} \)

   (d) \( \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2} \) **Hint:** take the log.

6. (a) Let \( X_n = \sum_{k=1}^{n} \frac{1}{k} - \ln(n + 1) \) for \( n \geq 1 \). Express \( X_n \) as an area. Hence show that \( X_n < X_{n+1} < 1 \) for all \( n \geq 1 \). Deduce that this sequence has a limit, usually called \( \gamma \).

   (b) Use (a) to obtain a useful expression for \( \sum_{k=1}^{p} \frac{1}{2k-1} = \sum_{k=1}^{q} \frac{1}{2k} \).

   **Hint:** \( \sum_{k=1}^{p} \frac{1}{2k-1} = \sum_{k=1}^{2p} \frac{1}{k} - \sum_{k=1}^{p} \frac{1}{2k} \).

   (c) Consider a simple rearrangement of the alternating harmonic series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \) that contains the odd terms in order and the even terms in order, but the first \( n \) term of the series consists of \( p_n \) odd terms and \( q_n \) even terms. Prove that the series converges if and only if the limit \( \lim_{n \to \infty} \frac{p_n}{q_n} = \alpha \) exists. Express the limit as a function of \( \alpha \).