PM 450 Solutions to Assignment 2

1. We have $x = r\cos\theta$ and $y = r\sin\theta$. Therefore

$$u_r = u_x\frac{\partial x}{\partial r} + u_y\frac{\partial y}{\partial r} = u_x\cos\theta + u_y\sin\theta$$

$$u_{rr} = \frac{\partial}{\partial r}(u_x\cos\theta + u_y\sin\theta) = \frac{\partial u_x}{\partial r}\cos\theta + \frac{\partial u_y}{\partial r}\sin\theta = (u_{xx}\frac{\partial x}{\partial r} + u_{xy}\frac{\partial y}{\partial r})\cos\theta + (u_{yx}\frac{\partial x}{\partial r} + u_{yy}\frac{\partial y}{\partial r})\sin\theta$$
$$= u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\,\sin\theta + u_{yy}\sin^2\theta$$

$$u_\theta = u_x\frac{\partial x}{\partial\theta} + u_y\frac{\partial\theta}{\partial r} = -u_x r\sin\theta + u_y r\cos\theta$$

$$u_{\theta\theta} = \frac{\partial}{\partial\theta}(-u_x r\sin\theta + u_y r\cos\theta) = -\frac{\partial u_x}{\partial\theta}r\sin\theta - u_x r\cos\theta + \frac{\partial u_y}{\partial\theta}r\cos\theta - u_y r\sin\theta$$
$$= -\big(u_{xx}(-r\sin\theta) + u_{xy}(r\cos\theta)\big)(r\sin\theta) - u_x r\cos\theta + \big(u_{yx}(-r\sin\theta) + u_{yy}(r\cos\theta)\big)(r\cos\theta) - u_y r\sin\theta$$
$$= u_{xx}r^2\sin^2\theta - 2u_{xy}r^2\cos\theta\,\sin\theta + u_{yy}r^2\cos^2\theta - u_x r\cos\theta - u_y r\sin\theta.$$

Combining we get: $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = u_{xx} + u_{yy}$.

2. (a) $\hat{f}(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\theta\,d\theta = 0$ because f is an odd function. For $n \neq 0$, integrate by parts:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta e^{-in\theta}\,d\theta = \frac{1}{2\pi}\theta\frac{1}{-in}e^{-in\theta}\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{-in}e^{-in\theta}\,d\theta = \frac{(-1)^n i}{n}.$$

Thus $f \sim \displaystyle\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n i}{n}e^{in\theta} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}(-2)\frac{e^{in\theta}-e^{-in\theta}}{2i} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2}{n}\sin n\theta.$

(b) This series does not converge uniformly. Since f is continuous on $[-\pi+\delta, \pi-\delta]$ for any $\delta > 0$, the series converges uniformly on $[-\pi+\delta, \pi-\delta]$ to f. Thus it converges to f on $(-\pi, \pi)$. At $\theta = \pi$, you can see that the sin series vanishes, and thus it converges to 0 at $\theta = \pi$. If convergence were uniform, since the uniform limit of continuous functions is continuous, the limit would be continuous—but there is a jump discontinuity at $\theta = \pi \equiv -\pi \pmod{2\pi}$.

(c) $u(r,\theta) = \sum_{n=1}^{\infty}(-1)^{n-1}r^n\frac{2}{n}\sin n\theta$. Hence $\lim_{r\to 1^-}u(r,\pi) = \lim_{r\to 1^-}0 = 0$.

3. Suppose that $\{A_n\}$ is bounded by C. Let $u_n(r,\theta) = A_n r^{|n|}e^{in\theta}$. Then

$$\frac{\partial^{j+k}}{\partial r^j\partial\theta^k}u_n(r,\theta) = A_n|n|\ldots(|n|-j+1)r^{|n|-j}(in)^k e^{in\theta}.$$

Therefore for $r \leq R < 1$,

$$\Big\|\frac{\partial^{j+k}}{\partial r^j\partial\theta^k}u_n(r,\theta)\Big\|_{\mathbb{D}_R} = \sup_{\substack{0\leq r\leq R\\ -\pi\leq\theta\leq\pi}}\Big|\frac{\partial^{j+k}}{\partial r^j\partial\theta^k}u_n(r,\theta)\Big| \leq C|n|^{k+j}R^{|n|-j}.$$

