PM 450 Solutions to Assignment 5

1. (a) Let $g_n = \inf\{f_i : i \geq n\}$. Then $0 \leq g_n \leq f_n$ are non-negative measurable functions which are monotone increasing to $\liminf f_n = \lim g_n =: g$. By the Monotone Convergence Theorem applied to $\{g_n\}_{n\geq 1}$,
$$\int \liminf f_n = \int g = \lim \int g_n \leq \liminf \int f_n.$$

(b) Since $f_n \to f$ a. e., we have $f_n \chi_E \to f\chi_E$ a. e. and $f_n\chi_{X\setminus E} \to f\chi_{X\setminus E}$ a. e. By Fatou's Lemma, we obtain
$$\int_E f = \int f\chi_E \leq \liminf \int f_n\chi_E = \liminf \int_E f_n$$
and
$$\int_{X\setminus E} f = \int f\chi_{X\setminus E} \leq \liminf \int f_n\chi_{X\setminus E} = \liminf \int_{X\setminus E} f_n.$$
Hence we have
$$\lim_{n\to\infty} \int f_n = \int f = \int_E f + \int_{X\setminus E} f \leq \liminf \int_E f_n + \liminf \int_{X\setminus E} f_n \leq \liminf \int f_n.$$
As the LHS equals the RHS and is finite, we must have that $\int_E f = \liminf \int_E f_n$ and $\int_{X\setminus E} f = \liminf \int_{X\setminus E} f_n$. Therefore
$$\limsup \int_E f_n = \limsup \int f_n - \int_{X\setminus E} f_n = \int f - \liminf \int_{X\setminus E} f_n = \int f - \int_{X\setminus E} f = \int_E f.$$
Consequently, $\int_E f = \limsup \int_E f_n = \liminf \int_E f_n = \lim \int_E f_n$.

(c) Define f_n on $(0,\infty)$ by $f_n(x) = \begin{cases} n & \text{if } \ x \in (0,\frac{1}{n}) \\ 1 & \text{if } \ x \in [1,n) \\ 0 & \text{otherwise} \end{cases}$. Then $\lim f_n = \chi_{[1,\infty)} =: f$. Note that $\int f_n = n \to \infty = \int f$. However taking $E = (0,1)$, we have $\int_E f_n = 1 \to 1 \neq 0 = \int_E f$.

2. (a) Let $f_n(x) = \dfrac{1+nx^2}{(1+x^2)^n}$ on $[0,\infty)$. Observe that for $x > 0$,
$$\frac{f_{n+1}(x)}{f_n(x)} = \frac{1+(n+1)x^2}{(1+nx^2)(1+x^2)} = \frac{1+\frac{x^2}{1+nx^2}}{1+x^2} < 1.$$
and that $\lim\limits_{n\to\infty} \dfrac{f_{n+1}(x)}{f_n(x)} = \dfrac{1}{1+x^2} < 1$. It follows that f_n are monotone decreasing and $\lim\limits_{n\to\infty} f_n(x) = 0$ for $x > 0$. So $f_n \to 0$ a. e. Note that $f_2(x) = \dfrac{1+2x^2}{1+x^2} \leq \dfrac{2}{1+x^2}$ is integrable on $[0,\infty)$. Thus by the LDCT (since $|f_n| = f_n \leq f_2$ for $n \geq 2$),
$$\lim_{n\to\infty} \int_0^\infty \frac{1+nx^2}{(1+x^2)^n}\,dx = \int_0^\infty 0 = 0.$$

(b) Let $f_n(x) := \dfrac{n\sin(x/n)}{x(1+x^2)} = \dfrac{\sin(x/n)}{x/n}\dfrac{1}{1+x^2}$. Observe that $|\sin t| \le |t|$ for all $t \in \mathbb{R}$, and therefore $|f_n| \le \dfrac{1}{1+x^2}$; and $\dfrac{1}{1+x^2}$ is integrable. Also $\lim\limits_{t\to 0}\dfrac{\sin t}{t} = 1$ and therefore $\lim\limits_{x\to 0} f_n(x) = \dfrac{1}{1+x^2}$. Thus by the LDCT,

$$\lim_{n\to\infty}\int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\,dx = \int_0^\infty \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_0^\infty = \frac{\pi}{2}.$$

3. First suppose that f_n are real valued. Then $g_n \pm f_n \ge 0$, and $g_n \pm f_n \to g \pm f$. Hence by Fatou's Lemma,

$$\int g \pm f \le \liminf \int g_n \pm f_n = \int g + \liminf \int \pm f_n.$$

Subtracting the finite value $\int g$, we obtain

$$\int f \le \liminf \int f_n \quad \text{and} \quad -\int f \le \liminf \int -f_n = -\limsup \int f_n.$$

Therefore $\limsup \int f_n \le \int f \le \liminf \int f_n$; whence $\lim\limits_{n\to\infty}\int f_n = \int f$. The complex case is obtained by considering the real and imaginary parts of f_n, which satisfy the same hypotheses.

