PM 450 Solutions to Assignment 1

1. (a) In standard form, the DE is $y' = \Phi(x,y) = xy + 1$. Thus we are seeking a fixed point of the map

$$Tf(x) = \int_0^x 1 + tf(t)\,dt \quad \text{for} \quad f \in \mathrm{C}[-b,b].$$

When $b = 1$, we compute for $f, g \in \mathrm{C}[-1,1]$

$$\begin{aligned} |Tf(x) - Tg(x)| &= \left| \int_0^x 1 + tf(t)\,dt - \int_0^x 1 + tg(t)\,dt \right| \le \left| \int_0^x t|f(t) - tg(t)|\,dt \right| \\ &\le \left| \int_0^x t\|f-g\|_\infty\,dt \right| = \frac{x^2}{2}\|f-g\|_\infty. \end{aligned}$$

Therefore $\|Tf - Tg\|_\infty = \sup_{|x|\le 1} |Tf(x) - Tg(x)| \le \frac{1}{2}\|f-g\|_\infty$. Thus T is a contraction mapping.

(b) This is a linear DE, and therefore satisfies a global Lipschitz condition on $[-b,b] \times \mathbb{R}$. (Alternatively, since $\frac{\partial}{\partial y}\Phi(x,y) = x$ is bounded by b on $[-b,b] \times \mathbb{R}$, this DE satisfies a global Lipschitz condition with constant b.) Therefore by the Global Picard Theorem, it has a unique solution f_b on $[-b,b]$. This is true for any value of b, and by uniqueness, we must have $f_c|_{[-b,b]} = f_b$ if $b < c$. Thus defining $f(x) = f_b(x)$ for any $|x| \le b$ uniquely defines the solution on $\mathbb{R}$.

(c) Compute the first few terms: $f_0 = 1$, $f_1(x) = Tf_0(x) = \int_0^x 1 + t\,dt = x + \frac{1}{2}x^2$, $f_2(x) = \int_0^x 1 + t(t + t^2/2)\,dt = x + \frac{1}{3}x^3 + \frac{1}{2\cdot 4}x^4$, $f_3(x) = \int_0^x 1 + t(t + t^3/3 + t^4/8)\,dt = x + \frac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5 + \frac{1}{2\cdot 4\cdot 6}x^6$. We claim that the general pattern is: $f_n(x) = \sum_{k=1}^n a_{2k-1}x^{2k-1} + a_{2n}x^{2n}$ where $a_{2k-1} = \frac{1}{1\cdot 3\cdot 5\cdot\dots\cdot(2k-1)}$ and $a_{2n} = \frac{1}{2\cdot 4\cdot\dots\cdot(2n)}$. We will verify this by induction. The calculations above verify it for $n = 0,1,2,3$. Assume that it is valid for $n-1$. Then

$$\begin{aligned} f_n(x) = Tf_{n-1}(x) &= \int_0^x 1 + t\Big(\sum_{k=1}^{n-1} a_{2k-1}t^{2k-1} + a_{2n-2}t^{2n}\Big)\,dt \\ &= x + \sum_{k=1}^{n-1} a_{2k-1}\int_0^x t^{2k}\,dt + a_{2n-2}\int_0^x t^{2n-1}\,dt \\ &= x + \sum_{k=1}^{n-1} a_{2k-1}\frac{1}{2k+1}x^{2k+1} + a_{2n-2}\frac{1}{2n}x^{2n} = \sum_{k=1}^{n} a_{2k-1}x^{2k-1} + a_{2n}x^{2n} \end{aligned}$$

The radius of convergence of series $f(x) = \sum_{k=1}^\infty a_{2k-1}x^{2k-1}$ can be computed by the ratio test:

$$\lim_{n\to\infty} \frac{|a_{2k+1}x^{2k+1}|}{|a_{2k-1}x^{2k-1}|} = \lim_{k\to\infty} \frac{x^2}{2k+1} = 0.$$

This shows that the radius of convergence is ∞ and the series converges uniformly on $[-b,b]$ for any $b > 0$. The annoying extra term $a_{2n}x^{2n}$ is bounded by $\sup_{|x|\le b} |a_{2n}x^{2n}| = a_{2n}b^{2n}$ on $[-b,b]$, and again by the ratio test $\lim_{n\to\infty} \dfrac{a_{2n+2}b^{2n+2}}{a_{2n}b^{2n}} = \lim_{n\to\infty} \dfrac{b^2}{2n+2} = 0$. Hence this term converges uniformly to 0 on $[-b,b]$. We conclude that $f_n(x)$ converges uniformly on $[-b,b]$ to $f(x) = \sum_{k=1}^\infty a_{2k-1}x^{2k-1}$ for every $b > 0$.

