Minimum-Phase Infinite-Dimensional Second-Order Systems

Birgit Jacob, Kirsten Morris and Carsten Trunk

Abstract

In general, better performance can be achieved with a controlled minimum-phase system than a controlled non-minimum-phase system. We show that a wide class of second-order infinite-dimensional systems with either velocity or position measurements are minimum-phase. The results are illustrated by two examples.

I. Introduction

There are a number of reasons why the minimum-phase property of a system is useful in controller design. For example, the sensitivity of a minimum-phase system can be reduced to an arbitrarily small level [41, e.g.], which implies good output disturbance rejection. On the other hand, the sensitivity of a non-minimum-phase system has a non-zero lower bound due to the non-minimum-phase part. A minimum-phase system that has relative degree no higher than two can be stabilized by high-gain control [25], [52, e.g.]. Furthermore, most adaptive controllers require the system to be minimum-phase. Thus, controller design for minimum-phase systems is in general much easier than for non-minimum-phase systems.

As for finite-dimensional systems, it is often desired that the system is minimum-phase. For instance, results on adaptive control and on high-gain feedback control of infinite-dimensional systems, see [37], [38], [39], [40], [42, e.g.], require the system to be minimum-phase. Moreover,

Birgit Jacob is with the Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands, b.jacob@ewi.tudelft.nl, Carsten Trunk is with the Institut für Mathematik, Technische Universität Berlin, Sekretariat MA 6-4, Straße des 17. Juni 136, D-10623 Berlin, Germany, trunk@math.tu-berlin.de, and Kirsten Morris is with the Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1, kmorris@uwaterloo.ca

the sensitivity of an infinite-dimensional minimum-phase system can be reduced to an arbitrarily small level and stabilizing controllers exist that achieve arbitrarily high gain or phase margin [17].

Due to the importance of the minimum-phase property, it is advantageous to establish conditions under which infinite-dimensional systems are minimum-phase. A stable finite-dimensional system is *minimum-phase* or *outer* if and only if its transfer function has no zeros in the right-half-plane. Unfortunately, determining the minimum-phase behaviour for infinite-dimensional systems is less straightforward than for finite-dimensional systems. There are aspects of the dynamics, besides zeros in the right-half-plane, that can lead to non-minimum-phase behaviour. For example, the transfer function of a pure delay, $\exp(-s)$, has no zeros, but is not minimum-phase. Also, in general, approximations to a model do not accurately predict the location of the transfer function zeros, or the system minimum-phase behaviour [9], [36],

There are some results for first-order systems guaranteeing that the transfer function is positive real [11], [12], [13], [44]. Positive-real systems are closely related to minimum-phase systems. We show that a stable, positive-real system with finite relative degree is also minimum-phase.

In this paper we will show that several large classes of second-order systems are minimum-phase. We study second-order systems of the form

$$\ddot{z}(t) + A_o z(t) + D\dot{z}(t) = B_o u(t), \tag{1}$$

equipped either with velocity measurements

$$y(t) = B_o^* \dot{z}(t), \tag{2}$$

or position measurements

$$y(t) = B_o^* z(t). \tag{3}$$

These systems have been studied in the literature for more than 20 years. Interest in this particular model is motivated by various problems such as stabilization, see for example [3], [33], [34], [53], solvability of the Riccati equations [20], and compensator problems with partial observations [21]. We will show that with this choice of output, and certain assumptions on the damping operator, these systems are well-posed and have an outer transfer function. In [60], [56] these systems have been studied with the damping $D = \frac{1}{2}B_o^* B_o$ and the output

$$y(t) = -B_o^* \dot{z}(t) + u(t). \tag{4}$$

This re-definition of the output is crucial. In one space dimension, the example in [60, Section 7] with output (4) has transfer function $\exp(-2s)$, an inner function. With measurements (2) we obtain the transfer function $1-\exp(-2s)$, an outer function. Note that the velocity measurement systems do have a positive real transfer function, whereas the position measurement systems do not have positive real transfer functions, even in the finite-dimensional case.

We proceed as follows. In Section II we introduce the classes of second-order systems considered in this article. General properties of these systems are shown in Sections III and IV. The minimum-phase property of these second-order systems is proven in Section V. Several examples at the end of this paper illustrate our results.

II. Second-order systems: Framework

We study second-order systems of the form (1) with either position measurements (3) or velocity measurements (2). As in [60], [56], we make the following assumptions throughout this paper.

(A1) The stiffness operator $A_o : \mathcal{D}(A_o) \subset H \to H$ is a self-adjoint, positive definite linear operator on a Hilbert space H such that zero is in the resolvent set of A_o. Here $\mathcal{D}(A_o)$ denotes the domain of A_o. Since A_o is self-adjoint and positive definite, A_o^α is well-defined for $\alpha \geq 0$. A scale of Hilbert spaces H_α is defined as follows: For $\alpha \geq 0$, we define $H_\alpha = \mathcal{D}(A_o^\alpha)$ equipped with the norm induced by the inner product

$$\langle x, y\rangle_{H_\alpha} = \langle A_o^\alpha x, A_o^\alpha y\rangle_H, \qquad x, y \in H_\alpha$$

and $H_{-\alpha} = H_\alpha^*$. Here the duality is taken with respect to the pivot space H, that is, equivalently $H_{-\alpha}$ is the completion of H with respect to the norm $\|z\|_{H_{-\alpha}} = \|A_o^{-\alpha} z\|_H$. Thus A_o extends (restricts) to $A_o : H_\alpha \to H_{\alpha-1}$ for $\alpha \in \mathbb{R}$. We use the same notation A_o to denote this extension (restriction).

We denote the inner product on H by $\langle\cdot,\cdot\rangle_H$ or $\langle\cdot,\cdot\rangle$, and the duality pairing on $H_{-\alpha} \times H_\alpha$ by $\langle\cdot,\cdot\rangle_{H_{-\alpha}\times H_\alpha}$. Note that for $(z', z) \in H \times H_\alpha$, $\alpha > 0$, we have

$$\langle z', z\rangle_{H_{-\alpha}\times H_\alpha} = \langle z', z\rangle_H.$$

(A2 i) Let $m \in \mathbb{N}$. The control operator B_o is a linear and bounded operator from $\mathbb{C}^m$ to $H_{-\frac{1}{2}}$.

(A2 ii) The damping operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is a bounded operator such that $A_o^{-1/2} D A_o^{-1/2}$ is a bounded self-adjoint non-negative operator on H. This implies that, in particular,

$$\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 0, \qquad z \in H_{\frac{1}{2}}.$$

Remark 2.1: If $z \in H_{\frac{1}{2}}$ with $\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = 0$, then $Dz = 0$. To see this, first note that since $A_o^{-1/2} D A_o^{-1/2}$ is a bounded self-adjoint non-negative operator on H, it has a self-adjoint non-negative square root, which we call S. It follows that

$$\begin{aligned} 0 &= \langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \\ &= \langle A_o^{-\frac{1}{2}} D A_o^{-\frac{1}{2}} A_o^{\frac{1}{2}} z, A_o^{\frac{1}{2}} z \rangle \\ &= \langle S A_o^{\frac{1}{2}} z, S A_o^{\frac{1}{2}} z \rangle \\ &= \| S A_o^{\frac{1}{2}} z \|^2. \end{aligned}$$

Thus $\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = 0$ implies that $S^2 A_o^{\frac{1}{2}} z = 0$. This implies that $A_o^{-\frac{1}{2}} Dz = 0$ and so $Dz = 0$.

Moreover we introduce one more assumption which we later need in Section IV and Section V to show well-posedness of the system (5), (6) below on the state space $H_{\frac{1}{2}} \times H$.

(A3) There exists a constant $\beta > 0$ such that

$$\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq \beta \| B_o^* z \|^2, \qquad z \in H_{\frac{1}{2}}.$$

The position control system (1), (3) is equivalent to the following standard first-order equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5}$$

$$y(t) = C_p x(t) \tag{6}$$

where $A : \mathcal{D}(A) \subset H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H$, $B : \mathbb{C}^m \to H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$ and $C_p : H_{\frac{1}{2}} \times H \to \mathbb{C}^m$ are given by

$$A = \begin{bmatrix} 0 & I \\ -A_o & -D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_o \end{bmatrix}, \quad C_p = \begin{bmatrix} B_o^* & 0 \end{bmatrix},$$

$$\mathcal{D}(A) = \left\{ \left[\begin{smallmatrix} z \\ w \end{smallmatrix} \right] \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_o z + Dw \in H \right\}.$$

The velocity control system (1), (2) has the equivalent first-order form (5) and

$$y(t) = C_v x(t), \quad t \geq 0, \tag{7}$$

where $C_v : H_{\frac{1}{2}} \times H_{\frac{1}{2}} \to \mathbb{C}^m$ is given by $C_v = \left[\begin{array}{cc} 0 & B_o^* \end{array}\right]$.

We will use the following notations throughout this article. We denote by $\mathcal{L}(X, Y)$ the set of linear, bounded operators from the Hilbert space X to the Hilbert space Y. We write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$.

We denote by $\mathcal{D}(S)$, $R(S)$, $N(S)$, $\sigma(S)$, and $\rho(S)$ the domain, range, null space, spectrum, and resolvent set, respectively, of an (unbounded) linear operator S on a Hilbert space X.

