PM 450 Solutions to Assignment 6

1. (a) Compute
$$\|f\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\theta^3 - \pi^2\theta)^2\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta^6 - 2\pi^2\theta^4 + \pi^4\theta^2\,d\theta$$
$$= \frac{1}{2\pi}\Big(\frac{\theta^7}{7} - \frac{2\pi^2\theta^5}{5} + \frac{\pi^4\theta^3}{3}\Big)\Big|_{-\pi}^{\pi} = \frac{8\pi^6}{105}.$$
Recall from Assignment 2 that $f \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}6i}{n^3}(e^{in\theta} - e^{-in\theta})$. Therefore
$$\|f\|_2^2 = \|\hat{f}\|_2^2 = \sum_{n\neq 0} \frac{36}{n^6} = 72\sum_{n=1}^{\infty}\frac{1}{n^6}.$$
Equating these two and solving, we obtain
$$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{8\pi^6}{(72)(105)} = \frac{\pi^6}{945}.$$

(b) Let $g = \frac{1}{4}\theta^4 - \frac{\pi^2}{2}\theta^2$, so that $g' = f$. As the series for f converges absolutely, it follows from term by term integration that
$$g \sim \hat{g}(0) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}6}{n^4}(e^{in\theta} + e^{-in\theta}).$$
We compute $\hat{g}(0) = \dfrac{-7\pi^4}{60}$. Also
$$\|g\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\tfrac{1}{4}\theta^4 - \tfrac{\pi^2}{2}\theta^2)^2\,d\theta = \frac{1}{32\pi}\int_{-\pi}^{\pi}\theta^8 - 4\pi^2\theta^6 + 4\pi^4\theta^4\,d\theta$$
$$= \frac{1}{32\pi}\Big(\frac{\theta^9}{9} - \frac{4\pi^2\theta^7}{7} + \frac{4\pi^4\theta^5}{5}\Big)\Big|_{-\pi}^{\pi} = \frac{107\,\pi^8}{(16)(315)}.$$
Therefore
$$\|g\|_2^2 = \|\hat{g}\|_2^2 = \frac{49\,\pi^8}{3600} + \sum_{n\neq 0}\frac{36}{n^8} = \frac{49\,\pi^8}{3600} + 72\sum_{n=1}^{\infty}\frac{1}{n^8}.$$
Equating these two and solving, we obtain
$$\sum_{n=1}^{\infty}\frac{1}{n^8} = \frac{107\,\pi^8}{(72)(16)(315)} - \frac{49\,\pi^8}{3600} = \frac{192\,\pi^8}{2^7 3^4 5^2 7} = \frac{\pi^8}{9450}.$$

2. Solution 1. Check by direct computation that $\{1, \sqrt{2}\cos n\theta : n \geq 1\}$ is an orthonormal set $L^2(0,\pi)$. To see that it is a basis, observe that the algebraic span is the set of even trig. polynomials. It is easy to see that the sum and product of even trig. polynomials are even trig. polynomials; and thus this set is an algebra. (It follows that the product of two cosines can be expressed as a sum of cosines, albeit in a somewhat complicated way.) Since $\cos\theta$ separates points of $[0,\pi]$, the Stone-Weierstrass Theorem shows that this algebra is dense in $\mathrm{C}[0,\pi]$ in the sup norm. Because $\|f\|_2 \leq \|f\|_\infty$, it follows that the L^2-closure of this algebra equals the L^2-closure of $\mathrm{C}[0,\pi]$, which is all of $L^2(0,\pi)$. Therefore this set is an orthonormal basis.

   Solution 2. Consider the map U of $L^2(0,\pi)$ into $L^2(-\pi,\pi)$ by $Uf(\theta) = \begin{cases} f(\theta) & \text{if } \theta \in (0,\pi) \\ f(-\theta) & \text{if } \theta \in (-\pi,0) \end{cases}$.
   Then it is clear that the range of U consists of all even functions. Moreover

$$\langle Uf, Ug\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} Uf(\theta)\overline{Ug(\theta)}\,d\theta = \frac{1}{\pi}\int_0^{\pi} f(\theta)\overline{g(\theta)}\,d\theta = \langle f, g\rangle.$$

   So U is a unitary map onto the subspace of even functions. This subspace is spanned by the orthonormal set $\{1, (e^{in\theta}+e^{-in\theta})/\sqrt{2} \ : n \geq 1\} = \{1, \sqrt{2}\cos n\theta : n \geq 1\}$. Thus U^{-1} will carry this set to an orthonormal basis of $L^2(0,\pi)$, namely $\{1, \sqrt{2}\cos n\theta : n \geq 1\}$.

