$\mathcal{H}^\infty$-Output Feedback of Infinite-Dimensional Systems via Approximation

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Abstract

As in the finite-dimensional case, a state-space based controller for the infinite-dimensional $\mathcal{H}^\infty$ disturbance-attenuation problem may be calculated by solving two Riccati equations. These operator Riccati equations can rarely be solved exactly. We approximate the original infinite-dimensional system by a sequence of finite-dimensional systems. The solutions to the corresponding finite-dimensional Riccati equations are shown to converge to the solution of the infinite-dimensional Riccati equations. Furthermore, the corresponding finite-dimensional controllers yield performance arbitrarily close to that obtained with the infinite-dimensional controller.

Keywords: infinite-dimensional systems, gap topology, $\mathcal{H}^\infty$, approximation, stability

1 Introduction

In this paper we discuss $\mathcal{H}^\infty$ control problems for the linear system in a separable Hilbert space X

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + B_1 v(t) + B_2 u(t), \quad x(0) = x_o \in X &(1)\\ y(t) &= C_1 x(t) + D_{12} u(t) &(2)\\ z(t) &= C_2 x(t) + D_{21} v(t)\,. &(3)\end{aligned}$$

The linear closed operator A generates the C_0-semigroup $S(t)$ on X. Let W, U, Y and Z be separable Hilbert spaces. The signal $v(t) \in L^2(0,\infty;W)$ is a $W-$ valued disturbance and $u(t) \in L^2(0,\infty;U)$ is the controlled input. We assume that the disturbance operator $B_1 \in \mathcal{L}(W,X)$ and the input operator $B_2 \in \mathcal{L}(U,X)$. The function y is the cost function and z is the controller input. Background on infinite-dimensional systems theory may be found in [4]. Let $C_1 \in \mathcal{L}(X,Y)$ be the reference output operator. The controlled output operator $C_2 \in \mathcal{L}(X,Z)$. Let G_{ij} be the transfer function with state-space realization (A, B_j, C_i, D_{ij}), and let G indicate the generalized plant transfer function

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}.$$

Let H be the transfer function of a controller so that the closed loop is well-posed. The closed loop transfer function from uncontrolled input v to cost y is

$$\Delta(G, H) = G_{11} + G_{12}H\ (I - G_{22}H)^{-1}G_{21}.$$

This paper is concerned with the problem of constructing a stabilizing feedback controller H so that

$$\|\Delta(G, H)\|_\infty < \gamma. \tag{4}$$

This is known as the H_∞*-disturbance attenuation problem.* Such problems arise in a variety of contexts; robust stabilization is one of the most important.

In general, for practical reasons, a finite-dimensional controller is wanted that achieves the required performance. One frequency domain approach is to design an infinite-dimensional controller for the infinite-dimensional system and then approximate this infinite-dimensional controller *e.g.* [6, 14]. This approach requires strong assumptions on the plant. Also, the required spectral factorizations are difficult numerically. Another frequency domain approach involves replacing the original system by a finite-dimensional approximation and then designing a controller for this approximation. It must then be verified that the controller stabilizes the orginal system, and that the required performance is achieved *e.g.* [2, 3, 12, 13].

In this paper a state-space approach to controller design is used. It is now well-known that, if the $\mathcal{H}^\infty$ disturbance-attenuation problem is solvable, then it can be solved by calculating the solutions to two algebraic Riccati equations. The finite-dimensional case was presented in [5] and generalized to infinite-dimensional systems in [1, 9]. The infinite-dimensional Riccati equations can rarely be solved exactly. In the special case of full-information control (C_2 is the identity operator) only one Riccati equation needs to be solved. For this case of full-information control it was shown in [7] that the sequence of solutions to the finite-dimensional Riccati equation converge to the solution of the infinite-dimensional Riccati equation. Furthermore, performance arbitrarily close to that obtained with infinite-dimensional state-feedback can be obtained using finite-dimensional state feedback. In this paper these results are extended to include output feedback. The gap topology used in [11, 12] is also used here. The resulting proofs are short.