For a closed linear operator S on X we need to introduce the following subsets of the spectrum $\sigma(S)$. We denote by $\sigma_c(S)$ the *continuous spectrum*, by $\sigma_r(S)$ the *residual spectrum* and by $\sigma_p(S)$ the point spectrum. The *approximate point spectrum*, $\sigma_{ap}(S)$, consists of all λ for which there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}(S)$ such that

$$\|x_n\| = 1 \text{ and } \|(\lambda I - S)x_n\| \to 0 \quad \text{as} \quad n \to \infty$$

(see, for example, [16, page 242] and [47, page 178]). Note that our definition of the approximate spectrum is different from the definition used in [56]. We point out that the point and continuous spectrum are subsets of the approximate point spectrum.

The notation $\mathcal{H}^2(\mathbb{C}_0; X)$ and $\mathcal{H}^\infty(\mathbb{C}_0; X)$, where $\mathbb{C}_0$ is the open right-half-plane, and X is a Hilbert space, indicate the Hardy spaces of X-valued functions on $\mathbb{C}_0$. If $X := \mathbb{C}$ we write for simplicity $\mathcal{H}^2(\mathbb{C}_0)$, and $\mathcal{H}^\infty(\mathbb{C}_0)$. The Lebesgue space $L^2(0, t_0; X)$ is the space of measurable, square integrable X-valued functions on the interval $(0, t_0)$, $0 < t_0 \leq \infty$, and $H^2(0, t_0; X)$ is the second-order Sobolev space of X-valued functions on the interval $(0, t_0)$.

III. Stability

For this section we always assume that (A1) and (A2) hold. The following theorem is well known, see e.g. [4], [32], [5], [8], [23] or [60].

Theorem 3.1: The operator A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ of contractions on the state space $H_{\frac{1}{2}} \times H$.

This guarantees that the spectrum of A is contained in the closed left half plane of $\mathbb{C}$. For the main result of this paper it is required that there is no spectrum on the imaginary axis. This is implied by the following well-known result (see e.g. [4], [5], [7], [8], [23], [24], [57], [56]).

Proposition 3.2: If there exists a constant $\beta > 0$ such that

$$\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq \beta \|z\|_H^2, \qquad z \in H_{\frac{1}{2}}, \tag{8}$$

then A generates an exponentially stable semigroup on $H_{\frac{1}{2}} \times H$.

One aim of this section is to give a different condition than (8) guaranteeing that $i\mathbb{R} \subset \rho(A)$. Concerning the spectrum of A on the imaginary axis we have the following proposition.

Proposition 3.3: The operator A has a bounded inverse, and for every $i\eta \in \sigma(A)$, $\eta \in \mathbb{R}$, we have $-i\eta \in \sigma(A)$, $\eta^2 \in \sigma(A_o)$, and $i\eta \in \sigma_{ap}(A)$. If, in addition,

$$\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} > 0 \text{ for any eigenvector } z \in H_1 \text{ of } A_o \tag{9}$$

holds, then the operator A has no eigenvalues on the imaginary axis and every $i\eta \in \sigma(A)$ satisfies $i\eta \in \sigma_c(A)$.

Proof: This result can be found partly in [56]. In [60, Formula (5.2)], $0 \in \rho(A)$ is proven and in [56, Proof of Lemma 4.5] it is shown that

$$A^* = JAJ, \qquad \text{with } J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{10}$$

This fact implies immediately that the spectrum of A is symmetric about the real axis, and thus $-i\eta \in \sigma(A)$. Since A generates a bounded C_0-semigroup, we have that $i\eta$ is an element of the boundary of $\sigma(A)$, which proves $i\eta \in \sigma_{ap}(A)$ [16, Proposition 1.10 on page 242]. The proof that $\eta^2 \in \sigma(A_o)$ can be found in [56, Lemma 4.6].

Now, suppose that A has an imaginary eigenvalue, $i\eta$. Since $0 \in \rho(A)$, we have $\eta \neq 0$. Then we can find a $\left(\begin{smallmatrix} z \\ w \end{smallmatrix} \right) \in \mathcal{D}(A) \backslash \{0\}$ such that

$$A \left(\begin{smallmatrix} z \\ w \end{smallmatrix} \right) = i\eta \left(\begin{smallmatrix} z \\ w \end{smallmatrix} \right),$$

which implies that

$$A_o z + i\eta D z = \eta^2 z. \tag{11}$$

Moreover, we have $z \neq 0$, since otherwise $\left(\begin{smallmatrix} z \\ w \end{smallmatrix} \right) = 0$. Taking the inner product with z,

$$\langle A_o z + i\eta D z, z\rangle = \eta^2 \|z\|^2.$$

Thus, the imaginary part is zero, which implies that $\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = 0$. Now Remark 2.1 implies that $Dz = 0$, and (11) shows that $A_o z = \eta^2 z$. Thus z is an eigenvector corresponding to the eigenvalue η^2. This implies that (9) does not hold. Thus, if (9) holds then A has no eigenvalues on the imaginary axis.

It remains to show that $i\eta \in \sigma_c(A)$. Equation (10) implies that $\sigma_p(A^*) = \sigma_p(A)$. Moreover, $\lambda \in \sigma_r(A)$ implies that $\overline{\lambda} \in \sigma_p(A^*)$, because $R(\lambda I - A)^\perp = N(\overline{\lambda} I - A^*)$. Since $i\mathbb{R} \cap \sigma_p(A) = \emptyset$, $i\mathbb{R} \cap \sigma_r(A) = \emptyset$, and therefore $i\eta \in \sigma_c(A)$. ■

If in addition to the assumption (9) of the previous proposition, we have that $\sigma_c(A) \cap i\mathbb{R}$ is countable, then A generates a strongly stable semigroup [1]. We need however that $i\mathbb{R} \subset \rho(A)$ to show the minimum-phase property of these systems. In the following theorem we give a different condition than (8) guaranteeing that $i\mathbb{R} \subset \rho(A)$. This also implies strong stability.

Theorem 3.4: [8, Lemma 4.1 and Theorem 4.4] Assume that A_o^{-1} is a compact operator. Then $i\eta I - A$ has a closed range for every $\eta \in \mathbb{R}$ and so $i\mathbb{R} \cap \sigma_c(A) = \emptyset$.

Moreover,

$$i\mathbb{R} \subset \rho(A) \tag{12}$$

if and only if

$$\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} > 0 \text{ for any eigenvector } z \in H_1 \text{ of } A_o. \tag{13}$$

In particular, (13) holds if $0 \notin \sigma_p(D)$.

We note that A does not necessarily have a compact resolvent if A_o has a compact resolvent. Indeed, if $A_o = D$ then the range of $-I - A$ is included in $\mathcal{D}(A_o) \times H$ and, if A_o is unbounded, $-1 \in \sigma_c(A)$ follows. Hence A has no compact resolvent.

The following example shows that the assumption of A_o having compact resolvent in Theorem 3.4 cannot be omitted.

Example 3.5: Let H be an infinite-dimensional Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ and $A_o := I$. Moreover, we define $D \in \mathcal{L}(H)$ by $Dz = \sum_{n=1}^{\infty} \frac{1}{n} \langle z, e_n \rangle e_n$. Then we have $i \in \sigma(A)$, because

$$\|(iI - A)\left(\begin{smallmatrix} e_n \\ ie_n \end{smallmatrix}\right)\| = \frac{1}{n} \to 0$$

as n tends to infinity.

Under the assumptions of Theorem 3.4 it is possible that the spectrum lies arbitrarily close to the imaginary axis.

Example 3.6: Let H be an infinite-dimensional Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$.

We define the operators $A_o : \mathcal{D}(A_o) \subset H \to H$ and $D \in \mathcal{L}(H)$ by

$$\begin{aligned} A_o z &:= \sum_{n=1}^{\infty} n^2 \langle z, e_n \rangle e_n, \quad z \in \mathcal{D}(A_o), \\ \mathcal{D}(A_o) &:= \{ z \in H \mid \sum_{n \in \mathbb{N}} n^4 |\langle z, e_n \rangle|^2 < \infty \}, \\ Dz &:= \sum_{n=1}^{\infty} \frac{1}{n} \langle z, e_n \rangle e_n. \end{aligned}$$

An easy calculation shows that $\lambda_n := -\frac{1}{2n} + i\sqrt{n^2 - \frac{1}{4n^2}}$, $n \in \mathbb{N}$, are eigenvalues of A with $\left(\begin{smallmatrix} e_n \\ \lambda_n e_n \end{smallmatrix} \right)$ as corresponding eigenvectors.

We conclude this section with the following proposition, which will be used in the next section.

Proposition 3.7: [60, Prop. 5.3] For every $s \in \rho(A)$,

1) $(sI - A)^{-1}$ is a bounded and invertible map from $H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$ to $H_{\frac{1}{2}} \times H_{\frac{1}{2}}$.
2) The operator $s^2 I + Ds + A_o \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ has a bounded inverse $V(s) \in \mathcal{L}(H_{-\frac{1}{2}}, H_{\frac{1}{2}})$,
$$V(s) = \left(s^2 I + Ds + A_o \right)^{-1}.$$
3) On $H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$, for every non-zero $s \in \rho(A)$,
$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} \left[I - V(s) A_o \right] & V(s) \\ -V(s) A_o & sV(s) \end{bmatrix}. \tag{14}$$

IV. System properties

A. Well-posed linear systems

In this subsection we review some definitions related to systems with unbounded control and observation operators. We remark that throughout this subsection A, B and C denote arbitrary operators between Hilbert spaces and not the specific operators introduced in Section II.

