PM 450 Solutions to Assignment 3

1. (a) Easy: $2\big(1-\frac{|k|}{2n+2}\big)-\big(1-\frac{|k|}{n+1}\big)=1$ for $|k|\le n+1$; and $2\big(1-\frac{|k|}{2n+2}\big)=\frac{2n+2-|k|}{n+1}$ for $n+2\le |k|\le 2n+1$. Thus $2K_{2n+1}(\theta)-K_n(\theta)=V_n(\theta)$.

   (b) Clearly V_n is even. Since the coefficient of 1 is always 1, we get $\frac{1}{2\pi}\int_{-\pi}^{\pi}V_n(\theta)\,d\theta=1$. By (a), we have

$$\|V_n\|_1\le 2\|K_{2n+1}\|_1+\|K_n\|_1=3.$$

   Thus V_n is a summability kernel.

2. (a) Note that $f(\theta)=\theta^3-\pi^2\theta$ is odd, so $\hat f(0)=0$. Compute for $n\ne 0$:

$$\begin{aligned}\hat f(n)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}(\theta^3-\pi^2\theta)e^{-in\theta}\,d\theta\\&=\frac{(\theta^3-\pi^2\theta)e^{-in\theta}}{2\pi(-in)}\bigg|_{-\pi}^{\pi}-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(3\theta^2-\pi^2)e^{-in\theta}}{-in}\,d\theta=\frac{3}{2\pi in}\int_{-\pi}^{\pi}\theta^2e^{-in\theta}\,d\theta\\&=\frac{3\theta^2}{2\pi in}\frac{e^{-in\theta}}{(-in)}\bigg|_{-\pi}^{\pi}-\frac{3}{2\pi n^2}\int_{-\pi}^{\pi}2\theta e^{-in\theta}\,d\theta=\frac{-3}{\pi n^2}\int_{-\pi}^{\pi}\theta e^{-in\theta}\,d\theta\\&=\frac{-3}{\pi n^2}\frac{\theta e^{-in\theta}}{(-in)}\bigg|_{-\pi}^{\pi}+\frac{3}{2\pi in^3}\int_{-\pi}^{\pi}e^{-in\theta}\,d\theta=\frac{3(-1)^n}{\pi in^3}(2\pi)=\frac{(-1)^{n-1}6i}{n^3}\end{aligned}$$

   Therefore

$$f\sim\sum_{n=1}^{\infty}\frac{(-1)^{n-1}6i}{n^3}(e^{in\theta}-e^{-in\theta})=\sum_{n=1}^{\infty}\frac{(-1)^n12}{n^3}\sin n\theta.$$

   (b) Since $\sum_{n\ne0}\frac{6}{n^3}<\infty$, the Fourier series converges absolutely and uniformly by the Weierstrass M-test to a 2π-periodic continuous function. Moreover the Fourier series of this limit coincides with the Fourier series of f. Two continuous functions with the same series are the same. So it converges uniformly to f.

   Alternatively, you could observe that $\hat f(n)=O(n^{-3})=O(\frac{1}{n})$. By Fejer's Theorem, the Cesáro means converge to f. Thus by Hardy's Tauberian Theorem, the Fourier series converges uniformly to f. Since f in C^1, you could instead quote the Dirichlet-Jordan Theorem.

   (c) If we substitute $\theta=\pi/2$, we obtain

$$f(\tfrac{\pi}{2})=\frac{-3\pi^3}{8}=12\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{(2n+1)^3}(-1)^n.$$

   Therefore $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{\pi^3}{32}.$

3. (a) First, $\hat{f}(0) = \frac{1}{2\pi}\int_{-2\pi/3}^{2\pi/3} dt = \frac{2}{3}$. For $n \neq 0$, compute:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-2\pi/3}^{2\pi/3} e^{-in\theta}\,d\theta = \frac{e^{-in\theta}}{-2\pi in}\Big|_{-2\pi/3}^{2\pi/3} = \frac{1}{\pi n}\sin\frac{2n\pi}{3} = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \\ \frac{1}{2\pi n} & \text{if } n \equiv 1 \pmod 3 \\ \frac{-1}{2\pi n} & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

