PM 450 Solutions to Assignment 4

1. Let $\varepsilon > 0$, and write $I_i = (a_i, b_i)$. I claim that $I_i' = (a_i - \varepsilon, b_i + \varepsilon)$ covers $[0,1]$. Indeed let $x \in [0,1]$. Pick a rational point $y \in (x-\varepsilon, x+\varepsilon) \cap \mathbb{Q} \cap [0,1]$. For some i, $y \in (a_i, b_i)$ by hypothesis. Hence $x \in I_i'$. Since $m^*([0,1]) = 1$, it follows that
$$1 \le \sum_{i=1}^{n} \ell(I_i') = \sum_{i=1}^{n} \ell(I_i) \ + 2n\varepsilon.$$
Let ε decrease to 0 to get $\sum_{i=1}^{n} \ell(I_i) \ge 1$.

   Second proof. If finitely many $I_i = (a_i, b_i)$ cover $\mathbb{Q} \cap [0,1]$, then their closures $[a_i, b_i]$ have closed union and thus cover the closure of $\mathbb{Q} \cap [0,1]$ which is $[0,1]$. Therefore $\sum_{i=1}^{n} b_i - a_i \ge m^*([0,1]) = 1$.

2. (a) Recall that $\{0 = r_0, r_1, r_2, \dots\}$ is an enumeration of $\mathbb{Q} \cap [0,1)$, and that
$$E_k = E_0 + r_k \pmod 1 := \big(E_0 \cap [0, 1-r_k)\big) + r_k \ \cup \ \big(E_0 \cap [1-r_k, 1)\big) + r_k - 1.$$
Define translates of F inside each E_k by
$$F_k = F + (r_k - r_n) \pmod 1$$
$$= \begin{cases} \big(F \cap [0, 1-r_k+r_n)\big) + r_k - r_n \ \cup \ \big(F \cap [1-r_k+r_n, 1)\big) + r_k - r_n - 1 & \text{if } r_k \ge r_n \\ \big(F \cap [0, r_n - r_k)\big) + 1 - r_n + r_k \ \cup \ \big(F \cap [r_n - r_k), 1)\big) - r_n + r_k & \text{if } r_k < r_n. \end{cases}$$
As above, this splits F into two pieces and translates each of them. It follows that F_n is measurable and $m(F_k) = m(F)$ for all $k \ge 0$. Since $F_k \subset E_k$, these sets are disjoint. Therefore by countable additivity,
$$1 = m([0,1)) \ge m\Big(\dot{\bigsqcup}_{k\ge 0} F_k\Big) = \sum\nolimits_{k \ge 0} m(F_k) = \sum\nolimits_{k\ge 0} m(F).$$
Therefore $m(F) = 0$.

   (b) Since $F = \dot{\bigsqcup}_{k \in \mathbb{Z}} F \cap [k, k+1)$ and $0 < m(F) = \sum_{k \in \mathbb{Z}} m(F \cap [k, k+1))$, there is an integer k so that $m(F \cap [k, k+1)) > 0$. We may translate this set into $[0,1)$. If we find a non-measureable set inside this set, then the translate back yields a non-measureable set inside F. So we may assume that $F \subset [0,1)$.
   Let $F_n = F \cap E_n$ for $n \ge 0$. If these sets are all measurable, then by 2(a), we have $m(F_n) = 0$. However $F = \dot{\bigsqcup}_{n \ge 0} F_n$, and hence $m(F) = \sum_{n\ge 0} m(F_n) = 0$. This is a contradiction, and thus at least one of the sets F_n must be non-measurable.

3. Since g is continuous, $g^{-1}((a,\infty)) = U$ is open for any $a \in \mathbb{R}$. Therefore
$$(g \circ f)^{-1}((a, \infty)) = f^{-1}(U)$$
is measurable. Hence $g \circ f$ is measurable.