Denote by U, X and Y Hilbert spaces. Let $A : \mathcal{D}(A) \subset X \to X$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on X, the state space. X_1 denotes the space $\mathcal{D}(A)$ equipped with the graph topology and X_{-1} is the completion of X with respect to the norm $\|x\|_{X_{-1}} := \|(\beta I - A)^{-1} x\|_X$, where β is an arbitrary element of the resolvent set of A. The semigroup $(T(t))_{t \geq 0}$ can be extended or restricted to a strongly continuous semigroup on X_{-1} or X_1, respectively. We will denote this extension (restriction) again by $(T(t))_{t \geq 0}$.

Consider for $B \in \mathcal{L}(U, X_{-1})$ the following linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \quad x(0) = x_o, \tag{15}$$

where $x_o \in X$ and $u \in L^2_{\text{loc}}(0, \infty; U)$. The operator $T(t)$ defines the map from initial condition to state, that is, for zero input u we have

$$x(t) = T(t)x(0). \tag{16}$$

The mild solution of (15)

$$x(t) := T(t)x_o + \int_0^t T(t-s)Bu(s)\, ds, \qquad t \geq 0, \tag{17}$$

is well-defined on X_{-1}. For $u \in H^2(0, \infty; U)$, define

$$\mathcal{B}_t u = \int_0^t T(t-s)Bu(s)ds. \tag{18}$$

For such u, the mild solution $x(\cdot)$ is a continuous X-valued function. An operator B is called an *admissible control operator for the semigroup* $(T(t))_{t\geq 0}$, if for every $t > 0$ there is a constant $M_t > 0$ such that for all $u \in H^2(0, \infty; U)$,

$$\|\mathcal{B}_t u\| \leq M_t \|u\|^2_{L^2(0,\infty;U)}.$$

This allows us to extend $\mathcal{B}_t$ to a linear bounded operator from $L^2(0, \infty; U)$ to X. For an admissible control operator the solution (17) is as a continuous X-valued function. We call B an *infinite-time admissible control operator* if the constant M_t is independent of t.

We now add an output to our system (15). Let $C \in \mathcal{L}(X_1, Y)$. For initial conditions $x(0) = x_o$ in X_1, define the output operator $\mathcal{C}_t : X_1 \to L^2(0, t; Y)$ by

$$(\mathcal{C}_t x_o)(s) = CT(s)x_o, \qquad 0 \leq s \leq t.$$

The operator $C \in \mathcal{L}(X_1, Y)$ is called an *admissible observation operator for the semigroup* $(T(t))_{t\geq 0}$, if for every $t > 0$ there is a constant $N_t > 0$ such that for $x_o \in X_1$,

$$\int_0^t \|\mathcal{C}_t x_o\|^2 ds \leq N_t \|x_o\|^2_X.$$

This allows us to extend the operator $\mathcal{C}_t$ to a linear bounded operator from X to $L^2(0, t; Y)$. We call C an *infinite-time admissible observation operator* if the constant N_t is independent of t. Further information on admissible control and observation can be found in [26], [58], [59].

For an input $u \in L^2_{\text{loc}}(0,\infty;U)$ and zero initial condition the output y is given by

$$y(\tau) = (\mathcal{G}_t u)(\tau), \qquad \tau < t, \tag{19}$$

where $\mathcal{G}_t$ is a linear operator from $L^2(0,t;U)$ to $L^2(0,t;Y)$. Moreover, a certain functional equation expressing the causality and time-invariance of the system must hold, see [51] or [10]. Because of this, $\mathcal{G}_t u$ is the convolution of the input u with a distribution g.

We define $\mathcal{G} : L^2_{\text{loc}}(0,\infty;U) \to L^2_{\text{loc}}(0,\infty;Y)$ by

$$(\mathcal{G}u)(\tau) := (\mathcal{G}_t u)(\tau), \qquad \tau \le t.$$

The transfer function G of system (15), (19), which is an analytic $\mathcal{L}(U,Y)$-valued function on some right-half-plane $\{s \in \mathbb{C} \mid \operatorname{Re} s > \mu\}$, can be defined as follows. By ω_o we denote the growth bound of $(T(t))_{t\ge 0}$. Let $x_o = 0$. For $\omega > \omega_o$ and $e^{-\omega\cdot}u \in L^2(0,\infty;U)$ we have $e^{-\omega\cdot}y \in L^2(0,\infty;U)$, and the transfer function G is defined by

$$\hat{y}(s) = G(s)\hat{u}(s), \qquad \operatorname{Re} s > \omega,$$

where $\hat{\cdot}$ denotes the Laplace transform.

The system (15), (19) is *well-posed* on X if and only if the four maps from input and initial condition to state and output defined by $T(t), \mathcal{B}_t, \mathcal{C}_t$ and $\mathcal{G}_t$ are bounded for some $t > 0$ (and hence every $t > 0$). Boundedness of $\mathcal{G}_t$ is equivalent to the boundedness of the transfer function G on some right-half-plane.

The system (15), (19) is *regular*, if it is well-posed and if for some $E \in \mathcal{L}(U,Y)$, the transfer function G satisfies

$$\lim_{s\to+\infty} G(s)u = Eu, \qquad u \in U.$$

E is called the *feedthrough operator*. Moreover, if C is a linear bounded operator from X to Y, then the transfer function G is given by

$$G(s) = C(sI - A)^{-1}B + E, \quad \operatorname{Re} s > w_o, \tag{20}$$

and we have

$$y(t) = Cx(t) + Eu(t), \qquad t > 0. \tag{21}$$

For more information on well-posed linear systems, regular linear systems, and the corresponding transfer function and feedthough operators we refer the reader to [10] and [54].

B. Velocity/position measurement systems

We now return to the classes of systems introduced in Section II. We start with the velocity measurement system (5), (7), that is, we assume that the output is given by

$$y(t) = B_o^* \dot{z}(t) = C_v x(t), \tag{22}$$

where $C_v : H_{\frac{1}{2}} \times H_{\frac{1}{2}} \to \mathbb{C}^m$ with $C_v = \begin{bmatrix} 0 & B_o^* \end{bmatrix}$.

Proposition 4.1: If, in addition to assumptions (A1)-(A2), (A3) also holds then

1) The control operator B is infinite-time admissible for the semigroup generated by A.
2) The observation operator C_v is infinite-time admissible for the semigroup generated by A.
3) The system (5), (7) is well-posed.
4) The transfer function of (5), (7) is given by $G_v(s) = sB_o^* V(s) B_o$ and satisfies $G_v \in \mathcal{H}^\infty(\mathbb{C}_0, \mathcal{L}(\mathbb{C}^m))$.

Proof: The proof of this proposition uses the approach in [60]. Following the proof of Lemma 5.4 in [60] we obtain, that for $u \in H^2(0, \infty; \mathbb{C}^m)$ and $z_o, w_o \in H_{\frac{1}{2}}$ satisfying

$$A_o z_o + D w_o - B_o u(0) \in H$$

there exists a unique solution $z \in C^1(0, \infty; H_{\frac{1}{2}}) \cap C^2(0, \infty; H)$ of (1) with $z(0) = z_o$ and $\dot{z}(0) = w_o$. Moreover, following the proof of Proposition 5.5 in [60] the identity

$$\frac{1}{2}\frac{d}{dt}\left\| \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} \right\|^2 = -\langle D\dot{z}(t), \dot{z}(t)\rangle + \mathrm{Re}\langle B_o u(t), \dot{z}(t)\rangle$$

holds. Using (A3) and the standard inequality that, for any $\varepsilon > 0$,

$$2\mathrm{Re}\langle a, b\rangle \le \varepsilon \|a\|^2 + \frac{1}{\varepsilon}\|b\|^2,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\left\| \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} \right\|^2 \le (-\beta + \frac{\varepsilon}{2})\|B_o^* \dot{z}(t)\|^2 + \frac{1}{2\varepsilon}\|u(t)\|^2.$$

Choosing $\varepsilon < 2\beta$, there are constants $c_1, c_2 > 0$ such that

$$\frac{d}{dt}\left\| \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} \right\|^2 \le c_1\|u(t)\|^2 - c_2\|B_o^* \dot{z}(t)\|^2.$$

Rearranging and writing $y(t) = B_o^* \dot{z}(t)$,

$$c_2 \|y(t)\|^2 + \frac{d}{dt} \left\| \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} \right\|^2 \leq c_1 \|u(t)\|^2. \tag{23}$$

Integrating this inequality over time and using Theorem 3.1 we obtain that B and C_v are (infinite-time) admissible and that system (5), (7) is a well-posed linear system. This inequality also implies that the system (5), (7) is L^2-stable, that is, the input/output operator $\mathcal{G}$ is a linear bounded operator from $L^2(0,\infty;\mathbb{C}^m)$ to $L^2(0,\infty;\mathbb{C}^m)$. Hence the transfer function G_v is in $\mathcal{H}^\infty(\mathbb{C}_0, \mathcal{L}(\mathbb{C}^m))$. Using Theorem 1.3 in [60] we obtain that G_v is given by $G_v(s) = sB_o^* V(s) B_o$, $s \in \mathbb{C}_0$. ■

The following example shows that, in general, if condition (A3) does not hold, the system (5), (7) is not well-posed.

