1. Abel's Theorem

The purpose of this section is to demonstrate that when a power series converges at some point on the boundary of the disk of convergence, then the sum equals the limit of the function along the radius approaching the point. Because convergence at the boundary may be conditional, this is a subtle fact.

First we need a new convergence test. The proof utilizes a rearrangement technique called *summation by parts*, which is analogous to integration by parts.

Lemma 1.1 (Summation by Parts Lemma)**.** *Suppose (x_n) and (y_n) are sequences of complex numbers and define $X_n = \sum_{k=1}^{n} x_k$ and $Y_n = \sum_{k=1}^{n} y_k$. Then*

$$\sum_{n=1}^{m} x_n Y_n + \sum_{n=1}^{m} X_n y_{n+1} = X_m Y_{m+1}.$$

Proof. The argument is essentially an exercise in reindexing summations. Let $X_0 = 0$ and notice that the left-hand side (LHS) equals

$$\begin{aligned} \text{LHS} &= \sum_{n=1}^{m} (X_n - X_{n-1}) Y_n + \sum_{n=1}^{m} X_n (Y_{n+1} - Y_n) \\ &= \sum_{n=1}^{m} X_n Y_n - \sum_{n=1}^{m} X_{n-1} Y_n + \sum_{n=1}^{m} X_n Y_{n+1} - \sum_{n=1}^{m} X_n Y_n. \\ &= -X_0 Y_1 + X_m Y_{m+1} = X_m Y_{m+1}. \end{aligned}$$

∎

Thus, provided that $\lim_{m\to\infty} X_m Y_{m+1}$ exists, the two series $\sum x_n Y_n$ and $\sum X_n y_n$ either both converge or both diverge.

Theorem 1.2 (Dirichlet's Test)**.** *Suppose that $(a_n)_{n\geq 1}$ is a sequence of complex numbers with bounded partial sums:*

$$\Big| \sum_{k=1}^{n} a_k \Big| \leq M < \infty \quad \textit{for all} \quad n \geq 1.$$

If $(b_n)_{n\geq 1}$ is a sequence of positive numbers decreasing monotonically to 0, then the series $\sum_{n=1}^{\infty} a_n b_n$ converges. Moreover, $\big| \sum_{n=1}^{\infty} a_n b_n \big| \leq 2Mb_1$.

Proof. We use the Summation by Parts Lemma to rewrite $a_n b_n$. Let $x_n = a_n$ for all n; and set $y_1 = b_1$ and $y_n = b_n - b_{n-1}$ for $n > 1$. Define X_n and Y_n as in the lemma. Note that $y_n < 0$ for $n > 1$, and that there is a telescoping sum

$$Y_n = b_1 + (b_2 - b_1) + \cdots + (b_n - b_{n-1}) = b_n.$$

Hence $a_n b_n = x_n Y_n$.

Notice that $|X_n| = \big| \sum_{k=1}^{n} a_k \big| \leq M$ for all n. Since $|X_n Y_{n+1}| \leq M |b_{n+1}|$, the Squeeze Theorem shows that $\lim_{n\to\infty} X_n Y_{n+1} = 0$. Furthermore,

$$\begin{aligned} \sum_{k=1}^{n} |X_k y_{k+1}| &\leq \sum_{k=1}^{n} M |y_{k+1}| = M\big(b_1 + \sum_{k=1}^{n} b_k - b_{k+1}\big) \\ &= M(2b_1 - b_{n+1}) \leq 2Mb_1. \end{aligned}$$

Thus $\sum_{k=1}^{\infty} X_k y_{k+1}$ converges absolutely. Using the Summation by Parts Lemma, convergence follows from

$$\sum_{n=1}^{\infty} a_n b_n = \lim_{m\to\infty} \sum_{n=1}^{m} x_n Y_n = \lim_{m\to\infty} X_m Y_m - \sum_{n=1}^{m} X_n y_{n+1} = -\sum_{k=1}^{\infty} X_k y_{k+1}.$$

Moreover, $\left|\sum_{n=1}^{\infty} a_n b_n\right| \le 2Mb_1$. ■

Now we apply this to a power series. We first consider a disk around 0 with radius 1 where the power series also converges at the point $z = 1$.

Lemma 1.3. *Suppose that the power series $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence 1, and that $\sum_{k=0}^{\infty} a_k$ converges. Then this series converges uniformly on $[0,1]$ to a continuous function $f(x)$.*

Proof. Since $\sum_{k=0}^{\infty} a_k$ converges, $\lim_{k\to\infty} a_k = 0$. Hence $\lim_k |a_k|^{1/k} \le 1$. So by Hadamard's Theorem, the power series $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence at least 1. Let $f(x)$ be the sum of this series for $0 \le x \le 1$. We know that convergence is uniform on $[0,r]$ for $r < 1$, but Hadamard's Theorem does not tell us about convergence near $z = 1$.

