PM 450 **Assignment 5** **Due Wednesday March 15.**

1. (a) Prove that if $f_n \geq 0$ are measurable functions on X, then $\int \liminf f_n \leq \liminf \int f_n$.
   (b) Suppose that $f_n \geq 0$ are measurable functions on X and $f_n \to f$ a.e. Moreover suppose that $\lim_{n\to\infty} \int f_n = \int f < \infty$. Prove that for every measurable $E \subset X$, $\lim_{n\to\infty} \int_E f_n = \int_E f$.
   (c) Construct an example to show that the previous result can fail if $\int f = \infty$.

2. Evaluate the following limits and justify your argument with appropriate convergence theorems.

   (a) $\lim_{n\to\infty} \int_0^\infty \frac{1+nx^2}{(1+x^2)^n}\,dx$ (b) $\lim_{n\to\infty} \int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\,dx$

3. Suppose that $f_n \to f$ a.e. and $g_n \to g$ a.e. are all integrable functions and $|f_n| \leq g_n$ for $n \geq 1$. Suppose $\lim_{n\to\infty} \int g_n = \int g$. Prove that $\lim_{n\to\infty} \int f_n = \int f$. **Hint:** rework the proof of LDCT.

4. (a) Show that the improper Riemann-integrable function $f(x) = \frac{\sin x}{x}$ for $x \geq 0$ is not Lebesgue integrable on $[0,\infty)$.
   (b) Show that if f is a bounded Lebesgue integrable function on $[0,\infty)$ and is also improper Riemann integrable, then the two integrals agree.

5. Suppose that $f,\ f_n \in L^p$ for some $1 \leq p < \infty$ and $f_n \to f$ a.e. and $\|f_n\|_p \to \|f\|_p$. Prove that f_n converges to f in the L^p norm.

6. Let $1 \leq p < r \leq \infty$.
   (a) Suppose that $X \subset \mathbb{R}$ is measurable, and $m(X) < \infty$. Prove that $L^r(X) \subset L^p(X)$.
   Hint: for f measurable on X, show that $\|f\|_p \leq \|f\|_r m(X)^{\frac{1}{p}-\frac{1}{r}}$.
   (b) Find a function f so that $f \in L^q((0,\infty))$ if and only if $p < q \leq r$. **Hint:** consider combinations of functions of the form $f(x) = x^{-a}|\log x|^{-b}$ on various domains.

7. Let $A_n = \big\{x \in [0,1] : x = (0.x_1x_2\ldots)_{\text{base } 2} \text{ and } x_n = 0\big\}$ for $n \geq 1$.
   Prove that $\lim_{n\to\infty} \int_{A_n} f = \frac{1}{2}\int_{[0,1]} f$ for all integrable functions on $[0,1]$.
   Hint: first prove it for step functions with jumps at diadic rationals.