Example 4.2: Let H be an infinite-dimensional Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. We define the operators $A_o : \mathcal{D}(A_o) \subset H \to H$, $D \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ and $B_o \in \mathcal{L}(\mathbb{C}, H_{-\frac{1}{2}})$ by

$$\begin{aligned} A_o z &:= \sum_{n=1}^{\infty} n^2 \langle z, e_n \rangle e_n, \quad z \in \mathcal{D}(A_o) \\ \mathcal{D}(A_o) &:= \{z \in H \mid \sum_{n=1}^{\infty} n^4 |\langle z, e_n \rangle|^2 < \infty\}, \\ Dz &:= \sum_{n=1}^{\infty} n \langle z, e_n \rangle e_n, \\ B_o &:= \sum_{n=1}^{\infty} n^{\frac{1}{4}} e_n. \end{aligned}$$

Assumptions (A1)-(A2) are satisfied, while (A3) is not satisfied. Using Theorem 3.4 we obtain that the transfer function $G_v(s) = sB_o^* V(s) B_o$ is a holomorphic function on an open set containing the closed right-half-plane. We now show that the transfer function is not bounded on the right-half-plane, and hence the control system is not well-posed. For positive integers ω,

$$\begin{aligned}
|G_v(i\omega)| &= |\omega B_o^* V(i\omega) B_o| \\
&= \left| \sum_{n=1}^{\infty} \frac{\omega n^{1/2}}{n^2 + i\omega n - \omega^2} \right| \\
&\geq \left| \mathrm{Im} \sum_{n=1}^{\infty} \frac{\omega n^{1/2}}{n^2 + i\omega n - \omega^2} \right| \\
&\geq \sum_{n=\omega}^{\infty} \frac{n^{3/2}\omega^2}{n^4 + \omega^4} \\
&\geq \int_{\omega}^{\infty} \frac{t^{3/2}\omega^2}{t^4 + \omega^4} dt \\
&= \sqrt{\omega} \int_1^{\infty} \frac{x^{3/2}}{x^4 + 1} dx.
\end{aligned}$$

Thus the function $sB_o^* V(s) B_o$ is not bounded on the right-half-plane. This implies that the system (5), (7) is not well-posed on any state-space [51].

We now study the properties of the position measurement system (5), (6).

Proposition 4.3: If, in addition to the standard assumptions (A1)-(A2), (A3) also holds then

1) The observation operator C_p is a bounded operator from $H_{\frac{1}{2}} \times H$ to $\mathbb{C}^m$ and is thus admissible for the semigroup generated by A.
2) The system (5), (6) is regular with feedthrough 0.
3) The transfer function of (5), (6) is given by $G_p(s) = B_o^* V(s) B_o$ and satisfies $G_p \in \mathcal{H}^\infty(\mathbb{C}_0, \mathcal{L}(\mathbb{C}^m))$.

Proof: The observation operator C_p is a bounded operator on the state space $H_{\frac{1}{2}} \times H$, and thus C_p is an admissible observation operator for the semigroup generated by A. In Proposition 4.1 we showed that B is an infinite-time admissible control operator for the semigroup generated by A. Using Proposition 3.7, (20) and (21) we see that the corresponding transfer function is given by $G_p(s) = B_o^* V(s) B_o$, $s \in \mathbb{C}_0$. In Proposition 4.1 we proved that G_v, given by $G_v(s) = sG_p(s)$, $s \in \mathbb{C}_0$, is a bounded holomorphic function on the right-half-plane. Thus the transfer function G_p is an analytic function on the right-half-plane. It remains to show that G_p is bounded on the right-half-plane. The function G_p has an analytic extension to a neighborhood of 0, since $0 \in \rho(A)$. (See Proposition 3.3.) Thus the boundedness of G_p on the right-half-plane follows from the fact that G_v is bounded on $\mathbb{C}_0$. Thus system (5), (6) is a well-posed linear system.

Further, the boundedness of G_v in the right-half-plane implies that

$$\lim_{s \to +\infty} G_p(s) = 0,$$

and therefore (5), (6) is a regular linear system with zero feedthrough. ■

V. Minimum-Phase Behaviour of Second-Order Systems

In this section we give some conditions under which the classes of second-order systems introduced in Section II are minimum-phase. We first introduce a definition of a minimum-phase system that is appropriate for infinite-dimensional systems. For any function $g \in \mathcal{H}^\infty(\mathbb{C}_0; \mathbb{C}^{m\times m})$ define the operator $\Lambda_g : \mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m) \to \mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m)$ by $\Lambda_g f = gf$ for any $f \in \mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m)$.

Definition 5.1: [49, page 94] A bounded, holomorphic function $g : \mathbb{C}_0 \to \mathbb{C}^{m\times m}$ is called *minimum-phase* or *outer* if the range of Λ_g is dense in $\mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m)$.

Thus, outer functions correspond to operators that have inverses defined on a dense subset of $\mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m)$. This explains their importance in controller design- such a system has an inverse defined on a dense subset of $\mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m)$. In particular, this implies that a scalar outer function has no zeros in the open right-half-plane, and it can be shown that a bounded rational function is outer if and only if the function has no zeros in the open right-half-plane. For more information on outer functions we refer the reader to [49, Chap. 5]. The following test will be helpful, see [43, page 22] for more details.

Theorem 5.2: **[Helson-Lowdenslager Theorem]** Let $g : \mathbb{C}_0 \to \mathbb{C}^{m\times m}$ be bounded and holomorphic. Then g is outer if and only if $\det(g(\cdot))$ is a scalar outer function.

It is difficult to establish that a given function is outer. Therefore we will use the following well-known factorization result: Every bounded holomorphic function $g : \mathbb{C}_0 \to \mathbb{C}$ can be factored as $g(s) = \tau(s)h(s)$, $s \in \mathbb{C}_0$, where τ is an *inner function*, that is, $|\tau(s)| \leq 1$ for $s \in \mathbb{C}_0$, and $|\tau(i\eta)| = 1$ for almost every $\eta \in \mathbb{R}$, and h is an outer function. We note that $|h(i\eta)| = |g(i\eta)|$ for almost every $\eta \in \mathbb{R}$, and that $|h(s)| \geq |g(s)|$ on $\mathbb{C}_0$. All the right-half-plane zeros of a function are included in the inner function. For more results on the inner-outer factorization of bounded, holomorphic functions we refer the reader to [15, page 192 ff.], [22, page 132 ff.].

We will show that the transfer functions discussed in previous sections have inner factor 1 and hence are outer or minimum-phase. We summarize some results on inner functions. Let $\{\beta_n\}_{n\in\mathbb{N}}$

be a sequence of points in $\mathbb{C}_0$ satisfying the *Blaschke condition*

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re}\beta_n}{1+|\beta_n|^2} < \infty. \tag{24}$$

Then the *Blaschke product* Θ corresponding to the sequence $\{\beta_n\}_{n\in\mathbb{N}}$ is given by

$$\Theta(s) = \prod_{n\in\mathbb{N}} \frac{|1-\beta_n^2|}{1-\beta_n^2} \frac{s-\beta_n}{s+\overline{\beta}_n}, \quad s \in \mathbb{C}_0, \tag{25}$$

where $\frac{|1-\beta_n^2|}{1-\beta_n^2}$ is assumed to be 1 if $\beta_n = 1$. The function Θ is in $\mathcal{H}^\infty(\mathbb{C}_0)$ and the zeros of Θ are precisely the points β_n, each zero having multiplicity equal to the number of times it occurs in the sequence. Moreover, $|\Theta(s)| \leq 1$ for all s with positive real part, and $|\Theta(i\eta)| = 1$ for almost all real η's. Thus every Blaschke product is an inner function. However, not every inner function can be written as a Blaschke product. Another class of inner functions are the singular functions. A *singular function* is a holomorphic function $S : \mathbb{C}_0 \to \mathbb{C}$ that can be written as

$$S(s) = e^{-\rho s} \exp\left[-\int_{\mathbb{R}} \frac{ts+i}{t+is}\, d\mu(t)\right], \quad s \in \mathbb{C}_0, \tag{26}$$

where μ is a finite singular positive measure on $\mathbb{R}$ and ρ is a non-negative real number. Every inner function τ can be uniquely written as

$$\tau(s) = e^{i\alpha}\Theta(s)S(s), \quad s \in \mathbb{C}_0, \tag{27}$$

where $\alpha \in \mathbb{R}$, Θ is a Blaschke product and S is a singular function, see for example [15, page 192ff.]. In the following proposition we formulate sufficient conditions for functions to be minimum-phase.

Definition 5.3: A function $g \in \mathcal{H}^\infty(\mathbb{C}_0)$ has *finite relative degree*, if there exists $n \in \mathbb{N}$ such that for real s,

$$\lim_{s\to\infty} s^n g(s) \neq 0. \tag{28}$$

The smallest $n_0 \in \mathbb{N}$ satisfying (28) is called the *relative degree* of g.

Proposition 5.4: Assume that $g \in \mathcal{H}^\infty(\mathbb{C}_0)$ has finite relative degree and is analytic on some open set Ω containing the closed right-half-plane. Then g is minimum-phase if and only if g has no zeros in the open right-half-plane.

