PM 450 **Assignment 3** **Due Wednesday February 15.**

1. Define $V_n(\theta) = \displaystyle\sum_{k=-n-1}^{n+1} e^{ik\theta} + \sum_{k=n+2}^{2n+1} \tfrac{2n+2-k}{n+1}\big(e^{ik\theta} + e^{-ik\theta}\big)$.
   (a) Prove that $V_n(\theta) = 2K_{2n+1}(\theta) - K_n(\theta)$ (where K_n is the Féjer kernel).
   (b) Hence prove that V_n is an even summability kernel.

2. Let $f(\theta) = \theta^3 - \pi^2\theta$ on $[-\pi, \pi]$.
   (a) Compute the Fourier series for f.
   (b) What can be said about the convergence of this series?
   (c) Evaluate at $\theta = \pi/2$ to evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$.

3. Let $f(\theta) = \begin{cases} 1 & \text{if} \quad 0 \le |\theta| \le \frac{2\pi}{3} \\ 0 & \text{if} \quad \frac{2\pi}{3} \le |\theta| \le \pi \end{cases}$.
   (a) Compute the Fourier series for f.
   (b) What can be said about the convergence of this series?
   (c) Evaluate at $\theta = 0$ and hence compute $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)}$.

4. For $f \in \mathrm{RI}(\mathbb{T})$, and define the translation of f by t be $f_t(\theta) = f(\theta - t)$.
   (a) Prove that $\lim_{t\to 0} \|f - f_t\|_2 = 0$. **Hint:** First prove it for $g \in \mathrm{C}(\mathbb{T})$, and then approximate f by continuous functions in the L^2 norm.
   (b) Prove that if $f, g \in \mathrm{RI}(\mathbb{T})$, then $f * g$ is continuous. **Hint:** Cauchy-Schwarz inequality for the L^2 inner product.

5. Let $p(\theta) = \sum_{k=-n}^{n} a_k e^{ik\theta}$ be a trig polynomial of degree n.
   (a) Compute the Fourier series of $g_n(\theta) = -2nK_{n-1}(\theta)\sin(n\theta)$.
   (b) Hence obtain a formula for $p'(\theta)$ as a convolution.
   (c) Prove that $\|p'\|_\infty \le 2n\|p\|_\infty$.

6. Suppose that $f(\theta)$ is a monotone increasing real valued function on $[-\pi, \pi]$.
   (a) Show that if $\{k_n\}$ is an even summability kernel, then $f * k_n$ converges pointwise, and find the limit.
   (b) Prove that $|\hat{f}(n)| \le C/|n|$ for $n \ne 0$.
   Hint: WLOG $f(-\pi) = 0$ (why?). Prove it first when f is a (monotone) step function by integrating over each interval and rearranging the sum. Then approximate a general monotone f.
   (c) Hence show that the Fourier series of f converges at every point in $[-\pi, \pi]$.