We will approximate the original system (1-3) by a sequence of finite-dimensional systems and consider the corresponding finite-dimensional disturbance-attenuation problems. It will be shown that the sequences of solutions to the finite-dimensional Riccati equations converge strongly to the solutions to the infinite-dimensional Riccati equations. Furthermore, and more importantly, the corresponding finite-dimensional feedback controllers yield performance arbitrarily close to that obtained with the infinite-dimensional controller.

2 Approximation Framework

Let X^N be a finite-dimensional subspace of X and p^N be the orthogonal projection of X onto X^N. The space X^N is equipped with the induced norm from X. Consider a sequence of operators $A^N \in \mathcal{L}(X^N, X^N)$, $B_i^N = p^N B_i$, $C_i^N =$ the restriction of C_i onto X^N, for $i = 1, 2$. The operator A^N can be extended to all of X by $A^N p^N x$.

Approximation Assumptions:

(A1) For each $x \in X$, we have

$$e^{A^N t} p^N x \to S(t)x,$$

$$(e^{A^N t})^* p^N x \to S^*(t)x,$$

uniformly in t on bounded intervals.

(A2) (i) The family of pairs (A^N, B_2^N) is uniformly exponentially stabilizable, *i.e.*, there exists a uniformly bounded sequence of operators $K^N \in \mathcal{L}(X^N, U)$ such that

$$\left| e^{(A^N - B_2^N K^N)t} p^N \, x \right|_X \leq M_1 \, e^{-\omega_1 t} \, |x|_X$$

for some positive constants $M_1 \geq 1$ and ω_1.

(ii) The family of pairs (A^N, C_1^N) is uniformly exponentially detectable, *i.e.*, there exists a uniformly bounded sequence of operators $F^N \in \mathcal{L}(Y, X^N)$ such that

$$\left| e^{(A^N - F^N C_1^N)t} p^N \, x \right|_X \leq M_2 \, e^{-\omega_2 t}, \quad t \geq 0,$$

for some positive constants $M_2 \geq 1$ and ω_2.

(iii) The family of pairs (A^N, B_1^N) is uniformly exponentially stabilizable (as in (A2i)).

(iv) The family of pairs (A^N, C_2^N) is uniformly exponentially detectable(as in (A2ii)).

(A3) (i) The disturbance operator B_1 is compact.

(ii) The input operator B_2 is compact.

(iii) The observation operator C_2 is compact.

Remarks:

1. Note that (A1) implies that $p^N x \to x$ for $x \in X$.
2. Assumptions (A2) are trivial for stable systems with uniformly stable approximations.
3. Since X^N is finite dimensional, Assumption (A3i) is equivalent to

$$\lim_{N \to \infty} |p^N B_1 - B_1| = 0.$$

   Similarly, if (A3ii) and (A3iii) hold, B_2^N and C_2^N converge in norm. Assumption (A3) follows trivially if the input spaces U and W and output space Z are finite-dimensional.

Assumptions (A1)-(A2) are similar to those required to show that the solutions to the Riccati equations arising in the approximation theory for linear quadratic problem converge *e.g.* [8]. Assumption (A3) is not required in the standard LQR problem for the existence of solutions to a family of approximating finite-dimensional Riccati equations. However, this assumption is required to ensure continuity of performance measure and to guarantee that the approximating controllers stabilize the infinite-dimensional system *e.g.* [7, 11, 12]. Assumption (A3) could be replaced by an assumption that the semigroup $S(t)$ is compact. Assumptions (A1) - (A3) are weaker than similar assumptions used in [16]. In particular, it is shown here, and not assumed, that if the original system is stabilizable with attenuation γ, then so are the approximating systems.

3 Main Results

We make the simplifying assumptions that $D_{22} = 0$, $D_{11} = 0$ (as written in equations (2-3)) and also that

$$D_{12}^* \left[\begin{array}{cc} C_1 & D_{12} \end{array} \right] = \left[\begin{array}{cc} 0 & I \end{array} \right], \qquad \left[\begin{array}{c} B_1 \\ D_{21} \end{array} \right] D_{21}^* = \left[\begin{array}{c} 0 \\ I \end{array} \right]. \tag{5}$$

We will assume throughout this paper that (A, B_1) and (A, B_2) are stabilizable and that (A, C_1) and (A, C_2) are detectable. These assumptions ensure that an internally stabilizing controller exists; and that internal and external stability are equivalent for the closed loop if the controller realization is stabilizable and detectable.