Proof: Due to the inner-outer factorization the function g can be written as

$$g(s) = e^{i\alpha}\Theta(s)S(s)h(s), \quad s \in \mathbb{C}_0,$$

where Θ is a Blaschke product, S is a singular function of the form (26) and h is an outer function. We note that the function g is outer if and only if the functions S and Θ are identically 1. The function g is holomorphic on Ω and the closed right-half-plane is contained in Ω. This implies that the measure μ in (26) is zero [49, page 142]. The finite relative degree property shows that $\rho = 0$, and thus the singular function $S(s)$ is identically 1. The Blaschke product will be the identity if and only if g has no zeros in the open right-half-plane. This proves the proposition. ■

There are some results for first-order systems guaranteeing that the transfer function is positive real [11], [12], [13], [44]. A function $g \in \mathcal{H}^\infty(\mathbb{C}_0)$ is called *positive real*, if $g(\overline{s}) = \overline{g(s)}$ and $\operatorname{Re} g(s) \geq 0$ for all $s \in \mathbb{C}_0$. An adaptive controller for a class of positive-real second-order systems with relative degree one is constructed in [27], [28], [29]. In [11] the positive real property, together with exponential stability of the semigroup, and a relative-degree assumption, is shown to imply convergence and stability of an adaptive compensator. The following proposition relates minimum-phase and positive real functions.

Proposition 5.5: Assume that $g \in \mathcal{H}^\infty(\mathbb{C}_0)$ has finite relative degree and is analytic on some open set Ω containing the closed right-half-plane. If g is also positive real then g is minimum-phase.

Proof: Due to Proposition 5.4, it is enough to show that g has no zeros in $\mathbb{C}_0$. Assume that there is a $s_o \in \mathbb{C}_0$ such that $g(s_o) = 0$. The function g is non constant, since $g(s_o) = 0$ and g has finite relative degree. Thus the open mapping theorem for analytic functions implies that we can find an element $s \in \mathbb{C}_0$ near s_o such that Re $g(s) < 0$. This implies that g is not positive real. ■

In particular, every positive real system that is exponentially stable and has finite relative degree is minimum-phase.

We now return to the class of systems introduced in Section II. We first show that if B_o is not the zero operator, $s^2\|G_p(s)\|_{\mathcal{L}(\mathbb{C}^m)} \not\to 0$ as s tends to $+\infty$ and so both control systems have finite relative degree.

Lemma 5.6: Assume that assumptions (A1)-(A3) are satisfied. We have that for $s \geq 1$ and $u \in \mathbb{C}^m$

$$\langle u, s^2 G_p(s)u\rangle_{\mathbb{C}^m} \geq M\|B_o u\|^2_{H_{-\frac{1}{2}}}. \tag{29}$$

Proof: We recall that $G_p(s) = B_o^* V(s) B_o$ for $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$, see Proposition 4.3, and we define the operator $X(s) \in \mathcal{L}(H)$, $s \in [0, \infty)$, by

$$X(s) = s^2 A_o^{-1} + s A_o^{-1/2} D A_o^{-1/2} + I.$$

$X(s)$ is a self-adjoint operator satisfying the estimate

$$\|X(s)\| \leq s^2 \|A_o^{-1}\| + s \|A_o^{-1/2} D A_o^{-1/2}\| + 1.$$

This implies that

$$\langle z, (X(s))^{-1} z \rangle \geq \frac{\|z\|^2}{s^2 \|A_o^{-1}\| + s \|A_o^{-1/2} D A_o^{-1/2}\| + 1}, \qquad s \in [0, \infty), z \in H,$$

and thus we have for $s \geq 1$ and $u \in \mathbb{C}^m$

$$\begin{aligned}
\langle u, s^2 G_p(s) u \rangle_{\mathbb{C}^m} &= \langle u, s^2 B_o^* A_o^{-1/2} (X(s))^{-1} A_o^{-1/2} B_o u \rangle_{\mathbb{C}^m} \\
&= \langle A_o^{-1/2} B_o u, s^2 (X(s))^{-1} A_o^{-1/2} B_o u \rangle_H \\
&\geq \frac{\|A_o^{-1/2} B_o u\|_H^2}{\|A_o^{-1}\| + s^{-1} \|A_o^{-1/2} D A_o^{-1/2}\| + s^{-2}} \\
&\geq \frac{\|A_o^{-1/2} B_o u\|_H^2}{\|A_o^{-1}\| + \|A_o^{-1/2} D A_o^{-1/2}\| + 1} \\
&\geq M \|B_o u\|_{H_{-\frac{1}{2}}}^2,
\end{aligned}$$

for some constant $M > 0$. ■

Lemma 5.7: Assume that assumptions (A1)-(A3) are satisfied and that B_o is injective. Then for every $s \in \mathbb{C}_0$ the matrices $G_p(s)$ and $G_v(s)$ are invertible.

Proof: It is sufficient to show invertibility of G_p. The proof is similar to the proof for finite-dimensional second-order systems in [35]. From Theorem 3.1, it follows that G_p is well-defined in the open right-half-plane.

We first show that if for some $s_o \in \mathbb{C}_0$, the nullspace of $G_p(s_o)$ contains a non-zero element then B_o is not injective. Suppose $s_o \in \mathbb{C}_0$ is such that there exists non-zero $u_o \in \mathbb{C}^m$ with $G_p(s_o) u_o = 0$. Using Prop. 4.3 for the representation of the transfer function,

$$B_o^* V(s_o) B_o u_o = 0.$$

Define $z_o = V(s_o)B_o u_o$. If $z_o = 0$ then $B_o u_o = 0$. Since u_o is non-zero, this implies that B_o is not injective. Assume now that $z_o \neq 0$. Noting that $z_o \in H_{\frac{1}{2}}$ we can write

$$\begin{aligned} (s_o^2 I + s_o D + A_o)z_o - B_o u_o &= 0 \\ B_o^* z_o &= 0 \end{aligned}$$

where the first equation holds in $H_{-\frac{1}{2}}$ and the second in $\mathbb{C}^m$. Thus,

$$\langle (s_o^2 I + s_o D + A_o)z_o - B_o u_o, z_o \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = 0.$$

Using $B_o^* z_o = 0$, this becomes

$$\langle (s_o^2 I + s_o D + A_o)z_o, z_o \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = 0. \tag{30}$$

Using the positivity of A_o, the non-negativity of D and decomposing s_o into real and imaginary parts, $s_o = \sigma + i\eta$ where $\sigma > 0$, the imaginary part of (30) is:

$$\eta \langle [2\sigma I + D]z_o, z_o \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = 0$$

and (A2) implies that $\eta = 0$. The real part of (30) is

$$\langle [(\sigma^2 - \eta^2)I + \sigma D + A_o]z_o, z_o \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = 0.$$

Since $\eta = 0$, this equation is not satisfied for any non-zero z_o. Thus, $G_p(s_o)u_o = 0$ implies that $u_o = 0$ or B_o is not injective. ■

Remark 5.8: Suppose that $\rho(A)$ includes the closed right-half-plane so that G_p can be extended to a set including the imaginary axis. Then Lemma 5.7 can be strengthened to include the imaginary axis for G_p. It is only necessary to consider the case where $s_o = i\eta$. Then equation (30) implies that

$$\begin{aligned} \eta \langle D z_o, z_o \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} &= 0 \\ \langle [-\eta^2 I + A_o] z_o, z_o \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} &= 0. \end{aligned}$$

These equations have a non-trivial solution for z_o if and only if η^2 is an eigenvalue of A_o with eigenvector z_o and $\langle D z_o, z_o \rangle = 0$. Thus, if B_o is injective, $\rho(A)$ includes the imaginary axis and $\langle Dz, z \rangle > 0$ for any eigenvector of A_o, then $G_p(s)$ is invertible for every $s \in \overline{\mathbb{C}_0}$. The same conclusion holds for G_v exept that $G_v(0) = 0$.

We give in Theorem 5.9 below sufficient conditions for the minimum-phase property of G_p and G_v.

Theorem 5.9: Assume that assumptions (A1)-(A3) are satisfied. If in addition, the resolvent of A contains the imaginary axis and the operator B_o is injective, then G_p and G_v are minimum-phase functions.

Proof: It is sufficient to show the result for G_p, and due to the Definition 5.1 it is sufficient to show that $\det G_p$ is a scalar outer function. In Lemma 5.7 it was shown that $G_p(s)$ is invertible for all $s \in \mathbb{C}_0$. This implies that $\det G_p(s) \neq 0$ for all $s \in \mathbb{C}_0$.

Since the resolvent set of A contains the imaginary axis, the transfer function G_p is analytic on an open set Ω containing the closed right-half-plane and belongs to $\mathcal{H}^\infty(\mathbb{C}_0, \mathcal{L}(\mathbb{C}^m))$ (cf. Proposition 4.3). Thus, $\det G_p$ is analytic on Ω and belongs to $\mathcal{H}^\infty(\mathbb{C}_0)$.

We show next that $\det G_p$ has finite relative degree. Lemma 5.6 implies that for some constant $M > 0$,

$$\langle s^2 G_p(s)u, u\rangle_{\mathbb{C}^m} \geq M\|B_o u\|^2_{H_{-\frac{1}{2}}}, \quad s \geq 1.$$

Since B_o is an injective bounded mapping from $\mathbb{C}^m$ to $H_{-\frac{1}{2}}$, this implies that

$$\langle s^2 G_p(s)u, u\rangle_{\mathbb{C}^m} \geq c\|u\|^2, \quad s \geq 1, u \in \mathbb{C}^m,$$

for some constant $c > 0$. This shows that the eigenvalues of $s^2 G_p(s)$ are uniformly bounded away from zero as s approaches infinity, which implies that $s^{2m} \det G_p(s) \not\to 0$ as s tends to infinity. Now Theorem 5.9 follows from Proposition 5.4. ∎

The following result is now immediate.