Since $\sum_{k=0}^{\infty} a_k$ converges, given any $\varepsilon > 0$, there is an integer N so that

$$\Big|\sum_{k=n+1}^{\infty} a_k\Big| < \frac{\varepsilon}{4} \quad \text{for all} \quad n \ge N.$$

Thus we have bounded partial sums

$$\Big|\sum_{k=n+1}^{m} a_k\Big| = \Big|\sum_{k=n+1}^{\infty} a_k - \sum_{k=m+1}^{\infty} a_k\Big| < \frac{\varepsilon}{2} \quad \text{for all} \quad m > n \ge N.$$

We make use of the Dirichlet Test. Fix $n \ge N$ and $x \in [0,1)$. Let $b_k = x^{n+k}$ for $k \ge 1$. Since this sequence decreases monotonically to 0, Dirichlet's Test shows that $\sum_{k=1}^{\infty} a_{n+k} b_k$ converges, say to a function

$$f_n(x) = \sum_{k=1}^{\infty} a_{n+k} b_k = \sum_{k=1}^{\infty} a_{n+k} x^{n+k}.$$

In addition, Dirichlet's Test provides an estimate for the size of the sum, namely

$$|f_n(x)| \le 2\,\frac{\varepsilon}{2}\,b_1 \le \varepsilon.$$

This estimate is independent of x, so we obtain

$$\sup_{0\le x<1} \Big|f(x) - \sum_{k=0}^{n} a_k x^k\Big| = \sup_{0\le x<1} |f_n(x)| \le \varepsilon.$$

We also have

$$\Big|f(1) - \sum_{k=0}^{n} a_k\Big| = \Big|\sum_{k=n+1}^{\infty} a_k\Big| < \frac{\varepsilon}{4}.$$

This establishes that the power series converges uniformly to $f(x)$ on the whole interval $[0,1]$. In particular, $f(x)$ is continuous. ■

Theorem 1.4 (Abel's Theorem)**.** *Let $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ be a power series with finite radius of convergence R. Suppose that there is a point $z_1 = z_0 + Re^{i\theta}$ for which $\sum_{k=0}^{\infty} a_k(z_1-z_0)^k$ converges. Then*

$$\sum_{k=0}^{\infty} a_k(z_1-z_0)^k = \lim_{r\to R^-} f(z_0+re^{i\theta}).$$

Proof. Clearly we may assume that $z_0 = 0$. Consider the power series

$$g(z) = \sum_{k=0}^{\infty} (a_k R^k e^{ik\theta}) z^k.$$

This has radius of convergence 1 since

$$\limsup_{k\to\infty} |a_k R^k e^{ik\theta}|^{1/k} = R \limsup_{k\to\infty} |a_k|^{1/k} = 1.$$

Also $\sum_{k=0}^{\infty} a_k R^k e^{ik\theta}$ converges by hypothesis. Therefore Lemma 1.3 applies. So $g(x) = \sum_{k=0}^{\infty} (a_k R^k e^{ik\theta}) x^k$ is continuous on $[0,1]$. In particular

$$\sum_{k=0}^{\infty} a_k(z_1-z_0)^k = g(1) = \lim_{x\to 1^-} g(x) = \lim_{r\to R^-} f(z_0+re^{i\theta}). \qquad \blacksquare$$

Example 1.5. Consider the power series $f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$. By the ratio test, this has radius of convergence 1.

Consider convergence on the boundary circle at $z = 1+e^{i\theta}$. At $z = 0$, this is the harmonic series $-\sum_{n\ge 1}\frac{1}{n}$, which diverges. However at $z = 1+e^{i\theta}$ for $-\pi < \theta < \pi$, we will show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n e^{in\theta}}{n}$ converges. The reason is that the sequence $a_n = (-1)^{n-1}e^{in\theta}$ for $n \ge 1$ has bounded partial sums because we have a geometric series:

$$\Big|\sum_{k=1}^{n} (-1)^{k-1} e^{ik\theta}\Big| = \Big|\frac{(-1)^n e^{i(n+1)\theta} - e^{i\theta}}{e^{i\theta}+1}\Big| \le \Big|\frac{2}{e^{i\theta/2}+e^{-i\theta/2}}\Big| = \sec\tfrac{\theta}{2}.$$

Now the sequence $b_n = \frac{1}{n}$ is positive and monotone decreasing to 0. So Dirichlet's Test applies to the series $\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}$, and we conclude that the power converges at every point on the boundary circle except $z = 1$.

If we apply the Term by Term Differentiation Theorem, we obtain that

$$f'(z) = \sum_{n=1}^{\infty} (-1)^{n-1}(z-1)^{n-1} = \frac{1}{1+(z-1)} = \frac{1}{z} \quad \text{for} \quad |z-1| < 1.$$

Consider the analytic function $g(z) = f(e^z)$ defined on an open set Ω about 0 which is mapped by e^z one-to one and onto $B_1(1)$. By the chain rule, $g'(z) = f'(e^z)e^z = 1$. Since $g(0) = 0$, we conclude that $g(z) = z$. Therefore f is a branch of the logarithm on $B_1(1)$ satisfying $f(1) = 0$.

Abel's Theorem tells us that $f(z)$ extends to be a continuous function on the set $\overline{B_1(1)} \setminus \{0\}$. Some trigonometry shows that $1+e^{i\theta} = 2\cos\theta e^{i\theta/2}$. So we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{in\theta} = f(1+e^{i\theta}) = \lim_{r\to 1^-} f(1+re^{i\theta}) = \log(2\cos\theta/2) + i\theta/2$$

for $-\pi < \theta < \pi$. In particular, for $\theta = 0$ we obtain

$$\log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} - \tfrac{1}{5} + \dots .$$

And for $\theta = \pi/2$, we split the series into even and odd terms to get

$$\begin{aligned} \frac{1}{2}\log 2 + i\frac{\pi}{4} = \log(2\cos\pi/4) + i\pi/4 &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} i^n \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} + i\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \end{aligned}$$

Thus

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{9} + \dots .$$