2. (a) It is easy to check that if $f(x) = x$, then $f'' - x^{-1}f' + x^{-2}f = 0 - x^{-1} + x^{-2}x = 0$. Now substitute $f(x) = xg(x)$ into the DE to get

$$0 = (xg)'' - x^{-1}(xg)' + x^{-2}(xg) = 2g' + xg'' - x^{-1}(g + xg') + x^{-1}g = xg'' + g'.$$

Hence $\dfrac{g''}{g'} = -\dfrac{1}{x}$. Integrating yields $\log g'(x) = -\log x + a$, whence $g'(x) = \dfrac{b}{x}$. Thus $g(x) = b\log x + c$. We obtain the solutions $f(x) = bx\log x + cx$.

(b) The set of solutions is $V = \operatorname{span}\{x, x\log x\}$, which is a 2-dimensional vector subspace of $\mathrm{C}[1,3]$ because x and $x\log x$ are linearly independent.

(c) Note that if $f = bx\log x + cx$ is a function in V, then

$$Af := \begin{bmatrix} f(1) \\ f'(1) \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ b \end{bmatrix}.$$

This is a bijection from V onto $\mathbb{R}^2$, and shows that every set of initial values is obtained.

Remark: The DE is linear, and thus satisfies a global Lipshitz condition. So there is a unique solution for each set of initial values. Our set V of solutions contains a solution for each pair of initial values. Therefore we have found the complete set of solutions.

3. (a) We have $y' = \Phi(x,y)$ where $\Phi(x,y) = 4xy - 4x^2 - y^2 + 2$ and $y(0) = 0$. Since Φ is C^1, it satisfies a local Lipschitz condition on $[-b,b] \times [-R,R]$ for $b, R > 0$. Thus by the Local Picard Theorem, there is a local solution.

(b) Substitute $f(x) = g(x) + 2x$ into the DE. We obtain

$$\begin{aligned} g' + 2 = f' &= 4xf - 4x^2 - f^2 + 2 \\ &= 4xg + 8x^2 - 4x^2 - (g^2 + 4xg + 4x^2) + 2 = -g^2 + 2. \end{aligned}$$

Therefore $g' = -g^2$ and $g(0) = 2$. Integrating $-g^{-2}g' = 1$ on $[0,x]$ yields

$$x = \int_0^x 1\,dt = \int_0^x -g^{-2}(t)g'(t)\,dt = g^{-1}\Big|_0^x = \frac{1}{g(x)} - \frac{1}{2}.$$

Therefore $g(x) = \dfrac{2}{2x+1}$. Hence $f(x) = 2x + \dfrac{2}{2x+1}$.

(c) The local Picard Theorem only yields a solution on some small interval around $x = 0$. But the Continuation Theorem shows that the solution continues until it goes off to infinity. In this case, we can see that the maximal solution is valid in $(-0.5, \infty)$.

Remark: the solution does not continue to $(-\infty, 0.5)$. The reason is that there are many solutions, depending on initial values, say at $x = -1$. Any of these solutions would be valid, as the initial value at $x = 0$ does not affect what happens here.

4. The DE is $y' = \Phi(x,y) = \sin\left(\dfrac{x^5 + 3x^2 - 1}{\sqrt{219 - 2y^2}}\right)$ and $y(2) = 3$. Observe that $|y'(x)| \le 1$, and therefore *if* there is a solution on $[-5,9] = [2-7, 2+7]$ with $y(2) = 3$, then by the MVT, $\dfrac{|y(x) - y(2)|}{|x-2|} \le 1$; and thus $|y(x)| \le 3 + |x-2| \le 10$. On the set $D = [-5,9] \times [-10,10]$, we have

$$\frac{\partial}{\partial y}\Phi(x,y) = \cos\left(\frac{x^5 + 3x^2 - 1}{\sqrt{219 - 2y^2}}\right)\frac{(x^5 + 3x^2 - 1)(-2y)}{(219 - 2y^2)^{3/2}}.$$

Thus since $|y| \le 10$, we obtain

$$\left| \frac{\partial}{\partial y} \Phi(x,y) \right| \le \frac{20}{\sqrt{19}} \sup_{x \in [-5,9]} |x^5 + 3x^2 - 1| < 3 \cdot 10^5.$$

Therefore the function Φ is Lipschitz on D. So now we may apply the Local Picard Theorem and the Continuation Theorem to obtain a solution on the largest interval which keeps the solution inside D. As we have observed that $|y| \le 10$, the solution cannot leave D before reaching the endpoints -5 and 9. Therefore there is a solution on the whole interval.