Corollary 5.10: Assume that assumptions (A1)-(A3) are satisfied. If A_o has a compact resolvent, $\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} > 0$ for any eigenvector of A_o, and B_o is injective, then the transfer functions G_p and G_v are minimum-phase functions.

VI. Examples

In this section we apply our results to some well-known models with position measurements. We will first study an Euler-Bernoulli beam with Kelvin-Voigt damping, and then a damped plate on a bounded connected domain. We show that both control systems have minimum-phase transfer functions.

A. Euler-Bernoulli Beam

Consider a beam with a thin film of piezoelectric polymer applied to one side. A spatially uniform voltage $u(t)$ is applied to the film to control the vibrations. Consider only transverse vibrations, and let $z(r,t)$ denote the deflection of the beam from its rigid body motion at time t and position r. The beam is clamped at the end $r=0$ and free at the other end $r=1$. Use of the Euler-Bernoulli model for the beam deflection and the Kelvin-Voigt damping model leads to the following description of the vibrations [2], [6]:

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^2}{\partial r^2}\left[E\frac{\partial^2 z}{\partial r^2} + C_d \frac{\partial^3 z}{\partial r^2 \partial t}\right] = 0, \qquad r \in (0,1), t > 0. \tag{31}$$

Here E and C_d are positive physical constants, representing a weighted average of the properties of the beam and of the piezoelectric film. For all $t>0$ the boundary conditions are, for some constant $c>0$,

$$\begin{array}{rcl} z(0,t) & = & 0, \\ \frac{\partial z}{\partial r}|_{r=0} & = & 0, \\ \left[E\frac{\partial^2 z}{\partial r^2} + C_d \frac{\partial^3 z}{\partial r^2 \partial t}\right]_{r=1} & = & cu(t), \\ \left[E\frac{\partial^3 z}{\partial r^3} + C_d \frac{\partial^4 z}{\partial r^3 \partial t}\right]_{r=1} & = & 0. \end{array} \tag{32}$$

A position sensor is used at the tip:

$$y(t) = \frac{\partial z}{\partial r}(1,t).$$

We will put this control system into the framework of this paper. The analysis is quite standard. See [31, Sect. 5.3] for the generalization to a plate.

Here H is $L^2(0,1)$ and $A_o = E\frac{d^4}{dr^4}$ with $\mathcal{D}(A_o)$ given by

$$\left\{z \in H^4(0,1) : z(0) = z'(0) = z''(1) = z'''(1) = 0\right\}.$$

For $z, v \in D(A_o)$,

$$\langle A_o^{\frac{1}{2}} z, A_o^{\frac{1}{2}} v\rangle = \langle A_o z, v\rangle = E\langle v'', v''\rangle.$$

The space $H_{\frac{1}{2}}$ is therefore the closure of $D(A_o)$ with respect to $E\langle z'', v''\rangle$ and so

$$H_{\frac{1}{2}} = \left\{z \in H^2(0,1) : z(0) = z'(0) = 0\right\}$$

with inner product $\langle z, v\rangle_{H_{\frac{1}{2}}} = E\langle z'', v''\rangle$. Let $x(t) = (z(\cdot,t), \dot{z}(\cdot,t))$. The damping operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is

$$\langle Dz, \phi\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \frac{C_d}{E}\langle z, \phi\rangle_{H_{\frac{1}{2}}}$$

for $z, \phi \in H_{\frac{1}{2}}$. Hence $D = \frac{C_d}{E} A_o$. The weak formulation of the boundary value problem (31), (32) is

$$\langle \ddot{z}(t), \phi \rangle + \langle A_o z(t), \phi \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + \langle D\dot{z}(t), \phi \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = c\phi'(1)u(t),$$

for all $\phi \in H_{\frac{1}{2}}$. It follows that $B_o u = c\delta'(1)u$. Sobolev's Inequality implies that evaluation of ϕ' at a point is bounded on $H_{\frac{1}{2}}$ and so the control operator B_o is bounded from $\mathbb{C}$ to $H_{-\frac{1}{2}}$. The dual operator is given by $B_o^* z = cz'(1)$. Assumptions (A1)-(A2) are satisfied. Notice that this choice of D and B_o is not included in the special class covered in [60], [56]. The inequality

$$\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \frac{C_d}{E} \|z\|^2_{H_{\frac{1}{2}}} \geq \alpha \|z\|^2, \quad z \in H_{\frac{1}{2}},$$

for a positive constant α, implies the well-known result that A generates an exponentially stable semigroup on $H_{\frac{1}{2}} \times L^2(0,1)$ (see Proposition 3.2). Furthermore, for $z \in H_{\frac{1}{2}}$,

$$\langle Dz, z \rangle = \frac{C_d}{E} \|z\|^2_{H_{\frac{1}{2}}} \geq \beta |z'(1)|^2 = \frac{\beta}{c} |B_o^* z|^2$$

for some $\beta > 0$ by Sobolev's Inequality. Thus (A3) is also satisfied, implying well-posedness of the control system with measurements $z'(1)$ (Proposition 4.3). Theorem 5.9 implies that the transfer function is minimum-phase.

If the position measurement is replaced by velocity measurement, the same conclusions hold.

Even for this simple example, the transfer function is quite complicated and it is not easy to determine from direct analysis of the transfer function that there are no right-hand-plane zeros and no singular part.

Also, we still have well-posedness of the system and the minimum-phase property if we consider weaker damping than Kelvin-Voigt damping. The damping must satisfy $\langle Dz, z \rangle \geq \beta |z'(1)|^2$ and also $\langle Dz, z \rangle > 0$ for any eigenvector of A_o (cf. Corollary 5.10).

B. Plate with Boundary Damping

We show that a problem consisting of a wave equation with damping, as well as control, occurring through the boundary has the minimum-phase property. This problem occurs in, for example, vibrations of a plate or membrane that is fixed on part of the boundary. The wave equation with boundary damping and control on the boundary has been studied many times in the literature. See for instance, [20], [30], [48], [55] for the state-space formulation of similar systems and [30], [46], [50] for stability analysis. The control system in [60] is similar to that studied here although we consider a more general control input and a different observation. In [60] the partial differential equation is placed into the framework used in this paper and in [60]. We include full details for completeness.

Consider a bounded connected region Ω with boundary Γ. The region $\Omega \subset \mathbb{R}^n$ has Lipschitz boundary Γ, where $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$ and Γ_0, Γ_1 are disjoint open subsets of Γ with both Γ_0 and Γ_1 not empty and Γ_1 is such that the interior sphere condition holds at least one point in Γ_1. Assume also that Ω is such that the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact. Then the Poincaré inequality is satisfied [14, pg. 127-130]. That is, there is a constant $c > 0$ such that for all $f \in H^1(\Omega)$ with $f|_{\Gamma_0} = 0$,

$$\int_\Omega |\nabla f(x)|^2 dx \geq c \int_\Omega |f(x)|^2 dx.$$

We use the following system description

$$\begin{array}{ll}
\ddot{z} = \nabla^2 z, & \Omega \times (0, \infty), \\
z(x,0) = z_0, \ \dot{z}(x,0) = z_1, & \Omega, \\
z(x,t) = 0, & \Gamma_0 \times (0,\infty), \\
\frac{\partial z(x,t)}{\partial n} + d(x)^2 \dot{z}(x,t) = b(x)u(t), & \Gamma_1 \times (0,\infty), \\
y(t) = \int_{\Gamma_1} b(x) z(x,t) dx, & [0,\infty).
\end{array} \tag{33}$$

We also assume that $b, d \in C(\Gamma_1) \cap L^2(\Gamma_1)$ with $\inf_{x\in\Gamma_1} d(x) > 0$ and b not identically zero. The Sobolev spaces $H^s(\Omega)$, $s = 1/2, 1, 2$, and the boundary spaces $H^{1/2}(\Gamma)$, $L^2(\Gamma_1)$ are defined as usual, see [19]. The Dirichlet trace operator, $\gamma g = g|_\Gamma$, is a linear bounded operator from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$. We then define

$$\gamma_1 g = g|_{\Gamma_1} = P_{\Gamma_1} \gamma g.$$

Define

$$H^1_{\Gamma_0}(\Omega) = \{g \in H^1(\Omega) \mid g|_{\Gamma_0} = 0\}$$

and define $\tilde{H}^{1/2}(\Gamma_1)$ to be traces $\gamma_1 g$ where $g \in H^1_{\Gamma_0}(\Omega)$ with the usual trace norm. Thus, $\gamma_1 \in \mathcal{L}(H^1_{\Gamma_0}(\Omega), \tilde{H}^{1/2}(\Gamma_1))$. The space $\tilde{H}^{1/2}(\Gamma_1)$ is dense in $L^2(\Gamma_1)$, see e.g. [19]. We will consider γ_1 as a map $\gamma_1 : H^1_{\Gamma_0}(\Omega) \to L^2(\Gamma_1)$. For $f \in C^1(\Omega)$ we define the Neumann trace by

$$\alpha_1 f = \frac{\partial f}{\partial n}|_{\Gamma_1}.$$

Using the following Green formula [19, Lem. 1.5.3.7], we can extend the Neumann trace: For $f \in H^2(\Omega)$ and $g \in H^1(\Omega)$,

$$\int_\Omega (\nabla^2 f) g dx = -\int_\Omega \nabla f \cdot \nabla g dx + \int_\Gamma \gamma(\frac{\partial f}{\partial \eta}) \gamma(g) dx.$$

For $g \in H^1_{\Gamma_0}(\Omega)$ we obtain

$$\int_\Omega (\nabla^2 f) g dx = -\int_\Omega \nabla f \cdot \nabla g dx + \int_{\Gamma_1} \alpha_1 f \, \gamma_1 g \, dx. \tag{34}$$

Using this, we can define $\alpha_1 f$ as an element of $\tilde{H}^{-1/2}(\Gamma_1)$ for all $f \in H^2(\Omega)$. Here $\tilde{H}^{-1/2}(\Gamma_1)$ denotes the dual space of $\tilde{H}^{1/2}(\Gamma_1)$. Note that $L^2(\Gamma_1)$ is densely and continuously embedded in $\tilde{H}^{-1/2}(\Gamma_1)$.

