PM 950 H^p Spaces **Assignment 3** **Due Thursday November 23.**

1. (a) Let V be an isometry on a Hilbert space $\mathcal{H}$. Let $\mathcal{W} = \operatorname{Ran}(V)^\perp$ and $\mathcal{K} = \bigcap_{k\geq 1} \operatorname{Ran}(V^k)$. Show that $\mathcal{W}$ is a wandering space (i.e. $\langle V^k x, V^l y\rangle = 0$ for all $x, y \in \mathcal{W}$ and $k > l \geq 0$), and that $\mathcal{K}^\perp = \sum_{k\geq 0}^{\oplus} V^k \mathcal{W}$.

   (b) Show that $\mathcal{K}$ is a reducing subspace for V, $V|_{\mathcal{K}}$ is unitary and $V|_{\mathcal{K}^\perp}$ is unitarily equivalent to a direct sum of $\dim \mathcal{W}$ copies of the unilateral shift.

   (c) Let $0 \neq x \in \mathcal{K}^\perp$, and let $\mathcal{M}$ be the cyclic invariant subspace for V determined by x. Prove that $V|_{\mathcal{M}}$ is unitarily equivalent to the unilateral shift. **Hint:** show that the wandering space for $V|_{\mathcal{M}}$ is 1-dimensional and there is no unitary part.

2. (a) A weak-$*$ linear functional on H^∞ is given by $\Phi(h) = \int_{\mathbb{T}} hf\,dm$ for some $f \in L^1$. Find f_1, f_2 in L^2 so that $\Phi(h) = \langle M_h f_1, f_2\rangle$. Hence for any $\varepsilon > 0$, find g_1, g_2 in H^2 so that $\Psi(h) = \langle T_h g_1, g_2\rangle$ satisfies $\|\Phi - \Psi\| < \varepsilon$.

   (b) Find sequences $G_i = (g_i^{(1)}, g_i^{(2)}, \dots)$ in H^2 so that

   $$\textstyle\sum_{k\geq 1} \|g_i^{(k)}\|_2^2 < \infty \text{ for } i = 1, 2 \quad \text{and} \quad \Phi(h) = \sum_{k\geq 1} \langle M_h g_1^{(k)}, g_2^{(k)}\rangle.$$

   (c) Think of G_i as vectors in the direct sum of countably many copies of H^2. Let $\mathcal{M}$ be the cyclic invariant subspace for shift generated by G_1. Use 1(c) to find g_1, g_2 in H^2 so that $\Phi(h) = \langle T_h g_1, g_2\rangle$.

   (d) Show that the weak-$*$ closed ideals of H^∞ are of the form ωH^∞ for ω inner. **Hint:** if $\mathcal{I}$ is a weak-$*$ closed ideal, its closure in H^2 is an invariant subspace, say ωH^2. If $I \neq \omega H^\infty$, separate I from ω with a weak-$*$ continuous functional.

3. (a) Suppose A is a function algebra on X, $f \in A$, and E is a (clopen) subset of X so that $\operatorname{Re} f \geq 1$ on E and $\operatorname{Re} f \leq -1$ on $X \setminus E$. Prove that χ_E belongs to A. **Hint:** use Runge's Theorem to find polynomials p_n that converge uniformly on $(\|f\| + 1)\overline{\mathbb{D}} \cap \big\{|\operatorname{Re} z| \geq 1/2\big\}$ to a characteristic function.

   (b) Show that $H^\infty + \overline{H^\infty}$ is not dense in L^∞. **Hint:** First show this is equivalent to $\operatorname{Re} H^\infty$ being dense in $L^\infty_{\mathbb{R}}$. Let E be a measureable subset of $\mathbb{T}$ with $0 < |E| < 2\pi$, and set $f = 2\chi_E - 1$. Suppose that there is an $h \in H^\infty$ such that $\|f - \operatorname{Re} h\| < 1/2$. Apply part (a) with $X = \mathcal{M}(L^\infty)$.

4. Let A be a closed subalgebra of L^∞ containing H^∞.

   (a) Show that the restriction map from $\mathcal{M}(A)$ to $\mathcal{M}(H^\infty)$ is injective. Thus $\mathcal{M}(A)$ can be considered as a subset of $\mathcal{M}(H^\infty)$.

   (b) Show that if $H^\infty \subsetneq A \subset L^\infty$, then $A \supset H^\infty + C$. **Hint:** follow the strategy of Wermer's theorem, either evaluation at 0 is in $\mathcal{M}(A)$ or it isn't.

   (c) If g is invertible in A, show that there is an invertible h in H^∞ and an invertible unimodular function $u \in A$ so that $g = uh$. Hence show that A is generated by H^∞ and $\mathcal{U} = \{u \in A^{-1} : |u| = 1 \text{ a.e. on } \mathbb{T}\}$.

   (d) Show that if A is generated as a closed algebra by H^∞ and a *group* $\mathcal{U}$ of unimodular functions, then $\mathcal{M}(A) = \{\varphi \in \mathcal{M}(H^\infty) : |\varphi(u)| = 1 \text{ for all } u \in \mathcal{U}\}$.

   (e) What is $\mathcal{M}(H^\infty + C)$?