Thus

$$f \sim \frac{2}{3} + \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{e^{i(3n+1)\theta} + e^{-i(3n+1)\theta}}{3n+1} - \frac{e^{i(3n+2)\theta} + e^{-i(3n+2)\theta}}{3n+2}.$$

(b) Since f is piecewise C^1, the Dirichlet Jordan Theorem (or Hardy's Tauberian Theorem) says that $s_n(f)(\theta)$ converges to $f(\theta)$ uniformly on $[-\frac{2\pi}{3}+\delta, \frac{2\pi}{3}-\delta]\cup[\frac{2\pi}{3}+\delta, \frac{4\pi}{3}-\delta]$ for any $\delta > 0$. Also $s_n(f)(\pm\frac{2\pi}{3})$ converge to $\frac{1}{2}$. As the function is discontinuous at $\pm\frac{2\pi}{3}$, convergence cannot be uniform on any interval containing $\pm\frac{2\pi}{3}$.

(c) If we substitute $\theta = 0$, we obtain

$$1 = f(0) = \frac{2}{3} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{3n+1} - \frac{1}{3n+2}.$$

However $\frac{1}{3n+1} - \frac{1}{3n+2} = \frac{1}{(3n+1)(3n+2)}$. Therefore $\sum_{k=0}^{\infty} \frac{1}{(3n+1)(3n+2)} = \frac{\pi}{3}$.

4. (a) If g is a continuous 2π-periodic function, then it is uniformly continuous. That is, for any $\varepsilon > 0$, there is a $\delta > 0$ so that $|\theta - \eta| < \delta$ implies that $|g(\theta) - g(\eta)| < \varepsilon$. This means that if $|t| < \delta$, then $\|g - g_t\|_\infty < \varepsilon$. Therefore $\|g - g_t\|_2 \le \|g - g_t\|_\infty < \varepsilon$. Now if $f \in \mathrm{RI}(\mathbb{T})$ and $\varepsilon > 0$, use Assignment 2, Q.4 to find $g \in \mathrm{C}(\mathbb{T})$ so that $\|f - g\|_2 < \varepsilon$. Find $\delta > 0$ as above. Then for $|t| < \delta$,

$$\|f - f_t\|_2 \le \|f - g\|_2 + \|g - g_t\|_2 + \|(g - f)_t\|_2 < 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that $\lim_{t\to 0} \|f - f_t\|_2 = 0$.

(b) Let $\theta, \eta \in [-\pi, \pi]$ and set $s = \theta - \eta$. Then

$$\begin{aligned} |f * g(\theta) - f * g(\eta)| &= \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} (f(\theta - t) - f(\eta - t))g(t)\,dt\Big| \\ &= \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} \big(f_{-\theta}(-t) - f_{-\eta}(-t)\big)g(t)\,dt\Big| \\ &\le \|f_{-\theta} - f_{-\eta}\|_2\|g\|_2 = \|f - f_s\|_2\|g\|_2 \end{aligned}$$

where we applied the Cauchy-Schwarz inequality in the last step. Thus by (a), $f * g$ is (uniformly) continuous. *This looks like there could be a problem at $\pm\pi$, but one can just translate the functions.*

5. (a)

$$\begin{aligned} g_n(\theta) &= -2nK_{n-1}(\theta)\sin(n\theta) = -2n\frac{e^{in\theta} - e^{-in\theta}}{2i}\frac{1}{n}\sum_{k=-n}^{n} (n - |k|)e^{ik\theta} \\ &= i\sum_{k=-n}^{n} (n - |k|)(e^{i(k+n)\theta} - e^{i(k-n)\theta}) \\ &= \sum_{k=-n}^{n} ike^{ik\theta} + \sum_{k=n+1}^{2n} i(2n - k)(e^{ik\theta} - e^{-ik\theta}) \end{aligned}$$

(b) Therefore: $p * g_n = \sum_{k=-n}^{n} ika_k e^{ik\theta} = p'(\theta)$.

