PM 950 H^p Spaces **Assignment 1** **Due Thursday October 5.**

1. (a) Show that if $u \in h_p$ for $p \geq 1$, then $|u(z)| \leq \Big(\dfrac{1+|z|}{1-|z|}\Big)^{1/p} \|u\|_p$.

   (b) Show that there is a universal constant $C_{1,p}$ so that if $f \in H^p$ for $p \geq 1$, then
$$|f'(z)| \leq C_{1,p}(1-|z|)^{-1-1/p}\|f\|_p.$$
   Hint: consider $g(z) = \frac{f(z)-f(z_0)}{z-z_0}$.

   (c) Generalize to find a similar estimate for the nth derivative of $f \in H^p$.

2. (a) Show that if $K \subset \mathbb{C}$ is compact and $v : K \to [-\infty, +\infty)$ is u.s.c., then there is a decreasing sequence u_n of continuous functions decreasing to v. **Hint:** $u_n(x) = \sup_{y\in K} v(y) - n|x-y|$.

   (b) Show that is v is subharmonic on a connected open set Ω and is not constantly $-\infty$, then for every $\overline{D_r(z_0)} \subset \Omega$,
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} v(z_0 + re^{it})\,dt > -\infty.$$

   (c) Show that if v_1 and v_2 are subharmonic on Ω, then so is $\max\{v_1, v_2\}$.

   (d) Prove that if v is C^2 and $\Delta v \geq 0$ on Ω, then v is subharmonic.

3. (a) Rewrite the formula for the conjugate function of $u = f * P$ for $f \in L^1$ as follows:
$$\tilde{u}(re^{i\theta}) = \frac{1}{2\pi}\int_0^{\pi} \Big(\frac{4r\sin^2(t/2)}{(1-r)^2 + 4r\sin^2(t/2)}\Big)\frac{f(\theta-t) - f(\theta+t)}{\tan(t/2)}\,dt.$$
   Hence show that if $\displaystyle\frac{1}{2\pi}\int_0^{\pi}\Big|\frac{f(\theta-t)-f(\theta+t)}{\tan(t/2)}\Big|\,dt < \infty$, then $\tilde{u}$ has a radial limit $\displaystyle\lim_{r\to 1^-}\tilde{u}(re^{i\theta})$.

   (b) Hence show that if f is C^1 on an arc $I \subset \mathbb{T}$, then for every compact arc $K \subset I$, $\tilde{u}(re^{i\theta})$ converges to $\frac{1}{2\pi}\int_0^{\pi}\frac{f(\theta-t)-f(\theta+t)}{\tan(t/2)}\,dt$ *uniformly* as $z \in \overline{\mathbb{D}}$ approaches $e^{i\theta} \in K$.

   (c) In particular, show that $\tilde{u}$ extends to be continuous on $\mathbb{D} \cup I$.

4. Let K be a compact subset of $\mathbb{T}$ of Lebesgue measure zero.

   (a) Find a positive function $f \in L^1$ such that f is C^∞ on $\mathbb{T} \setminus K$ and $\displaystyle\lim_{t\to t_0} f(t) = +\infty$ for every $t_0 \in K$.

   (b) Define $u = f * P$ and let $\tilde{u}$ be its harmonic conjugate. Set $g(z) = \dfrac{u + i\tilde{u}}{1 + u + i\tilde{u}}$.
   Prove that $g \in \mathrm{A}(\mathbb{D})$ (i.e. g extends to be continuous on the closed disc $\overline{\mathbb{D}}$), that $g|_K = 1$ and $|g| < 1$ on $\overline{\mathbb{D}} \setminus K$.

   (c) If $\mu \in \mathrm{M}(\mathbb{T})$ is of analytic type, evaluate $\displaystyle\lim_{k\to\infty}\int g^k\,d\mu$ in two ways. Hence provide another proof of the F. and M. Riesz Theorem.

5. (a) Show that if $f \in H^p$ and w is an analytic function from $\mathbb{D}$ into itself, then $f(w(z))$ belongs to H^p. **Hint:** use the least harmonic majorant.

   (b) Show that $f(z) = (1-z)^{-1}$ is in H^p for $0 < p < 1$, but not $p = 1$.

   (c) Show that if f is analytic on $\mathbb{D}$ and $\operatorname{Re} f(z) > 0$, then $f \in H^p$ for $0 < p < 1$.

   (d) Generalize to analytic functions with values in a sector $S = \{z : \alpha < \arg f(z) < \beta\}$..