   Remark. We could instead have used odd functions. Then we would find that $\{\sqrt{2}\sin n\theta : n \geq 1\}$ is also an orthonormal basis for $L^2(0,\pi)$.

3. Compute $\hat{f}(n) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{i(a-n)\theta}\,d\theta = \dfrac{e^{i(a-n)\theta}}{2\pi i(a-n)}\Big|_{-\pi}^{\pi} = \dfrac{(-1)^n}{\pi(a-n)}\,\dfrac{e^{ia\pi}-e^{-ia\pi}}{2i} = \dfrac{(-1)^n\sin a\pi}{\pi(a-n)}$.
   Applying Parseval's Theorem, we get

$$1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |e^{ia\theta}|^2\,d\theta = \|f\|_2^2 = \|\hat{f}\|_2^2 = \frac{1}{\pi^2}\sum_{n=\infty}^{\infty}\frac{\sin^2 a\pi}{(a-n)^2}.$$

   Rearranging, this yields $\displaystyle\sum_{n=\infty}^{\infty}\frac{1}{(a-n)^2} = \frac{\pi^2}{\sin^2(a\pi)}$.

4. (a) Let $f \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}$. Since f is continuous and 2π-periodic, $\widehat{f'}(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f'(\theta)\,d\theta = 0$.
   Integration by parts or term by term integration can be used to show that $\widehat{f'}(n) = ina_n$ for $n \neq 0$ (and thus for all n). Thus we have (since $a_0 = \hat{f}(0) = 0$)

$$\|f'\|_2^2 = \|\widehat{f'}\|_2^2 = \sum_{n=-\infty}^{\infty} n^2|a_n|^2 \geq \sum_{n=-\infty}^{\infty} |a_n|^2 = \|\hat{f}\|_2^2 = \|f\|_2^2.$$

   (b) Simllarly, we obtain that $\widehat{f''}(0) = 0$ and $\widehat{f''}(n) = -n^2 a_n$ for $n \neq 0$ (and thus for all n). Therefore by the Cauchy-Schwarz inequality,

$$\|f'\|_2^2 = \|\widehat{f'}\|_2^2 = \sum_{n=-\infty}^{\infty} n^2|a_n|^2 = \sum_{n=-\infty}^{\infty} a_n(n^2\bar{a}_n) = |\langle f, f''\rangle| \leq \|f\|_2\,\|f''\|_2.$$

5. (a) $\left|\int_X fg - \int_X f_n g\right| = \left|\int_X (f-f_n)g\right| \le \|f-f_n\|_p\,\|g\|_q$ by Hölder's inequality. By hypothesis, the RHS converges to 0, establishing that $\lim_{n\to\infty}\int_X f_n g = \int_X fg$.

(b) Since $f \in L^p(\mathbb{T})$ for $p<\infty$, we know that the Cesàro means $\sigma_n(f)$ converge to f in the $L^p(\mathbb{T})$ norm. By (a), we have

$$\frac{1}{2\pi}\int_{\mathbb{T}} fg = \lim_{n\to\infty}\frac{1}{2\pi}\int_{\mathbb{T}}\sigma_n(f)g = \lim_{n\to\infty}\frac{1}{2\pi}\int_{\mathbb{T}}\sum_{k=-n}^{n}(1-\tfrac{|k|}{n+1})\hat{f}(k)e^{ik\theta}g$$
$$= \lim_{n\to\infty}\sum_{k=-n}^{n}(1-\tfrac{|k|}{n+1})\hat{f}(k)\frac{1}{2\pi}\int_{\mathbb{T}}e^{ik\theta}g = \lim_{n\to\infty}\sum_{k=-n}^{n}\big(1-\tfrac{|k|}{n+1}\big)\hat{f}(k)\hat{g}(-k).$$

Remark: if $f\in L^\infty(\mathbb{T})$, this formula is still valid, but one must use the Cesàro means of g in the proof.