(c) Now $\|g_n\|_1 = 2n\|K_{n-1} \sin n\theta\|_1 \le 2n\|K_{n-1}\|_1 = 2n$. Hence

$$\|p'\|_\infty = \|p * g_n\|_\infty \le \|p\|_\infty \|g_n\|_1 \le 2n\|p\|_\infty.$$

6. (a) Monotone functions are Riemann integrable. Moreover at every point, the Monotone Convergence Theorem for sequences of real numbers shows that left and right limits $f(\theta^-)$ and $f(\theta^+)$ exist. Hence since k_n is an *even* summability kernel,

$$\lim_{n\to\infty} f * k_n(\theta) = \frac{1}{2}(f(\theta^-) + f(\theta^+)).$$

(b) Suppose first that g is a piecewise constant monotone function, say given by a sequence
$-\pi = a_0 < a_1 < \cdots < a_p = \pi$ such that $g(t) = b_k$ on (a_{k-1}, a_k), where $b_{k-1} \le b_k$, for $1 \le k \le p$. Then, for $n \ne 0$,

$$\hat{g}(n) = \sum_{k=1}^{p} \frac{1}{2\pi} \int_{a_{k-1}}^{a_k} b_k e^{-in\theta}\, d\theta = \sum_{k=1}^{p} \frac{b_k}{-2\pi ni}(e^{-ina_k} - e^{-ina_{k-1}})$$

$$= \sum_{k=1}^{p} \frac{(b_{k+1} - b_k)}{2\pi ni} e^{-ina_k}.$$

Here $b_{p+1} = b_0$. Hence

$$|\hat{g}(n)| \le \frac{\sum_{k=1}^{p-1}(b_{k+1} - b_k) + (b_p - b_0)}{2\pi|n|} = \frac{2(b_p - b_0)}{2\pi|n|} = \frac{2\|g\|_\infty}{\pi|n|}.$$

Now every monotone function can be approximated uniformly by monotone step functions. Since translation by a contant only changes $\hat{f}(0)$, we may translate so that $f(-\pi) = 0$. Let $M = f(\pi)$. For each $m \ge 1$, let $a_0 = -\pi$, $b_0 = f(-\pi) = 0$, and let $a_k = \sup\{t : f(t) < kM2^{-m}\}$ for $1 \le k \le 2^m$. Define

$$g_m(t) = \begin{cases} kM2^{-m} & \text{for} \quad t \in (a_k, a_{k+1}),\ 0 \le k < 2^m \\ f(a_k) & \text{for} \quad t = a_k, 0 \le k \le 2^m. \end{cases}$$

It is easy to check that g_m is monotone, piecewise constant, and

$$0 = f(-\pi) \le g_m \le f \le g_m + 2^{-m}.$$

Hence g_m converges uniformly to f on $[-\pi, \pi]$.

Let g_m be a sequence of monotone, piecewise constant functions just constructed. Then

$$|\hat{f}(n)| \le \inf_{m\ge1} \|f - g_m\|_\infty + |\widehat{g_m}(n)| \le \inf_{m\ge1} 2^{-m} + \frac{2\|f\|_\infty}{\pi|n|} = \frac{2\|f\|_\infty}{\pi|n|}.$$

(c) By (a), $\sigma_n(f)(\theta) = f * K_n(\theta)$ converges to $\frac{1}{2}(f(\theta^-) + f(\theta^+))$. By (b), $\hat{f}(n) = O(\frac{1}{n})$. Hence by Hardy's Tauberian Theorem, $s_n(f)(\theta)$ also converges to $\frac{1}{2}(f(\theta^-) + f(\theta^+))$.