If the $\mathcal{H}^\infty$ disturbance-attenuation problem is solvable, then it can be solved by calculating the solutions to two Riccati equations [1, 9].

Definition 1 *The system (1.1,1.2) is stabilizable with attenuation γ if and only if there is a stabilizing controller with transfer function H so that inequality (4) is satisfied.*

Theorem 3.1 *[1, Thm. 4.1] The system is stabilizable with attenuation $\gamma > 0$ if and only if the following two conditions are satisfied:*

1. *There exists a nonnegative self-adjoint operator Σ on X satisfying the Riccati equation*

$$A^*\Sigma + \Sigma A + \Sigma \left(\frac{1}{\gamma^2} B_1 B_1^* - B_2 B_2^* \right) \Sigma + C_1^* C_1 = 0 \tag{6}$$

such that $A + (\frac{1}{\gamma^2} B_1 B_1^* - B_2 B_2^*)\Sigma$ generates an exponentially stable semigroup on X.

2. *Define $\tilde{A} = A + \frac{1}{\gamma^2} B_1 B_1^* \Sigma$ and $\hat{K} = B_2^* \Sigma$. There exists a nonnegative self-adjoint operator $\tilde{\Pi}$ on X satisfying the Riccati equation*

$$\tilde{A}\tilde{\Pi} + \tilde{\Pi}\tilde{A}^* + \tilde{\Pi} \left(\frac{1}{\gamma^2} \hat{K}^* \hat{K} - C_2^* C_2 \right) \tilde{\Pi} + B_1 B_1^* = 0 \tag{7}$$

such that $\tilde{A} + \tilde{\Pi} \left(\frac{1}{\gamma^2} \hat{K}^* \hat{K} - C_2^* C_2 \right)$ generates an exponentially stable semigroup on X.

Moreover, if both conditions are satisfied, define $F = \tilde{\Pi} C_2^*$ *and* $A_c = A + \frac{1}{\gamma^2} B_1 B_1^* \Sigma - B_2 \hat{K} - \hat{F} C_2$*. The controller with state-space description*

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + \hat{F} z(t) \\ u(t) &= -\hat{K} x_c(t) \end{aligned} \tag{8}$$

solves the $\mathcal{H}^\infty$ *disturbance-attenuation problem.*

The solution to an infinite-dimensional algebraic Riccati equation is taken to exist in the usual sense [4, pg.293]. (For instance, (6) holds for $x \in dom(A)$.)

Condition (2) above is more often written as the following two equivalent conditions:

a) There exists a nonnegative, self-adjoint operator Π on X satisfying the Riccati equation

$$A\Pi + \Pi A^* + \Pi \left(\frac{1}{\gamma^2} C_1^* C_1 - C_2^* C_2 \right) \Pi + B_1 B_1^* = 0 \tag{9}$$

such that $A + \Pi(\frac{1}{\gamma^2} C_1^* C_1 - C_2^* C_2)$ generates an exponentially stable semigroup on X, and

b) $r(\Pi\Sigma) < \gamma^2$.

In the presence of condition (1) in Theorem 3.1, condition (2) is equivalent to conditions (a) and (b). Also $\tilde{\Pi} = (I - \frac{1}{\gamma^2}\Pi\Sigma)^{-1}\Pi = \Pi(I - \frac{1}{\gamma^2}\Sigma\Pi)^{-1}$.

These equations can rarely be solved exactly. The Riccati equations corresponding to the finite-dimensional approximations can be solved to obtain a finite-dimensional controller. The important questions are as follows:

1. If the original system is stabilizable with attenuation γ, do the approximation systems also have this property?

2. Does the designed finite-dimensional controller stabilize the original plant?

3. If the finite-dimensional controller does stabilize the original plant, is the attenuation close to that obtained with the infinite-dimensional controller?

