Math 148 Assignment 3

Due 2:00 p.m. Monday, February 9 in the Math 148 dropbox.

1. Evaluate the following improper integrals when they exist.

   (a) $\displaystyle\int_2^\infty \frac{dx}{x(\log x)^a}$ for $a > 0$.

   (b) $\displaystyle\int_0^{\pi/2} \log \sin x \, dx$. **Hint:** substitute $u = \frac{\pi}{2} - x$ and combine.

2. Which of the following improper integrals exist? (Do not try to evaluate them exactly.)

   (a) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}} \sin \tfrac{1}{x} \, dx$ (b) $\displaystyle\int_0^1 \frac{dx}{\ln x}$ (c) $\displaystyle\int_\pi^\infty \frac{\sin x}{\log x} \, dx$

3. Suppose that $f(x)$ and $g(x)$ are bounded continuous functions on $[0, \infty)$. If $\int_0^\infty f(x)\,dx$ exists as an improper integral, does it follow that $\int_0^\infty f(x)g(x)\,dx$ also exists? Give a proof or provide a counterexample.

4. (a) Consider the region R bounded by the curve $y = \dfrac{1}{\sqrt{7x - 10 - x^2}}$ for $2 < x < 5$ together with the lines $x = 2$, $x = 5$ and $y = 0$. Compute the volume of the solid obtained by rotating R about the y-axis.

   (b) Consider the region S bounded by the curve $y = \log x$ for $0 < x \leq 1$ together with the lines $x = 0$ and $y = 0$. Compute the volume of the solid obtained by rotating S about the x-axis.

5. Consider the parabola P given by $y = ax^2$ for $a > 0$. At each point (x_0, y_0) on P, construct the *normal* line through (x_0, y_0) perpendicular to the tangent line, and consider the area of the sector of P cut off by this line.

   (a) Find the minimal area of this sector.

   (b) What are the slopes of the normal lines that minimize this area?

6. (a) Compute the arc length of the curve $y = x^2$ from $x = 0$ to $x = 1$.

   (b) Show that the arc length of the curve $y = x^p$ from $x = 0$ to $x = 1$ is an increasing function of p for $p \geq 1$.
   Warning: as far as I know, this cannot be done using the arc length formula.
   A geometric argument is needed.