We define the self-adjoint operator A_o on $L^2(\Omega)$ by

$$A_o f = -\nabla^2 f, \quad \mathcal{D}(A_o) = \{f \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega), \ \alpha_1 f = 0\}.$$

It is well-known that this operator is positive definite. The existence of a bounded inverse follows from the Poincaré inequality and so it satisfies (A1) on $H = L^2(\Omega)$. The inner product on H will be indicated by $(\cdot, \cdot)$. The space $H_{\frac{1}{2}} = \mathcal{D}(A_o^{\frac{1}{2}})$ is the completion of $\mathcal{D}(A_o)$ in the norm $(A_o z, z)^{1/2}$ and so, using (34) we see that $H_{\frac{1}{2}} = H^1_{\Gamma_0}(\Omega)$. The inner product on $H_{\frac{1}{2}}$ is indicated by $(\cdot, \cdot)_{H_{\frac{1}{2}}}$. The inner product on $L^2(\Gamma_1)$ is indicated by $\langle \cdot, \cdot \rangle$. We define the extension of A_o to $H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ by

$$(A_o f, g) = (\nabla f, \nabla g)_{H^n}. \tag{35}$$

By the Riesz representation theorem, for any $v \in L^2(\Gamma_1)$, there is a unique $g \in H_{\frac{1}{2}}$ such that

$$(g, \phi)_{H_{\frac{1}{2}}} = \langle v, \gamma_1 \phi \rangle$$

for all $\phi \in H_{\frac{1}{2}}$. This defines a map $N : L^2(\Gamma_1) \to H_{\frac{1}{2}}$ with $Nv = g$. Alternatively,

$$(A_o g, \phi)_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle v, \gamma_1 \phi \rangle$$

where A_o is understood in the extended sense (35). Equivalently,

$$(A_o N v, \phi)_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle v, \gamma_1 \phi \rangle$$

and so $\gamma_1^* = A_o N$.

We define $D = \gamma_1^*(d(\cdot)^2 \gamma_1)$. Then D is a linear bounded operator from $H_{\frac{1}{2}}$ to $H_{-\frac{1}{2}}$ and $\langle Df, f \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 0$ for all $f \in H_{\frac{1}{2}}$. Further, we define $B_o : \mathbb{C} \to H_{-\frac{1}{2}}$ as $B_o = \gamma_1^* b$. The operators B_o and D satisfy assumption (A2i) and (A2ii), respectively, with $m = 1$.

We now write the boundary control problem (33) in the abstract second-order form (1). For $z \in H^2(\Omega)$, $g \in H^1(\Gamma_0)$, we have from (34) and the boundary condition

$$\begin{aligned} \left(\nabla^2 z, g\right) &= -(\nabla z, \nabla g)_{H^n} - \langle d(\cdot)^2 \gamma_1 \dot{z}, \gamma_1 g\rangle + \langle b(\cdot)u, \gamma_1 g\rangle \\ &= -(A_o z, g)_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} - (Dz, g)_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + (B_o u, g)_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} . \end{aligned}$$

We thus obtain the abstract second-order differential equation:

$$\ddot{z}(t) + A_o z(t) + Dz(t) = B_o u(t)$$

valid for all $z \in H_{\frac{1}{2}}$ as an equation in $H_{-\frac{1}{2}}$. We now show that assumption (A3) is also satisfied. For $z \in H_{\frac{1}{2}}$ we obtain

$$\|B_o^* z\|^2 = \langle b, \gamma_1 z\rangle^2 \leq \|b\|^2 \|\gamma_1 z\|^2 \leq \frac{\|b\|^2}{\inf |d(x)|^2} \|d(\cdot)\gamma_1 z\|^2.$$

Thus,

$$(Dz, z)_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle d(\cdot)\gamma_1 z, d(\cdot)\gamma_1 z\rangle \geq \frac{\inf d(x)^2}{\|b\|^2} \|B_o^* z\|^2 = \beta \|B_o^* z\|^2$$

for some $\beta > 0$. Thus, (A3) is satisfied. By Proposition 4.3 the position control system with $y(t) = B_o^* z(t)$, or

$$y(t) = \langle b(x), \gamma_1 z\rangle$$

is well-posed. Note that the damping operator does not satisfy the inequality

$$(Dz, z)_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq \beta \|z\|^2, \qquad z \in H_{\frac{1}{2}},$$

for some $\beta > 0$. Although the semigroup is a contraction, it is not exponentially stable for all geometries Γ_0, Γ_1 [46], [50].

We now show that the conditions of Theorem 3.4 are satisfied and so $iR \subset \rho(A)$ and also the system is strongly stable. It is well-known that A_o^{-1} is a compact operator. Suppose that z is an eigenvector of A_o with $Dz = 0$. In other words, we have $z \in H^2(\Omega)$ satisfying for some complex number λ,

$$\nabla^2 z - \lambda z = 0, \quad z|_\Gamma = 0, \quad \frac{\partial z}{\partial n}|_{\Gamma_1} = 0.$$

For all $\lambda \geq 0$, z must be the zero function [19, Thm. 2.2.3]. Consider now the case $\lambda < 0$ and define the sets in Ω,

$$\begin{aligned} \Omega^+ &= \{x \in \Omega;\ \nabla^2 z - \lambda z = 0,\ z(x) > 0\}, \\ \Omega^- &= \{x \in \Omega;\ \nabla^2 z - \lambda z = 0,\ z(x) < 0\}. \end{aligned}$$

If both sets are empty, this implies that z is the zero function. Let $x_o \in \Gamma_1$ be a point satisfying an interior sphere condition. Either

1) x_o is in the boundary of Ω^- or
2) x_o is in the boundary of Ω^+ or
3) $z(x) = 0$ for all x in some open set $W \subset \Omega$ with x_o in the boundary of W.

Since $\frac{\partial z}{\partial n}|(x_o) = 0$, Hopf's Maximum Principle [18, Lem. 3.4] implies that for every open set $W \subset \Omega$ with x_o on the boundary of W, there are points x_1, x_2 in W with $z(x_1) \geq 0$ and $z(x_2) \leq 0$. Thus neither alternative (1) or (2) is possible. Thus, the last condition must hold. Since z is analytic, we obtain that z is the zero function on Ω. Thus, for any eigenvector z of A_o we have $\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} > 0$. Theorem 3.4 then implies that the resolvent of A contains the imaginary axis and that the system is strongly stable. Theorem 5.9 then shows that the transfer function G of the boundary control system (33) is a minimum-phase function.

VII. Conclusions

In this paper we examined second-order control systems. The second-order structure of these systems was used to show that a wide class of control systems, with either position or velocity measurements, are minimum-phase. The class of systems with velocity measurements for which this property holds is slightly larger than previously shown. The major contribution of this work, however, is to establish the minimum-phase property for systems with position measurements. These systems do not usually have positive-real transfer functions. It was assumed that the damping is stronger than the control effort, see (A3). A counterexample illustrated that the system transfer function may be improper if this assumption fails to hold. Another assumption that was implicit in the framework is that the observation of position is given by $C_p = [B_o^* \; 0]$. Although this leads to a system that is not mathematically collocated, this represents a type of *physical* collocation condition in that it implies a relation between the location of the sensors and the actuators. The results were illustrated with several common applications.

Acknowledgement

The authors are grateful to David Siegel for helpful discussions on elliptic partial differential equations.

REFERENCES

[1] W. Arendt and C.J.K. Batty, "Tauberian theorems and stability of one-parameters semigroups", *Trans. Amer. Math. Soc.*, Vol. 306, pg. 837-841, 1988.

[2] T. Bailey and J.E. Hubbard, "Distributed piezoelectric-polymer active vibration control of a cantilever beam", *Jour. of Guidance, Control and Dynamics*, Vol. 8, pg. 605-611, 1985.

[3] C.D. Benchimol, "A note on weak stabilizability of contraction semigroups", *SIAM J. Control Optimization*, Vol. 16, No. 3, pg. 373-379, 1978.

[4] H.T. Banks and K. Ito, "A unified framework for approximation in inverse problems for distributed parameter systems", *Control Theory and Adv. Tech.*, Vol. 4, pg. 73-90, 1988.

[5] H.T. Banks, K. Ito and Y. Wang, "Well posedness for damped second order systems with unbounded input operators", *Differential Integral Equations*, Vol. 8, pg. 587-606, 1995.

[6] H.T. Banks, R.C. Smith and Y. Wang, "Modeling aspects for piezoelectric patch activation of shells, plates and beams", *Quart. Appl. Math.*, Vol. 53, pg. 353-381, 1995.