We will show that the answers to all these questions is yes. The main tool is the *gap topology e.g.* [15, Ch. 7]. The importance of the gap topology in controller design is due to the following important property of the gap topology: A sequence of plants G_n can be stabilized by a common controller H that stabilizes a plant G *if and and only if* G_n converges to G in the gap topology. Also, if $G_n \longrightarrow G$ in the gap topology, the closed loop responses converge uniformly. Details can be found in [12]. Use of the gap topology enables us to answer the questions above. Furthermore, the proofs are quite short.

Theorem 3.2 *Assume that (A1) to (A3) hold. Each approximating subsystem* G^n_{ij}*,* $i,j = 1,2$ *converges to* G_{ij} *in the gap topology.*

Proof: Consider first the sequence of approximations G^N_{11} to the subsystem (A, B_1, C_1, D_{11}). Assumptions (A1) and (A2iii) imply that there is a strongly convergent sequence K^N such that $e^{(A^N - B_1^N K^N)t}$ is uniformly exponentially stable [8, Thm 2.1]. This, together with assumption (A3) implies that the approximations converge in the gap topology to the original system [12, Thm. 4.2]. (The only difference between the framework here and that in [12] is that in [12] it is assumed that the input space is finite-dimensional, while here this assumption is replaced by compactness of the input operators B_1 and B_2. As mentioned in note 3 above, this is sufficient to guarantee the required uniform convergence of B_1^N and B_2^N.) Convergence of the other subsystems follows identically.□

Let H^N be any sequence of finite-dimensional approximations to the controller H (8) that converge in the gap topology to H. Since the approximating systems converge in the gap topology to the infinite-dimensional system, H^N will stabilize G and provide γ-attenuation for sufficiently large N. However, we cannot generally solve the infinite-dimensional Riccati equations and so this is not useful for design.

Theorem 3.3 *Assume that (A1) to (A3) hold. If the original system is stabilizable with attenuation* γ *then the approximating systems are stabilizable with attenuation* γ *for all* N *sufficiently large.*

Proof: Since G^N_{22} converges to G_{22} in the gap topology, the infinite-dimensional controller (8) with transfer function H stabilizes the approximating systems for sufficiently large N. The closed loop of the approximation with this controller has transfer function

$$\Delta(G_N, H) = G^N_{11} + G^N_{12} H\ (I - G^N_{22} H)^{-1} G^N_{21}.$$

This function converges in the gap topology to $\Delta(G, H)$. This implies that it converges in the $\mathcal{H}^\infty$-norm for N large enough that the closed loop system is stable. Hence, for large enough N, $\Delta(G_N, H)$ has norm less than γ. Thus, the approximating system is stabilizable with attenuation γ. □

The proof of this theorem showed that for large N, the infinite-dimensional controller (8) provides γ-attenuation for the approximating systems.(Note however that Theorem 3.1 or *e.g.* [10, Thm. 8.15] yields a finite-dimensional state-space based controller.) We now show the more useful converse, that for large N, the controllers for the approximating systems provide γ-attenuation for the infinite-dimensional system. Suppose that the algebraic Riccati equations (6,7) with A replaced by A^N etc. have solutions $\Sigma^N$and $\tilde{\Pi}^N$ respectively. Define $\hat{K}^N = (B_2^N)^* \Sigma^N$, $\hat{F}^N = \tilde{\Pi}^N (C_2^N)^*$ and $A_c^N = A^N + \frac{1}{\gamma^2} B_1^N (B_1^N)^* \Sigma^N - B_2^N \hat{K}^N - \hat{F}^N C_2^N$. The corresponding finite-dimensional state-space based controller is

$$\begin{aligned} \dot{x}_c(t) &= A_c^N x_c(t) + \hat{F}^N z(t) \\ u(t) &= -\hat{K}^N x_c(t)\,. \end{aligned} \tag{10}$$