Since $\displaystyle\lim_{n\to\infty}\frac{C(|n|+1)^{k+j}R^{|n|+1-j}}{C|n|^{k+j}R^{|n|-j}} = R < 1$, the ratio test guarantees that $\displaystyle\sum_{n=-\infty}^{\infty}\Big\|\frac{\partial^{j+k}}{\partial r^j\partial\theta^k}u_n(r,\theta)\Big\|_{\mathbb{D}_R}$ converges. Hence, by the Weierstrass M-test, $\displaystyle\sum_{n=-\infty}^{\infty}\frac{\partial^{j+k}}{\partial r^j\partial\theta^k}u_n(r,\theta)$ converges uniformly on each $\mathbb{D}_R$. Appealing repeatedly to the Term by Term Differentiation Lemma shows that this series converges to $\dfrac{\partial^{j+k}}{\partial r^j\partial\theta^k}u(r,\theta)$. Therefore u is C^∞.

4. Assume that f is real valued. Set $M = \|f\|_\infty$, and let $\varepsilon > 0$ be given. Since f is Riemann integrable, there is a partition $\Delta = \{-\pi = t_0 < t_1 < \cdots < t_n = \pi\}$ so that with

$$l_k = \inf_{t_{k-1} \le t \le t_k} f(t) \quad \text{and} \quad u_k = \sup_{t_{k-1} \le t \le t_k} f(t),$$

we obtain

$$\frac{1}{2\pi}\sum_{k=1}^{n} l_k(t_k - t_{k-1}) \le \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt \le \frac{1}{2\pi}\sum_{k=1}^{n} u_k(t_k - t_{k-1}) \quad \text{and} \quad \frac{1}{2\pi}\sum_{k=1}^{n}(u_k - l_k)(t_k - t_{k-1}) < \varepsilon.$$

Define $g(t)$ to be piecewise linear on $[t_{k-1}, t_k]$ with $g(t_k) = f(t_k)$. Then it follows that g is continuous including $g(-\pi) = g(\pi)$, $\|g\|_\infty \le \|f\|_\infty$ and $l_k \le g(t) \le u_k$ for $t \in [t_{k-1}, t_k]$. Therefore $|f(t) - g(t)| \le u_k - l_k$ on $[t_{k-1}, t_k]$. We estimate

$$\begin{aligned}\|f - g\|_2^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t) - g(t)|^2 \le \frac{1}{2\pi}\sum_{k=1}^{n}(u_k - l_k)^2(t_k - t_{k-1}) \\ &= \frac{1}{2\pi}\sum_{k=1}^{n}(u_k + l_k)(u_k - l_k)(t_k - t_{k-1}) \le 2M\frac{1}{2\pi}\sum_{k=1}^{n}(u_k - l_k)(t_k - t_{k-1}) < 2M\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can approximate f by continuous functions in the L^2 norm as accurately as desired.

5. (a) $u(r,\theta) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta - t)P(r,t)dt \ge 0$ because the integrand is positive.

(b) $\dfrac{1-r}{1+r} = \dfrac{1-r^2}{1+2r+r^2} \le \dfrac{1-r^2}{1+2r\cos\theta + r^2} = P(r,\theta) \le \dfrac{1-r^2}{1-2r+r^2} = \dfrac{1+r}{1-r}.$

(c) Note that $u(0,0) = u(0,\theta) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta - t)P(0,t)dt = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta - t)dt$. Therefore, by (b),

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - t)P(r,t)dt \ge \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - t)\frac{1-r}{1+r}dt = \frac{1-r}{1+r}u(0,0).$$

Similarly,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - t)P(r,t)dt \le \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - t)\frac{1+r}{1-r}dt = \frac{1+r}{1-r}u(0,0).$$

6. (a) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t)P(s,t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta - t)}\sum_{m=-\infty}^{\infty} s^{|m|}e^{imt}\,dt$$