6. Let $g_n(\theta)=g(n\theta)$. We first need to compute the Fourier coefficients of g_n. Then for $0<|k|<n$,

$$\widehat{g_n}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(n\theta)e^{-ik\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(n(\theta+\tfrac{2\pi}{n}))e^{-ik\theta}\,d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(nt)e^{-ik(t-2\pi/n)}\,d\theta = e^{i2k\pi/n}\widehat{g_n}(k).$$

It follows that $\widehat{g_n}(k)=0$. Therefore

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} g(n\theta)e^{-ik\theta}\,d\theta = \lim_{n\to\infty}\widehat{g_n}(k) = 0 \quad \text{for all} \quad k\neq 0.$$

On the other hand, using the 2π-periodicity if g,

$$\widehat{g_n}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(n\theta)\,d\theta = \frac{1}{2\pi n}\int_{-n\pi}^{n\pi} g(t)\,dt = \frac{n}{2\pi n}\int_{-\pi}^{\pi} g(t)\,dt = \hat{g}(0).$$

<u>Alternate proof.</u> *For $k\notin n\mathbb{Z}$, using the 2π-periodicity of g,*

$$\widehat{g_n}(k) = \frac{1}{2\pi}\int_{0}^{2\pi} g(n\theta)e^{-ikt}\,d\theta = \frac{1}{2\pi n}\int_{0}^{-2n\pi} g(t)e^{-ikt/n}\,dt$$
$$= \sum_{p=0}^{n-1}\frac{1}{2\pi n}\int_{2p\pi}^{(2p+2)\pi} g(t)e^{-ikt/n}\,dt = \frac{1}{2\pi n}\int_0^{2\pi} g(t)\sum_{p=0}^{n-1}e^{-ik/n(t+2p\pi)}\,dt$$
$$= \frac{1}{2\pi n}\int_0^{2\pi} g(t)e^{-ikt/n}\sum_{p=0}^{n-1}e^{-ik2p\pi/n}\,dt = \frac{1}{2\pi n}\int_0^{2\pi} g(t)e^{-ikt/n}\Big(\frac{e^{-ik2n\pi/n}-1}{e^{-ik2\pi/n}-1}\Big)\,dt = 0$$

Therefore, if $h(\theta)=\sum_{k=-N}^{N} a_k e^{ik\theta}$ is a trig. polynomial, and $n>N$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)g(n\theta)\,d\theta = \sum_{k=-N}^{N} a_k\,\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}g(n\theta)\,d\theta = a_0\hat{g}(0) = \hat{h}(0)\hat{g}(0).$$

For a general $f \in L^p(\mathbb{T})$, pick a trig. polynomial h with $\|f - h\|_p < \varepsilon$ and $\hat{h}(0) = \hat{f}(0)$. Use Hölder's inequality to estimate

$$\begin{aligned}\Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)g(n\theta)\,d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)g(n\theta)\,d\theta\Big| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta) - h(\theta)||g(n\theta)|\,d\theta \\ &\le \|f - h\|_p\,\|g_n\|_q < \varepsilon\|g\|_q.\end{aligned}$$

Therefore for $n > \deg h$, we have

$$\Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)g(n\theta)\,d\theta - \hat{f}(0)\hat{g}(0)\Big| < \varepsilon\|g\|_q.$$

As $\varepsilon > 0$ is arbitrary, , we obtain

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)g(n\theta)\,d\theta = \hat{f}(0)\hat{g}(0).$$

7. (a) Observe that (3) implies (2) is trivial. The implication (2) implies (1) is an immediate consequence of the Uniform Boundedness Principle. We will prove that (1) implies (3). Let $C = \sup_{n\ge 1}\|S_{p,n}\|$. Fix $f \in L^p(\mathbb{T})$. For $\varepsilon > 0$, pick a trig. polynomial g such that $\|f - g\| < \varepsilon$. Then $S_{p,n}(g) = g$ for $n \ge \deg g$. Therefore for $n \ge \deg g$,

$$\|S_{p,n}f - f\|_p \le \|S_{p,n}S(f - g)\|_p + \|S_{p,n}g - g\|_p + \|g - f\|_p \le (C + 1)\varepsilon.$$

It follows that $S_{p,n}f$ converges to f in the $L^p(\mathbb{T})$ norm.

(b) For $f \in L^1(\mathbb{T})$,

$$\|S_{1,n}f\| = \|f * D_n\| \le \|f\|_1\,\|D_n\|_1;$$

so that $\|S_{1,n}\| \le \|D_n\|_1$. On the other hand, if K_m is the Féjer kernel, then $\|K_m\|_1 = 1$. We have

$$\lim_{m\to\infty} S_{1,n}K_m = \lim_{m\to\infty} K_m * D_n = \lim_{m\to\infty} \sigma_m(D_n) = D_n.$$

It follows that

$$\|S_{1,n}\| \ge \lim_{m\to\infty} \|\sigma_m(D_n)\|_1 = \|D_n\|_1.$$

Hence $\|S_{1,n}\| = \|D_n\|_1$. This is unbounded as $n \to \infty$, and therefore by (a), we deduce that there is some function $f \in L^1(\mathbb{T})$ such that $\|S_{1,n}(f)\|$ is unbounded, and therefore diverges.