PM 450 **Assignment 4** **Due Friday March 3.**

1. Show that if $\mathbb{Q} \cap [0,1]$ is covered by *finitely many* intervals $I_1, \ldots, I_n$, then $\sum_{i=1}^{n} \ell(I_i) \geq 1$.

2. Let $\{E_n\}_{n\geq 0}$ be the non-measurable sets constructed in class which partition $[0,1)$.
   (a) Show that if F is measurable and $F \subset E_n$, then $m(F) = 0$.
   (b) Show that if F is measurable and $m(F) > 0$, then F contains a non-measurable set.
   Hint: WLOG $F \subset [0,1)$. Consider $F = \dot{\bigsqcup}_{n\geq 0} F \cap E_n$.

3. Show that if $f : \mathbb{R} \to \mathbb{R}$ is measurable and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g \circ f$ is measurable.

4. Let $f_n : \mathbb{R} \to \mathbb{C}$ be a sequence of measurable functions. Define $A = \{x : \lim_{n\to\infty} f_n(x) \text{ exists}\}$. Prove that A is measurable.

5. A sequence $f_n : [0,1] \to \mathbb{C}$ of measurable functions *converges in measure* to f if for all $\varepsilon > 0$, $\lim_{n\to\infty} m(\{x : |f(x) - f_n(x)| \geq \varepsilon\}) = 0$. Prove that if f_n converges in measure to f, then there is a subsequence f_{n_i} which converges to f almost everywhere.

6. A generalized Cantor set can be constructed by removing an open interval of length a_1 from the middle of $[0,1]$, then removing open intervals of length a_2 from the middle of the two remaining intervals, etc. The only constraint is that a_{n+1} should be strictly smaller than the lengths of the intervals remaining at the nth stage.
   (a) Compute the measure of the usual Cantor set C.
   (b) Show that for any $0 \leq r < 1$, there is a generalized Cantor set K with $m(K) = r$.
   (c) Given a generalized Cantor set K, show that there is a homeomorphism h of $[0,1]$ onto itself such that $h(C) = K$.
   (d) Construct a measurable set $E \subset \mathbb{R}$ so that for every non-empty finite open interval $I = (a,b)$, one has $0 < m(E \cap I) < m(I)$.
   Hint: it is enough to do this on $[0,1]$ and replicate. Repeatedly fill the gaps of a generalized Cantor set with more generalized Cantor sets.

7. Let f be the Cantor function, which is a continuous monotone function on $[0,1]$ which takes the value $1/2$ on the middle third, values $1/4$ and $3/4$ on the middle thirds of the second level, etc. Let $g(x) = x + f(x)$.
   (a) Show that g is a homeomorphism of $[0,1]$ onto $[0,2]$. i.e. show that $h = g^{-1}$ is a continuous function.
   (b) Show that $m(g(C)) = 1$.
   (c) Show that there is a measurable subset $A \subset [0,1]$ so that $h^{-1}(A)$ is not measurable.
   (d) Hence show that if f is measurable and g is continuous, then $f \circ g$ need not be measurable. Compare with Q.3.