In many cases, the structure of the generalized plant guarantees that the finite-dimensional controllers will provide γ-attenuation for the original plant for sufficiently large approximation order. For instance, consider the common mixed sensitivity problem of finding a controller that stabilizes a plant in the usual feedback configuration and reduces the norm

$$\left\| \begin{bmatrix} W_1(I+PH)^{-1} \\ W_2H(I+PH)^{-1} \end{bmatrix} \right\|_\infty . \tag{11}$$

Here H is the controller transfer function and P is the plant transfer function. The weights W_1 and W_2 are typically matrices with entries that are rational functions in $\mathcal{H}^\infty$. We can assume that W_1 and W_2 have minimal realizations $(A_{w1}, B_{w1}, C_{w1}, 0)$ and $(A_{w2}, B_{w2}C_{w2}, D_{w2})$ respectively where A_{w1} and A_{w2} generate exponentially stable semigroups. The plant has a realization $(A_p, B_p, C_p, 0)$ where A_p generates a strongly continuous semigroup over an infinite-dimensional Hilbert space and this semigroup must in general be approximated. The approximation of the plant leads to the sequence of approximating generalized plants. If A_p^N satisfies assumptions (A1), (A_p^N, B_p^N) are uniformly stabilizable and (A_p^N, C_p^N) are uniformly detectable then there are strongly convergent sequences $K_p^N \to K_P$ and $F_p^N \to F_P$ such that $e^{(A_p^N - B_p^N K_p^N)t}$ and $e^{(A_p^N - F_p^N C_p^N)t}$ are uniformly exponentially stable [8, Thm. 2.1]. This implies that the approximations P^N converge in the gap topology to P[12]. It follows that for large N, P is included in the additive uncertainty set $\mathcal{A}(P^N, W_2)$ and so P is stabilized by the finite-dimensional controller. The attenuation level also converges [12]. This approach is developed in detail in [2, 3].

However, we will show a stronger result: the finite-dimensional controllers converge in the gap topology to the infinite-dimensional controller. This means that the performance of the finite-dimensional controllers (10) with the approximating plants will converge to the performance of the infinite-dimensional controller (8) with the original plant. Most importantly, closed loop performance arbitrarily close to that with the infinite-dimensional controller can be obtained by choosing a controller of sufficiently high order. Actual response is predicted by the simulated response in all respects. The solution to the full-information problem ($C_2 = I$) from [7] is key.

Theorem 3.4 *[7, Theorem 2.5, Cor. 2.6] Assume that (A1),(A2i), (A2ii) and (A3i) and (A3ii) hold. Assume that the original problem is stabilizable with attenuation γ. For sufficiently large N the Riccati equation*

$$(A^N)^*\Sigma^N + \Sigma^N A^N + \Sigma^N \left(\frac{1}{\gamma^2} B_1^N (B_1^N)^* - B_2^N (B_2^N)^* \right) \Sigma^N + (C_1^N)^* C_1^N = 0, \tag{12}$$

has a nonnegative, self-adjoint solution Σ^N and $\Sigma^N p^N x \to \Sigma x$ strongly in X as $N \to \infty$. Moreover, $\hat{K}^N = (B_2^N)^*\Sigma^N$ converges to $\hat{K} = B_2^*\Sigma$ in norm. Also, for such N there exist positive constants M_3 and ω_3 such that

$$|e^{(A^N + \frac{1}{\gamma^2} B_1^N (B_1^N)^* \Sigma^N - B_2^N (B_2^N)^* \Sigma^N)t} p^N x| \le M_3 e^{-\omega_3 t} |x|_X . \tag{13}$$

At this point convergence of the solution Π^N to the Riccati equation (9) will follow from this theorem and a straightforward duality argument if assumptions (A1), (A2iii), (A2iv) and (A3iii) hold, along with compactness of C_1. However, since we only have strong convergence of $\Sigma^N \longrightarrow \Sigma$ and of $\Pi^N \longrightarrow \Pi$, convergence of the inverse operator $(I - \frac{1}{\gamma^2}\Pi^N\Sigma^N)^{-1}$is not implied and so we do not have controller convergence. Convergence of the solution $\tilde{\Pi}^N$ to the estimation Riccati equation (7) will be shown, and this will lead to controller convergence.

