PM 450 **Assignment 6** **Due Wednesday March 29.**

1. (a) Using $f(\theta) = \theta^3 - \pi^2\theta$ from Assignment 2, Q3, evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$.

   (b) Integrate f and use it to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^8}$.

2. Prove that $\{1, \sqrt{2}\cos n\theta : n \geq 1\}$ is an orthonormal basis for $L^2(0,\pi)$ with the inner product
$$\langle f, g\rangle = \frac{1}{\pi}\int_{(0,\pi)} f\overline{g}.$$

3. Let $a \in \mathbb{R}\setminus\mathbb{Z}$. Let $f(\theta) = e^{ia\theta}$ for $\theta \in (-\pi,\pi]$. Evaluate $\|f\|_2^2$ in two ways and deduce that
$$\sum_{n=-\infty}^{\infty} \frac{1}{(a-n)^2} = \frac{\pi^2}{\sin^2 a\pi}.$$

4. (a) Show that if f is a 2π-periodic C^1 function and $\hat{f}(0) = 0$, then $\|f\|_2 \leq \|f'\|_2$.

   (b) Show that if f is a 2π-periodic C^2 function, then $\|f'\|_2^2 \leq \|f\|_2\,\|f''\|_2$.

5. Let $1 \leq p < \infty$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$.

   (a) Prove that if $f, f_n \in L^p(X)$ and $g \in L^q(X)$ and $\lim_{n\to\infty} \|f - f_n\|_p = 0$, then
$$\lim_{n\to\infty} \int_X f_n g = \int_X fg.$$

   (b) Show that if $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$, then
$$\frac{1}{2\pi}\int_{\mathbb{T}} fg = \lim_{n\to\infty} \sum_{k=-n}^{n} \left(1 - \tfrac{|k|}{n+1}\right)\hat{f}(k)\hat{g}(-k).$$

6. Let $1 \leq p < \infty$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$. Prove that
$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)g(n\theta)\,d\theta = \hat{f}(0)\hat{g}(0).$$
   Hint: prove it first when f is a trigonometric polynomial.

7. Fix $1 \leq p < \infty$. Define a linear map $S_{p,n} : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ given by $S_{p,n}f = s_n(f) = f * D_n$.

   (a) Prove that the following statements are equivalent:
      (1) $\sup_{n\geq 1} \|S_{p,n}\| < \infty$.
      (2) $\sup_{n\geq 1} \|S_{p,n}(f)\|_p < \infty$ for every $f \in L^p(\mathbb{T})$.
      (3) $S_{p,n}f$ converges to f in the $L^p(\mathbb{T})$ norm for every $f \in L^p(\mathbb{T})$.

   (b) Show that $\|S_{1,n}\| = \|D_n\|_1$. Hence show that there is a function $f \in L^1$ such that $s_n(f)$ diverges. **Hint:** look at $S_{1,n}(K_m)$ as $m \to \infty$.