[7] A. Bátkai and K.-J. Engel, "Exponential decay of 2×2 operator matrix semigroups", *J. Comp. Anal. Appl.*, Vol. 6, pg. 153-164, 2004.

[8] S. Chen, K. Liu and Z. Liu, "Spectrum and stability for elastic systems with global or local Kelvin-Voigt damping", *SIAM J. Appl. Math.*, Vol. 59, No. 2, pg. 651-668, 1998.

[9] A. Cheng and K.A. Morris, "Accurate zeros approximation for infinite-dimensional systems", *42nd IEEE Conference on Decision and Control*, Honolulu, Hawaii, 2003.

[10] R.F. Curtain, "The Salamon-Weiss class of well-posed infinite-dimensional linear systems: A survey", *IMA J. Math. Control Inform.*, Vol. 14, No.2, pg. 207-223, 1997.

[11] R. F. Curtain, M.A. Demetriou and K. Ito, "Adaptive compensators for perturbed positive real infinite-dimensional systems", *Int. J. Appl. Math. Comput. Sci.*, Vol. 13, No. 4, pg. 441-452, 2003.

[12] R.F. Curtain, "Linear operator inequalities for stable weakly regular linear systems", *Math. Control Signals Systems*, Vol. 14, No. 4, pg. 299-337, 2001.

[13] R.F. Curtain, "The Kalman-Yakubovich-Popov lemma for Pritchard-Salamon systems", *Systems Control Lett.*, Vol. 27, pg. 67-72, 1996.

[14] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. 2, Springer-Verlag, Berlin, 1988.

[15] P. L. Duren, *Theory of H^p Spaces*, Pure and Applied Mathematics, 38, Academic Press, New York - London, 1970.

[16] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.

[17] C. Foias, H. Özbay and A. Tannenbaum, *Robust Control of Infinite Dimensional Systems*, Lecture Notes in Control and InformationSciences, 209, Springer-Verlag, London, 1995.

[18] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order", Springer-Verlag, Heidelberg, 1983.

[19] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[20] E. Hendrickson and I. Lasiecka, "Numerical approximations and regularizations of Riccati equations arising in hyperbolic dynamics with unbounded control operators", *Comput. Optim. Appl.*, Vol. 2, No. 4, pg. 343-390, 1993.

[21] E. Hendrickson and I. Lasiecka, "Finite-dimensional approximations of boundary control problems arising in partially observed hyperbolic systems", *Dynam. Contin. Discrete Impuls. Systems* Vol. 1, No. 1, pg. 101-142, 1995.

[22] K. Hoffman, “Banach Spaces of Analytic Functions”, Dover Publications, Inc. , New York, 1988.

[23] R.O. Hryniv and A.A. Shkalikov, “Exponential stability of semigroups related to operator models in mechanics”, *Math. Notes*, Vol. 73, No. 5, pg. 618-624, 2003.

[24] R.O. Hryniv and A.A. Shkalikov, “Exponential decay of solution energy for equations associated with some operator models of mechanics”, *Functional Analysis and Its Applications*, Vol. 38, No. 3, pg. 163-172, 2004.

[25] A. Ilchmann, *Non-Identifier-Based High-Gain Adaptive Control*, Lecture Notes in Control and Information Sciences 189, Springer-Verlag, London, 1993.

[26] B. Jacob and J.R. Partington, “Admissibility of control and observation operators for semigroups: a survey”, In J.A. Ball, J.W. Helton, M. Klaus, and L. Rodman, editors, *Current Trends in Operator Theory and its Applications, Proceedings of IWOTA 2002*, Operator Theory: Advances and Applications, Vol. 149, Birkhäuser Verlag, Basel, pg. 199-221, 2004.

[27] T. Kobayashi, “Stabilization of infinite-dimensional second-order systems by adaptive PI-controllers”, *Math. Methods Appl. Sci.*, Vol. 24, pg. 513-527, 2001.

[28] T. Kobayashi, “Adaptive regulator design for a class if infinite-dimensional second-order systems”, *J. Math. Anal. Appl.*, Vol. 261, pg. 337-349, 2001.

[29] T. Kobayashi, “Low-gain adaptive stabilization of infinite-dimensional second-order systems”, *J. Math. Anal. Appl.*, Vol. 275, pg. 835-849, 2002.

[30] J. Lagnese, “Decay of Solutions of Wave Equations in a Bounded Region with Boundary Dissipation”, *Jour. of Diff. Equations*, Vol. 50, pg. 163-182, 1983.

[31] I. Lasiecka, *Mathematical Theory of Coupled PDE’s*, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, 2002.

[32] I. Lasiecka, “Stabilization of wave and plate equations with nonlinear dissipation on the boundary”, *Jour. of Diff. +Equations*, Vol. 79, pg. 340, 1989.

[33] I. Lasiecka and R. Triggiani, “Uniform exponential energy decay of wave equations in a bounded region with $L^2(0,\infty;L^2(\Gamma))$-feedback control in the Dirichlet boundary condition”, *J. Differential Equations*, Vol. 66, No. 3, pg. 340-390, 1987.

[34] N. Levan, “The stabilization problem: A Hilbert space operator decomposition approach”, *IEEE Trans. Circuits and Systems*, Vol. 25, No. 9, 721-727, 1978.

[35] J-L Lin and J-N Juang, “Sufficient conditions for minimum-phase second-order systems”, *J. Vib. Control*, Vol. 1, pg. 183-199, 1995.

[36] D.K. Lindner, K.M. Reichard, and L.M. Tarkenton, “Zeros of modal models of flexible structures”, *IEEE Trans. Automat. Control*, Vol. 38, No. 9, pg. 1384-1388, 1993.

[37] H. Logemann and D. H. Owens, “Robust High-gain feedback control of infinite-dimensional minimum-phase systems”, *IMA J. Math. Control Inform.*, Vol. 4, pg. 195-220, 1987.

[38] H. Logemann and S. Townley, “Adaptive control of infinite-dimensional systems without parameter estimation: an overview”, *IMA J. Math. Control Inform.*, Vol. 14, pg. 175-206, 1997.

[39] H. Logemann and S. Townley, “Adaptive low-gain integral control of multivariable well-posed linear systems”, *SIAM J. Control Optim.*, Vol. 41, No. 6, pg. 1722-1732, 2003.

[40] H. Logemann and H. Zwart, “On robust PI-control of infinite-dimensional systems”, *SIAM J. Control Optim.*, Vol. 30(3), pg. 573-593, 1992.

[41] K.A. Morris, *Introduction to Feedback Control Theory*, Harcourt/Academic Press, Burlington, MA, 2001.

[42] S. Nikitin and M. Nikitina, "High gain output feedbacks for systems with distributed parameters", *Math. Models Methods Appl. Sci.*, Vol. 9(6), pg. 933-940, 1999.

[43] N.K. Nikol'ski, *Treatise on the Shift Operator*, Fundamental Principles of Mathematical Sciences, 273, Springer-Verlag, Berlin, 1986.

[44] J. Oostveen, *Strongly Stabilizable Infinite-Dimensional Systems*, Ph.D. Thesis, Univ. Groningen, 2002.

[45] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, 44, Springer-Verlag, New York, 1983.

[46] J.P. Quinn and D.L. Russell, "Asymptotic stability and energy decay rates for solutions of hyperbolic equations with boundary damping", *Proc. Roy. Soc. Edinburgh Sect. A*, Vol. 77A, pg. 97-127, 1977.

[47] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press, New York-London, 1978.

[48] A. Rodriquez-Bernal and E. Zuazua, " Parabolic singular limit of a wave equation with localized boundary damping", *Discrete Contin Dynam. Systems*, Vol. 1, pg. 303-346, 1995.

[49] M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Dover Publications, Inc., Mineola, NY, 1997.

[50] D.L. Russell, "Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions", *SIAM Rev.*, Vol. 20, pg. 639-739, 1978.

[51] D. Salamon, "Realization theory in Hilbert space", *Math. Systems Theory*, Vol. 21, No. 3, pg. 147-164, 1989.

[52] S.M. Shinners, *Modern Control System Theory and Design*, John Wiley & Sons, Inc., New York, 1992.

[53] M. Slemrod, "Stabilization of boundary control systems", *J. Differential Equation*, Vol. 22, No. 2, pg. 402-415, 1976.

[54] O.J. Staffans, *Well-posed Linear Systems*, Encyclopedia of Mathematics and its Applications, 103, Cambridge University Press, Cambridge, 2005.

[55] R. Triggiani, "Wave equation on a bounded domain with boundary dissipation: an operator approach", *JMAA*, Vol. 137, pg. 438-461, 1989.

[56] M. Tucsnak and G. Weiss, "How to get a conservative well-posed system out of thin air", Part II, *SIAM J. Control Optim.*, Vol. 42, No. 3, pg. 907-935, 2003.

[57] K. Veselić, "Energy Decay of Damped Systems", ZAMM, Vol. 84, pg. 856-864, 2004.

[58] G. Weiss, "Admissibility of unbounded control operators", *SIAM J. Control Optim.*, Vol. 27, No.3, 527-545, 1989.

[59] G. Weiss, "Admissible observation operators for linear semigroups", *Isr. J. Math.*, Vol.65, No. 1, pg. 17-43, 1989.

[60] G. Weiss and M. Tucsnak, "How to get a conservative well-posed system out of thin air", Part I, *ESAIM Control Optim. Calc. Var.*, Vol. 9, pg. 247-274, 2003.