4. (a) Note that $\lim_{x\to 0} f(x) = 1$, so f is continuous on $[0,\infty)$, and thus is both Riemann and Lebesgue integrable on $[0,A]$ for any $A < \infty$. To set the stage, we recall the argument from first year calculus that the improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\,dx := \lim_{A\to\infty}\int_0^A \frac{\sin x}{x}\,dx$$

exists. Indeed let $a_n = \int_{n\pi}^{(n+1)\pi}\dfrac{\sin x}{x}\,dx$ for $n \ge 0$. We can make the estimates for $n \ge 1$: we have $a_n = (-1)^n|a_n|$ and

$$\int_{n\pi}^{(n+1)\pi}\frac{|\sin x|}{(n+1)\pi}\,dx \le |a_n| \le \int_{n\pi}^{(n+1)\pi}\frac{|\sin x|}{n\pi}\,dx.$$

This yields $\frac{2}{(n+1)\pi} \le |a_n| \le \frac{2}{n\pi}$. Thus $|a_n|$ decreases to 0 monotonely, and the terms alternate in sign, whence the series $\sum_{n\ge 0} a_n$ converges by the alternating series test. From that, it is easy to deduce that the limit exists as $A \to \infty$. These estimates also show that

$$\int_0^\infty |f(x)|\,dx = \sum_{n=0}^\infty |a_n| \ge \sum_{n=1}^\infty \frac{2}{(n+1)\pi} = +\infty$$

because the harmonic series diverges. So $|f|$ is not integrable. Therefore f is not Lebesgue integrable.

(b) Let $f_n = f\chi_{[0,n]}$. Then by assumption, the improper Riemann integral exists and equals

$$\int_0^\infty f(x) = \lim_{A\to\infty}\int_0^\infty f(x)\,dx = \lim_{n\to\infty}\int_0^n f(x)\,dx = \lim_{n\to\infty}\int f_n.$$

However since f is Lebesgue integrable, so is $|f|$. We have that $|f_n| \le |f|$ and f_n converges to f pointwise. Therefore by the LDCT, we have

$$\int f = \lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int_0^n f(x)\,dx.$$

Thus the two integrals agree.

5. <u>First proof.</u> Since $\varphi(x) = x^p$ is convex on $\mathbb{R}_+$, we have

$$\left(\frac{x+y}{2}\right)^p \le \frac{x^p+y^p}{2} \quad \text{or} \quad (x+y)^p \le 2^{p-1}(x^p+y^p) \quad \text{for} \quad x,y \ge 0.$$

Therefore for any complex numbers z, w, we have $|z-w|^p \le (|z|+|w|)^p \le 2^{p-1}(|z|^p + |w|^p)$. It follows that $|f-f_n|^p \le 2^{p-1}(|f|^p + |f_n|^p) =: g_n$. Moreover $\lim g_n = 2^p|f|^p =: g$. As these functions are all integrable, and $\lim \displaystyle\int g_n = \int g$ by hypothesis, we may apply Q.3 to obtain that

$$\lim_{n\to\infty} \|f-f_n\|_p^p = \lim_{n\to\infty} \int |f-f_n|^p = \int \lim_{n\to\infty} |f-f_n|^p = \int 0 = 0.$$

That is, f_n converges to f in L^p.