5. (a) If $f(x) = 1 - \sqrt{1-x^2}$ on $[-1,1]$, then $f'(x) = \dfrac{x}{\sqrt{1-x^2}}$ and therefore $f(0) = f'(0) = 0$ and

$$f''(x) = \frac{1}{(1-x^2)^{3/2}} = \Big(1 + \frac{x^2}{1-x^2}\Big)^{3/2} = (1 + (f'(x))^2)^{3/2}.$$

The DE is C^1 and thuse there is a local Lipschitz condition. Hence the solution in unique.

(b) To put this into the context of the Continuation Theorem, we must consider it as a first order vector valued DE. We set

$$F(x) = \begin{bmatrix} f_0(x) \\ f_1(x) \end{bmatrix} \quad \text{and} \quad F'(x) = \begin{bmatrix} f_1(x) \\ (1 + (f_1(x))^2)^{3/2} \end{bmatrix} \quad \text{and} \quad F(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution is $F(x) = \begin{bmatrix} 1 - \sqrt{1-x^2} \\ x(1-x^2)^{-1/2} \end{bmatrix}$. Because the derivative $f'(x)$ blows up at $x = \pm 1$, we do have $\lim_{x \to \pm 1} \|F(x)\| = \infty$. So the conclusion of the Continuation Theorem is satisfied.

6. (a) In standard form $y' = \Phi(x,y) = \dfrac{y^2-1}{xy}$. Then $\dfrac{\partial}{\partial y}\Phi(x,y) = \dfrac{2yxy - (y^2-1)x}{x^2y^2} = \dfrac{1}{x} + \dfrac{1}{xy^2}$.
If both $x \ge \varepsilon > 0$ and $y \ge \varepsilon > 0$, then $\left|\frac{\partial}{\partial y}\Phi(x,y)\right| \le \varepsilon^{-1} + \varepsilon^{-3}$; and therefore there is a local Lipschitz condition in the open first quadrant.

(b) By separation of variables, we have $\dfrac{yy'}{y^2-1} = \dfrac{1}{x}$. Integrating, we obtain

$$\int \frac{y}{y^2-1}\, y'dx = \frac{1}{2}\log|y^2-1| = \int \frac{1}{x}\,dx = \log x + c.$$

Thus $y^2 = 1 + cx^2$, or $y = \sqrt{1+cx^2}$ since $y > 0$. The initial value $y(1) = a$ yields $c = a^2 - 1$; whence $y = \sqrt{1 + (a^2-1)x^2}$. If $a \ge 1$, the function is defined on $(0,\infty)$; while if $0 < a < 1$, then the solution hits the x-axis at $x = (1-a^2)^{-1/2}$. At this point, we have $y'(x) = -\infty$. So the solution stops there. (*Indeed when* $a > 1$, *so* $c > 0$, *this curve* $y^2 - cx^2 = 1$ *is a hyperbola, and when* $0 < a < 1$, *we have* $c < 0$ *and the curve is* $y^2 + |c|x^2 = 1$, *which is an ellipse with a vertical tangent where it hits the* x-*axis.*)

(c) Note $\lim_{x\to 0^+} y(x) = \lim_{x \to 0^+} \sqrt{1+(a^2-1)x^2} = 1$ and $\lim_{x \to 0^+} y'(x) = \lim_{x\to 0^+} \dfrac{(a^2-1)x}{\sqrt{1+(a^2-1)x^2}} = 0$.
Thus all solutions pass through $(0,1)$ with slope 0. Formally all of these solutions continue into the second quadrant by symmetry. However we can pass from one solution in the first quadrant to some different solution in the second quadrant. Because of the limits above, any such continuation will be C^1 because the functions and their derivatives are continuous at $x = 0$. This does not contradict our theory because there is no Lipschitz condition valid in a neighbourhood of $x = 0$ since $\Phi(x,y) = \dfrac{y^2-1}{xy}$ is not even defined at $x = 0$.