PM 450 **Assignment 2** **Due Wednesday February 1.**

1. Convert the Laplacian $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$ to Cartesion coordinates.
Hint: Compute $u_r,\ u_{rr},\ u_\theta$ and $u_{\theta\theta}$ in terms of partials w.r.t. x and y.

2. Let $f(\theta) = \theta$ for $-\pi < \theta \le \pi$.
   (a) Compute the Fourier series of f, and convert it to a sum of sines and cosines.
   (b) Does this series converge uniformly on $[-\pi, \pi]$?
   (c) Evaluate $\lim\limits_{r\to 1^-} u(r, \pi)$.

3. Prove that if $\{A_n\}_{n\in\mathbb{Z}}$ is a bounded sequence, then $u(r,\theta) = \sum_{n\in\mathbb{Z}} A_n r^{|n|} e^{in\theta}$ is C^∞ on the open disc $\mathbb{D}$, i.e. show that all partial derivatives $\frac{\partial^{j+k}}{\partial r^j \partial\theta^k}u$ exist and are continuous.

4. Show that every Riemann integrable function f on $\mathbb{T}$ is a limit of continuous functions in the $L^2(\mathbb{T})$ norm. **Hint:** Assume that f is real valued first. Find a partition for which the upper and lower Riemann sums of $\|f\|_2^2 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |f(\theta)|^2\,d\theta$ are close. Use this partitition to define a continuous function g that fits within the same upper and lower bounds. Estimate $\|f - g\|_2^2$.

5. Let f be a *positive* Riemann integrable 2π-periodic function with harmonic extension $u(r,\theta)$.
   (a) Prove that $u(r,\theta) \ge 0$.
   (b) Prove that $\dfrac{1-r}{1+r} \le P(r,\theta) \le \dfrac{1+r}{1-r}$.
   (c) Hence prove that $\left(\dfrac{1-r}{1+r}\right)u(0,0) \le u(r,\theta) \le \left(\dfrac{1+r}{1-r}\right)u(0,0)$.

6. (a) Prove that $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} P(r,\theta - t)P(s,t)\,dt = P(rs,\theta)$.
   Hint: Use the series expansion of the Poisson kernel. Be careful about convergence issues.
   (b) Let $f(\theta)$ be a Riemann integrable 2π-periodic function, and let $u(r,\theta)$ be its harmonic extension. Fix $s \in (0,1)$, and define $g(\theta) = u(s,\theta)$. Prove that the harmonic extension of g is $u(rs,\theta)$.

7. Say that a 2π-periodic function f is in $C^k(\mathbb{T})$ if f has derivatives up to order k and all are continuous and 2π-periodic.
   (a) Show if $f \in C^1(\mathbb{T})$, then $|\hat{f}(n)| \le C/|n|$ for some constant C. **Hint:** Integrate by parts.
   (b) Show by induction that if $f \in C^k(\mathbb{T})$, then $|\hat{f}(n)| \le Cn^{-k}$ for some constant C.
   (c) In particular, if $f \in C^2(\mathbb{T})$, show that the Fourier series for f converges uniformly on $\mathbb{T}$ to a continuous function.
   (d) Conversely show that if $f \in C(\mathbb{T})$ satisfies $|\hat{f}(n)| \le Cn^{-k}$ for an integer $k \ge 2$, then $f \in C^{k-2}(\mathbb{T})$. **Hint:** Term-by-term differentiation and the M-test.
   (e) Hence give necessary and sufficient conditions on $\hat{f}$ for f to be in $C^\infty(\mathbb{T})$.