4. First assume that f_n are real valued. Then $g = \liminf f_n$ and $h = \limsup f_n$ are measurable. Notice that
$$A = \{x : \lim_{n\to\infty} f_n(x) \ \text{ exists}\} = \{x : g(x) = h(x)\} = (h-g)^{-1}(\{0\}),$$
which is measurable because $\{0\}$ is Borel. For the complex case, let
$$B = \{x : \lim_{n\to\infty} \operatorname{Re} f_n(x) \ \text{ exists}\} \quad \text{and} \quad C = \{x : \lim_{n\to\infty} \operatorname{Im} f_n(x) \ \text{ exists}\}.$$
Then by the real case, B and C are measurable. Since $A = B \cap C$, it is also measurable.

5. Let $n_0 = 1$. For each $i \geq 1$, recursively select $n_i > n_{i-1}$ so that

$$m(\{x : |f(x) - f_{n_i}(x)| \geq 2^{-i}\}) < 2^{-i}.$$

I claim that f_{n_i} converges to f almost everywhere. Indeed, let

$$A_i = \bigcup_{j \geq i} \{x : |f(x) - f_{n_j}(x)| \geq 2^{-i}\} \subseteq \bigcup_{j \geq i} \{x : |f(x) - f_{n_j}(x)| \geq 2^{-j}\}.$$

By subadditivity,

$$m(A_i) \leq \sum_{j \geq i} m(\{x : |f(x) - f_{n_i}(x)| \geq 2^{-i}\}) < \sum_{j \geq i} 2^{-j} = 2^{1-i}.$$

Observe that $A_i \supseteq A_{i+1}$ and define $A = \bigcap_{i \geq 1} A_i$. Then $m(A) = \lim_{i \to \infty} m(A_i) = 0$. So for $x \in [0,1] \setminus A$, there is some i_0 so that $x \notin A_{i_0}$. Therefore for every $i \geq i_0$, we have $|f(x) - f_{n_j}(x)| < 2^{-i}$ for all $j \geq i$. Thus $\lim_{i \to \infty} f_{n_i}(x) = f(x)$ for all $x \in [0,1] \setminus A$, which is almost everywhere.

Remark: it is not true that the whole sequence converges a.e. *Take a sequence of characteristic functions* $f_n = \chi_{A_n}$ *where* $A_n = [\log n, \log(n+1))] \pmod 1$, *by which I mean that the set is translated by an integer (or split and translated) so as to fit inside* $[0,1)$. *The because* $\log n \to +\infty$, *for every* x *in* $[0,1)$, $f_n(x) = 1$ *infinitely often, and* $f_n(x) = 0$ *infinitely often. So the series does not converge at any point. However* $m(\{x : f_n(x) \neq 0\}) = \log \frac{n+1}{n} \to 0$, *so the sequence converges to* 0 *in measure.*

6. (a) The usual Cantor set is obtained using $a_n = 3^{-n}$, and there are 2^{n-1} intervals removed at the nth stage. Therefore

$$m(C) = m([0,1]) - \sum_{n \geq 1} \frac{2^{n-1}}{3^n} = 1 - \frac{1/3}{1 - 2/3} = 0.$$

(b) Let $0 < t \leq 1/3$ and let $a_n = t^n$ to get a Cantor set K_t. Then arguing as in 6(a), this set has measure

$$m(K_t) = 1 - \sum_{n \geq 1} 2^{n-1} t^n = 1 - \frac{t}{1-2t} = \frac{1-3t}{1-2t}.$$

Solve $\dfrac{1-3t}{1-2t} = r$ to get $t = \dfrac{1-r}{3-2r}$. Then $m(K_t) = r$.

(c) We need some notation. Let the intervals removed at the nth stage for C be denoted $I_{n,i}$ for $1 \leq i \leq 2^{n-1}$ in increasing order; and let the corresponding intervals removed at the nth stage for K be denoted $J_{n,i}$ for $1 \leq i \leq 2^{n-1}$. We could obtain a formula for the endpoints of these intervals, but it is not necessary. Define h to be linear and increasing on each $I_{n,i}$ with range $J_{n,i}$. Observe that h is an increasing function from $U := \bigcup_{n,i} I_{n,i}$ onto $V := \bigcup_{n,i} J_{n,i}$, and that these are both dense open subsets of $[0,1]$. Then define

$$h(0) = 0 \quad \text{and} \quad h(x) = \sup\{h(t) : t < x,\ t \in U\} \text{ for } x \in [0,1] \setminus U.$$

Note that h is a monotone strictly increasing function from $[0,1]$ into $[0,1]$ and that $h(1) = 1$.