The double sum $\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} r^{|n|}s^{|m|} < \infty$, so this series converges uniformly by the M-test. Thus it is valid to interchange the order of the integral and summation:

$$\begin{aligned}&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} r^{|n|}s^{|m|}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in(\theta - t)}e^{imt}\,dt \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} r^{|n|}s^{|m|}e^{in\theta}\delta_{n,m} = \sum_{n=-\infty}^{\infty}(rs)^{|n|}e^{in\theta} = P(rs,\theta).\end{aligned}$$

Alternatively, observe that the LHS is $P_r * P_s$. Since $P(r,\theta) = \sum_{-\infty}^{\infty} r^{|n|}e^{in\theta}$ and $P(s,\theta) = \sum_{-\infty}^{\infty} s^{|n|}e^{in\theta}$, we have $\widehat{P_r * P_s}(k) = \widehat{P_r}(k)\widehat{P_s}(k) = r^{|n|}s^{|n|} = (rs)^{|n|}$. Thus

$$P_r * P_s(\theta) = \sum_{-\infty}^{\infty}(rs)^{|n|}e^{in\theta} = P_{rs}(\theta).$$

(b) Observe that $g = f * P_s$. The harmonic extension of g is

$$v(r,\theta) = (g * P_r)(\theta) = \big((f * P_s) * P_r\big)(\theta) = \big(f * (P_s * P_r)\big)(\theta) = (f * P_{rs})(\theta) = u(rs,\theta).$$

7. (a) Integrating by parts, we obtain

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta = \frac{1}{2\pi} f(\theta)\frac{e^{-in\theta}}{-in}\Big|_{-\pi}^{\pi} + \frac{1}{2\pi in}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}\,d\theta = \frac{\widehat{f'}(n)}{in}$$

where the first term is 2π-periodic and hence is zero. Therefore, $|\hat{f}(n)| = |\widehat{f'}(n)|/|n| \le \|f'\|_\infty/|n|$.

(b) Proceed by induction. Part (a) does the case $k = 1$. Assume that $|\hat{f}(n)| \le C|n|^{-k+1}$ for all $f \in C^{k-1}$. Then when f is C^k, f' is C^{k-1} and thus $|\widehat{f'}(n)| \le C|n|^{-k+1}$ for some constant C. Therefore $|\hat{f}(n)| = |\widehat{f'}(n)|/|n| \le C|n|^{-k}$.

(c) If f is C^2, there is a constant C so that $|\hat{f}(n)| \le Cn^{-2}$ for $n \ne 0$. Therefore $\|\hat{f}(n)e^{in\theta}\|_\infty \le Cn^{-2}$. Since $|\hat{f}(0)| + \sum_{\substack{n=-\infty \\ n\ne 0}}^{\infty} Cn^{-2} < \infty$, the Fourier series converges uniformly by the Weierstrass M-test.

(d) If $k \ge 2$, the series $\sum_{-\infty}^{\infty} |\hat{f}(n)| \le |\hat{f}(0)| + 2C\sum_{n=1}^{\infty} n^{-k} < \infty$ is summable. Therefore the series $\sum_{-\infty}^{\infty} \hat{f}(n)e^{in\theta}$ converges uniformly to f. If $m \le k-2$, the Fourier coefficients of $f^{(m)}$ are (by repeated use of (a)) $\widehat{f^{(m)}}(n) = \hat{f}(n)(in)^m$ which is bounded by $C|n|^{m-k} \le Cn^{-2}$. Since $\sum_{\substack{n=-\infty \\ n\ne 0}}^{\infty} Cn^{-2} < \infty$, the series $\sum_{-\infty}^{\infty} \hat{f}(n)(in)^m e^{in\theta}$ converges uniformly by the Weierstrass M-test to a continuous 2π-periodic function f_m. Therefore we can repeatedly apply the term by term differentiation lemma to see that $f_m = f^{(m)}$. Therefore f is C^{k-2}.

(e) Combining parts (b) and (d), we see that f is C^∞ if and only if $\hat{f}(n)$ is $O(|n|^{-k})$ for all $k \ge 1$.