PM 450 **Assignment 1** **Due Wednesday January 18.**

1. Consider the DE: $y' = 1 + xy$ and $y(0) = 0$ for $x \in [-b, b]$, where $b > 0$.
   (a) Reduce this to finding the fixed point of a mapping T. Show that when $b = 1$, the map T is a contraction mapping.
   (b) Prove that the DE has a unique solution on $[-b, b]$ for any $b > 0$. Hence deduce that there is a unique solution on the whole line $\mathbb{R}$.
   (c) Start with $f_0(x) = 1$ and compute $f_n(x) = T^n f_0$ by induction. Prove directly (rather than by quoting a theorem) that the sequence f_n converges uniformly on $[-b, b]$.

2. Consider the DE $y'' - x^{-1}y' + x^{-2}y = 0$ for $x \in [1, 3]$.
   (a) Check that $y = x$ is a solution. Look for a solution of the form $f(x) = xg(x)$ by showing that g' satisfies a 1st order DE, and solving it.
   (b) Show that the set of solutions that you obtain is a 2-dimensional vector space.
   (c) Show that the initial value conditions $y(1) = a_0$ and $y'(1) = a_1$ determine a unique solution from this set.

3. Consider $f'(x) + f(x)^2 = 4xf(x) - 4x^2 + 2$ for $x \in \mathbb{R}$ and $f(0) = 2$.
   (a) Show that this DE satisfies a local Lipschitz condition on some smaller region around $x = 0$; and hence deduce that there is a local solution.
   (b) Solve this DE explicitly. **Hint:** Find the DE satisfied by $g(x) = f(x) - 2x$ and solve it first.
   (c) Hence find the maximal continuation of the solution.

4. Consider $y' = \sin\left(\dfrac{x^5 + 3x^2 - 1}{\sqrt{219 - 2y^2}}\right)$ and $y(2) = 3$. Prove that there is a unique solution on $[-5, 9]$.
   Hint: Show that a solution must satisfy $|y| \le 10$. Obtain a Lipschitz condition valid in this range.

5. Consider the DE: $y'' = (1 + (y')^2)^{3/2}$ and $y(0) = 0$, $y'(0) = 0$.
   (a) Show that $f(x) = 1 - \sqrt{1 - x^2}$ is the unique solution on $[-1, 1]$.
   (b) This solution does not continue further, yet $|f(x)| \le 1$. Why does this not contradict the Continuation Theorem?

6. Consider the DE: $xyy' = y^2 - 1$ and $y(1) = a > 0$.
   (a) Show that this DE satisfies a local Lipschitz condition in y as long as x, y are both positive.
   (b) Solve the DE and find the largest interval on which a solution exists.
   (c) Observe that all solutions pass through $(0, 1)$ with the same slope. What happens when the solution is continued through this point into the second quadrant? Is this a problem for the theory?