<u>Second proof.</u> Let $\varepsilon > 0$. Using absolute continuity of the integrable function $|f|^p$, there is a $\delta > 0$ so that if $m(A) < \delta$, then $\displaystyle\int_A |f|^p < \varepsilon$. There is a measurable set X so that $m(X) < \infty$ such that $\displaystyle\int_X |f|^p > \|f\|_p^p - \varepsilon$; so $\displaystyle\int_{X^c} |f|^p < \varepsilon$. By Egorov's Theorem, there is a subset $E \subset X$ with $m(X \setminus E) < \delta$ so that $f_n \to f$ uniformly on E. Therefore, $\lim \displaystyle\int_E |f-f_n|^p = 0$. By Q.1b, since $|f_n|^p \to |f|^p$ a. e., and all are positive and integrable,

$$\lim \int_{E^c} |f_n|^p = \int_{E^c} |f|^p = \int_{X\setminus E} |f|^p + \int_{X^c} |f|^p < \varepsilon + \varepsilon = 2\varepsilon.$$

Therefore

$$\begin{aligned}
\limsup \|f-f_n\|_p^p &= \limsup \int |f-f_n|^p \\
&\le \limsup \int_E |f-f_n|^p + \int_{E^c} 2^{p-1}(|f|^p + |f_n|^p) \\
&< 0 + 2^{p-1}(2\varepsilon + 2\varepsilon) = 2^{p+1}\varepsilon.
\end{aligned}$$

Consequently, $\lim\limits_{n\to\infty} \|f-f_n\|_p = 0$; that is, $f_n \to f$ in L^p.

6. (a) Let $s = r/p$ and let t be the conjugate value so that $\frac{1}{s} + \frac{1}{t} = 1$. Apply Hölder's inequality:

$$\|f\|_p^p = \int_X |f|^p = \int_X |f|^p \cdot 1 \le \left(\int_X |f|^{ps}\right)^{1/s} \left(\int_X 1^t\right)^{1/t} = \|f\|_r^{r/s} m(X)^{1/t} = \|f\|_r^p\, m(X)^{1-\frac{p}{r}}.$$

Take the pth root, and get $\|f\|_p \le \|f\|_r m(X)^{\frac{1}{p}-\frac{1}{r}}$. Hence if $f \in L^r(X)$, it also belongs to $L^p(X)$. That is, $L^r(X) \subset L^p(X)$.

(b) Observe that $f_a(x) := x^{-a}\chi_{[1,\infty)} \in L^q \Leftrightarrow \displaystyle\int_1^\infty x^{-aq}\,dx < \infty \Leftrightarrow aq > 1 \Leftrightarrow q > \frac{1}{a}$. Likewise

$g_b(x) = x^{-b}|\log x|^{-2b}\chi_{(0,1/e]} \in L^q \Leftrightarrow \displaystyle\int_0^{1/e} x^{-bq}|\log x|^{-2bq}\,dx < \infty \Leftrightarrow bq \le 1 \Leftrightarrow q \le \frac{1}{b}$. So take $a = 1/p$ and $b = 1/r$. Then $f_{1/p} + g_{1/r}$ belongs to L^q if and only if $p < q \le r$.

7. First suppose that $f = \chi_{(2^{-k}(i-1),2^{-k}i]}$ is the characteristic function of some dyadic interval. Then for $n > k$, it is clear that $m(A_n \cap (2^{-k}(i-1), 2^{-k}i]) = 2^{-k-1}$. Hence

$$\int_{A_n} f = \frac{1}{2}\int_0^1 f(x)\,dx \quad \text{for} \quad n > k.$$

The same is true for a finite linear combination of such functions, namely step functions with discontinuities at dyadic rationals. Now every continuous function is uniformly approximable by such step functions. Thus these step functions are uniformly dense in $C[0,1]$. The continuous functions are dense in $L^1(0,1)$ and $\|f\|_1 \le \|f\|_\infty$, so the step functions are also dense in $L^1(0,1)$. Take $f \in L^1(0,1)$ and let f_k be step functions with dyadic discontinuities such that $f_k \to f$ in L^1. Then for any fixed k and n sufficiently large, we have

$$\begin{aligned}
\left|\int_{A_n} f - \frac{1}{2}\int_{[0,1]} f\right| &= \left|\int_{A_n} f - f_k + \int_{A_n} f_k - \frac{1}{2}\int_{[0,1]} f_k + \frac{1}{2}\int_{[0,1]} f_k - f\right| \\
&\le \int_{A_n} |f - f_k| + \Big(\int_{A_n} f_k - \frac{1}{2}\int_{[0,1]} f_k\Big) + \frac{1}{2}\int_{[0,1]} |f_k - f| \\
&\le \|f - f_k\|_1 + 0 + \frac{1}{2}\|f - f_k\|_1 = \frac{3}{2}\|f - f_k\|_1.
\end{aligned}$$

As $\|f - f_k\|_1$ is arbitrarily small, it follows that

$$\lim_{n\to\infty} \int_{A_n} f = \frac{1}{2}\int_{[0,1]} f$$