The only discontinuities that a monotone function can have are jump discontinuities, because there is a limit from the left and from the right at each point. After removing the intervals at the nth stage, there are 2^n intervals remaining of equal length, so that their lengths are at most 2^{-n}. It follows that the monotone function h has no jump discontinuities with a gap of more than 2^{-n}. Since n is arbitrary, h

has no jump discontinuities, and therefore is continuous. So h is a homeomorphism of $[0,1]$ onto itself. Since $h(U) = V$, we have

$$h(C) = h([0,1] \setminus U) = [0,1] \setminus V = K.$$

(d) This problem is tricky. Some care must be taken to ensure that the resulting set has less than full measure. Following the hint, consider Cantor sets K_n in $[0,1]$ with $m(K_n) = 2^{-n-1}$ as in 6(b). Work inside $[0,1]$. Let $E_1 = K_1$. In each open interval in the complement of E_1, put a scaled copy of K_2 to get a set E_2. Then in each interval of the complement of E_2, put a scaled copy of K_3 to get E_3, etc. At each stage, the complement of E_n in $[0,1]$ is a dense open set U_n. In the end, one obtains a set $E_\infty = \bigcup_{n\geq 1} E_n \subset [0,1]$.

Estimate $m(E_\infty)$. Clearly $m(E_\infty) > m(K_1) = 1/4$. On the other hand, at the nth stage, we are adding sets which have total measure equal to $2^{-n-1} m([0,1] \setminus E_{n-1}) < 2^{-n-1}$. Thus $1/4 < m(E_\infty) < \sum_{n\geq 1} 2^{-n-1} = 1/2$. Observe that the largest interval removed from any K_n has length less than $1/3$, and thus at the nth stage, the largest interval in the complement of E_n has length at most 3^{-n}. Finally we define $E = \bigcup_{k\in\mathbb{Z}} E_\infty + k$.

Given an interval I, translate it by an integer so that $I \cap (0,1) = (a,b)$ is non-empty. This does not affect $m(I \cap E)$. Choose n_0 so that $b - a > 3^{1-n_0}$. Since U_{n_0} is dense in $[0,1]$, it contains a point x within $\frac{1}{2}3^{-n_0}$ of the midpoint of I. The component J of U_n containing x has length less than 3^{-n_0}, so $J \subset I$. Hence I contains a scaled copy of K_{n_0+1} inside J. Thus $m(I \cap E) > 0$. On the other hand, within J, the argument of the previous paragraph shows that $m(E \cap J) < m(J)$. It follows that $0 < m(I \cap E) < m(I)$.

7. (a) Since $f(x)$ is monotone increasing and continuous, and $k(x) = x$ is monotone strictly increasing and continuous, $g(x) = f(x) + x$ is strictly increasing and continuous. Clearly $g(0) = 0$ and $g(1) = 2$, so g maps $[0,1]$ one-to-one and onto $[0,2]$. The inverse function $h = g^{-1}$ is thus a strictly increasing function of $[0,2]$ onto $[0,1]$. As the range of h has no gaps, it has no jump discontinuities, and therefore it is continuous. So g is a homeomorphism.

(b) Recall that $U = [0,1] \setminus C$ is a disjoint union of open intervals $I_i = (a_i, b_i)$ on which f is constant, so g translates each interval to $J_i = I_i + f(a_i)$. It follows from 6(a) that the open set $g(U)$ has measure

$$m(g(U)) = \sum_{i\geq 1} m(J_i) = \sum_{i\geq 1} m(I_i) = 1.$$

Therefore

$$m(g(C)) = m(g([0,1]) - m(g(U)) = 2 - 1 = 1.$$

(c) By 2(b), the set $g(C)$ contains a non-measurable set F. Let $A = g^{-1}(F) \subset C$. Then $m(A) = 0$ because all sets with outer measure 0 are measurable. However $h^{-1}(A) = F$ is not measurable.

(d) Let $f = \chi_A$ be the characteristic function of A, which is measurable because A is measurable. Then $f \circ h = \chi_F$, which is not measurable because $\chi_F^{-1}((.5,\infty)) = F$.