Theorem 3.5 *Assume that (A1)-(A3) hold, and that the infinite-dimensional problem is stabilizable with attenuation γ. Let N be large enough that the approximations are stabilizable with attenuation γ. Define $\hat{K}^N = B_2^{N*}\Sigma^N$ and*

$$\tilde{A}^N = A^N + \frac{1}{\gamma^2}B_1^N B_1^{N*}\Sigma^N .$$

For sufficiently large N the Riccati equation

$$\tilde{A}^N\tilde{\Pi}^N + \tilde{\Pi}^N\tilde{A}^{N*} + \tilde{\Pi}^N\left(\frac{1}{\gamma^2}(\hat{K}^N)^*\hat{K}^N - (C_2^N)^*C_2^N\right)\tilde{\Pi}^N + B_1^N(B_1^N)^* = 0 \tag{14}$$

has a nonnegative, self-adjoint solution $\tilde{\Pi}^N$ and $\tilde{\Pi}^N p^N x \to \tilde{\Pi}x$ strongly in X as $N \to \infty$. Moreover, $\hat{F}^N = \tilde{\Pi}^N(C_2^N)^*$converges to $\hat{F} = \tilde{\Pi}C_2^*$ in norm. Also, for such N there exist positive constants M_4 and ω_4 such that

$$|e^{(\tilde{A}^N + \frac{1}{\gamma^2}\tilde{\Pi}^N(\hat{K}^N)^*\hat{K}^N - \tilde{\Pi}^N(C_2^N)^*C_2^N)t}p^N x| \leq M_5 e^{-\omega_5 t}|x|_X . \tag{15}$$

Proof: For N large enough that the approximations are stabilizable, the finite-dimensional versions of the algebraic Riccati equations (6) and (7) have solutions Σ^N and $\tilde{\Pi}^N$ respectively.

Define

$$\tilde{C}_1 = \begin{bmatrix} B_1^*\Sigma \\ \hat{K} \end{bmatrix}, \qquad \tilde{C}_2 = \begin{bmatrix} D_{21}^*C_2 \\ \frac{1}{\gamma}B_1^*\Sigma \end{bmatrix},$$

and define similarly $\tilde{C}_1^N$ and $\tilde{C}_2^N$. Using (5), we can rewrite (7) as

$$\tilde{A}\tilde{\Pi} + \tilde{\Pi}\tilde{A}^* + \tilde{\Pi}\left(\frac{1}{\gamma^2}\tilde{C}_1^*\tilde{C}_1 - \tilde{C}_2^*\tilde{C}_2\right)\tilde{\Pi} + B_1^*B_1 = 0. \tag{16}$$

and equation (14) can be similarly rewritten. Now,

$$\tilde{A} - \begin{bmatrix} FD_{21} & \frac{1}{\gamma}B_1 \end{bmatrix}\tilde{C}_2 = A - FC_2,$$

and so by assumption (A2iv), $(\tilde{A}, \tilde{C}_2)$ is uniformly detectable. Also, by assumption (A2iii), $(\tilde{A}, B_1)$ is uniformly stabilizable. Assumption (A3) implies that $\tilde{C}_1$ and $\tilde{C}_2$ are compact, and this with the strong convergence of Σ^N to Σ (Thm. 3.4) implies that $\tilde{C}_1^N$ and $\tilde{C}_2^N$converge uniformly to $\tilde{C}_1$ and $\tilde{C}_2$ respectively. Apply Theorem 3.4 on the dual of the Riccati equation (16) to obtain that $\tilde{\Pi}^N p^N x$ converges strongly to $\tilde{\Pi}$. Uniform convergence of $\hat{F}^N$ to $\hat{F}$ and the exponential growth bound (15) follows.

Theorem 3.6 *Let γ be such that the infinite-dimensional problem is solvable. Assume that assumptions (A1)-(A3) hold. Then the finite-dimensional controllers (10) converge in the gap topology to the infinite-dimensional controller (8). For sufficiently large N, the finite-dimensional controllers (10) stabilize the infinite-dimensional system and provide γ attenuation.*

Proof: We have from Theorems 3.4 and 3.5 that $\hat{K}^N$ and $\hat{F}^N$ converge uniformly to $\hat{K}$ and $\hat{F}$ respectively. The operator A_c is a bounded perturbation of A and since $\frac{1}{\gamma^2}B_1^N B_1^{N*}\Sigma^N - B_2^N\hat{K}^N - \hat{F}^N C_2^N$ converges strongly to $\frac{1}{\gamma^2}B_1 B_1^*\Sigma - B_2\hat{K} - \hat{F}C_2$, $e^{A_c^N t}$ converges strongly to $e^{A_c t}$, uniformly on bounded intervals of time. From (13), $A_c^N + \hat{F}^N C_2^N$ is uniformly exponentially stable and so $(A_c^N, \hat{F}^N)$ is uniformly exponentially stabilizable (in the sense of assumption A2i). Recall that C_2^N converges uniformly to C_2. Also, $\hat{F}^N$ converges uniformly to $\hat{F}$ and $\hat{K}^N$ converges uniformly to $\hat{K}$. It follows that the controller sequence (10) converges to (8) in the gap topology [12, Thm. 4.2]. Thus, for sufficiently large N, these controllers stabilize the original system and provide γ-attenuation. □

The optimal attenuation problem for the infinite-dimensional system is to find

$$\hat{\gamma} = \inf \gamma$$

where the infinum is calculated over all γ such that the problem is stabilizable with attenuation γ. Let $\hat{\gamma}^N$ indicate the optimal attenuation for the corresponding approximating system.

Theorem 3.7 *Assume that (A1)-(A3) hold. Then*

$$\lim_{N \longrightarrow \infty} \hat{\gamma}^N = \hat{\gamma}.$$

Proof: Theorem 3.3 showed that

$$\limsup_{N \to \infty} \hat{\gamma}^N \leq \hat{\gamma}$$

so it only remains to show that

$$\liminf_{N \to \infty} \hat{\gamma}^N \geq \hat{\gamma}.$$

Suppose not. Then there is $\delta > 0$ such that for all N there is $M > N$ with $\hat{\gamma}^M < \hat{\gamma} - \delta$. We can construct a subsequence $\left\{\hat{\gamma}^M\right\}$ with $\hat{\gamma}^M < \hat{\gamma} - \delta$. We have a corresponding sequence of approximating systems, each of which is stabilizable with attenuation $\hat{\gamma} - \frac{\delta}{2}$.

This implies each full information problem is stabilizable with attenuation $\hat{\gamma} - \frac{\delta}{2}$ and so the infinite-dimensional full-information problem is stabilizable with attenuation $\hat{\gamma} - \frac{\delta}{2}$ [7, Thm. 2.8]. In other words, the Riccati equation (6) with $\gamma = \hat{\gamma} - \frac{\delta}{2}$ has a unique stabilizing solution Σ.

Now consider the corresponding sequence of estimation Riccati equations (14). Since the subsequence is stabilizable with attenuation $\hat{\gamma} - \frac{\delta}{2}$, these equations are solvable with

$\gamma = \hat{\gamma} - \frac{\delta}{2}$ (Thm. 3.1 or *e.g.*[10, Thm 8.15]). Each equation is the dual of the Riccati equation corresponding to the full-information problem with realization

$$\left[\begin{array}{c|cc} \tilde{A}^{N*} & \Sigma^N B_2^{N*} & C_2^{N*} \\ \hline B_1^{N*} & 0 & D_{21}^* \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right].$$

All assumptions of [7, Thm. 2.8] are satisfied and so arguing as before, the Riccati equation (7) with $\gamma = \hat{\gamma} - \frac{\delta}{2}$ has a unique stabilizing solution $\tilde{\Pi}$. Thus, by Theorem 3.1, the infinite-dimensional problem is solvable with attenuation $\hat{\gamma} - \frac{\delta}{2}$. This contradicts the optimality of $\hat{\gamma}$, completing the proof.□

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