Closed-Loop Position Control of Preisach Hystereses

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ABSTRACT

We discuss closed-loop control of hysteretic systems having Preisach representations. We apply dissipativity methods, the key component of which is the determination of an appropriate supply rate. The concept of cyclo-dissipation is used to identify a new supply rate for Preisach hystereses and dissipativity with respect to this supply rate is shown. The result is useful in the design of output feedback controllers, and complements a previous result using feedback of the output derivative. We consider the specific example of a shape memory alloy actuator and identify a class of controllers which provide stability of position feedback with current as the control variable. This demonstrates one way in which system dynamics that couple with hysteresis can be included in controller design.

Keywords: hysteresis, smart materials, position control, stability, Preisach, shape memory alloy, dissipativity

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1 Introduction

Hysteresis non-linearities are present in the dynamics of many physical systems, and notably in a class of materials known as “smart materials.” Smart materials such as shape memory alloys (SMA), piezoceramics, and magnetostrictive materials hold high promise for the design of a new generation of actuation systems. Although each has particular advantages, actuators made from these so-called smart materials are generally scalable, lightweight, smaller than more traditional alternatives, and have fewer moving parts. A key disadvantage of these actuators is the presence, to varying degrees in different materials, of a hysteresis non-linearity. This non-linearity can be complex in shape, introduces memory into the system, and is not generally sector-bounded. This behaviour, combined with uncertain identification of model parameters, can make design of reliable controllers for these systems difficult.

It is little surprise that piezoceramic actuators, which have the smallest hysteresis of any of these materials, are those found most often in commercial applications. While SMA have enjoyed reasonable success in “on-off” applications, SMA and other smart materials have yet to achieve broad acceptance as alternative options for controlled actuation. This is due at least in part to the complexity of their behaviour. In order to extract the full potential from closed-loop systems incorporating these materials, designers must take non-linear behaviour into account.

The prime consideration in the current work is proving the stability of the closed-loop control system. This stability is of the utmost importance in many proposed applications of smart actuators, especially in the medical and aerospace fields. A number of different researchers have worked with Preisach and related models, and have been successful in designing controllers for hysteretic systems with Preisach representations. Because of the non-linearities involved, however, few have considered the theoretical stability of the resulting control systems. We list first some of the common approaches to improving performance, then describes past work on closed-loop stability for Preisach models.

Under certain reasonable parameter assumptions, the Preisach model can be inverted in a relatively straightforward and computationally inexpensive manner. By implementing a model-based inverse as part of the closed-loop controller[14, e.g.] or as a feed-forward compensator[9, 23], the hysteresis can be (approximately) linearized. A linear feedback controller can be used to
reduce any residual error due to model uncertainty. While this approach has shown good results in the literature, no attempt has been made to demonstrate the stability of the resulting closed-loop system.

In [24], the authors introduce an adaptive inverse hysteresis model. Used to reduce the hysteresis and in conjunction with an adaptive feedback controller, the authors show good tracking performance for unknown hystereses. While the hysteresis inverse model appears quite general, stability results are available only for hystereses demonstrating a certain amount of symmetry. SMA actuators are one example of a system which would not satisfy the symmetry constraints.

A novel approach to hysteresis compensation is found in [5], where the authors model the hysteretic behaviour as a frequency-dependent delay. They introduce the concept of a “phaser,” a constant-phase, unity-gain linear controller. An approximation to this ideal phaser is used to cancel the identified system phase delay in the frequency band of interest, resulting in remarkable linearization for periodic inputs. If the range of operating frequencies is large and the delay varies in this region, multiple phasers can be combined to (mostly) linearize the hysteresis. This method works quite well, especially for piezoceramics, which have a relatively well-behaved hysteresis curve. Unfortunately, the method is only suited for periodic signals. Stability is not investigated.

In this work, we consider hysteresis non-linearities which can be represented by the Preisach hysteresis model. Few researchers have investigated the stability of closed-loop control for smart materials using this general hysteresis model. Closed-loop stability has been shown in the past for feedback using the derivative of the output[12, 21]. Other researchers have addressed the stability question for specific material models, for example, the one-dimensional energy-based model for SMA actuators in [20]. The focus of this paper being the Preisach model, the reader is referred to [2, 7, 20] for examples of non-Preisach work.

Dissipativity theory, which is a generalization of the well-known passivity theory, is applied. Mathematically, dissipativity is a property of the general model structure and is independent of any parameter errors or assumptions about linearity. Thus, controllers designed using these energy-based methods enjoy a large degree of robustness.

We provide two general results for use in the design of stabilizing controllers: the first for feedback of the output derivative, and the second for output position feedback. We then apply the second result to an SMA actuator, as
an example of how coupled dynamics (in this case, the heating dynamics) can be accounted for in the controller design. This is the first work to our knowledge to demonstrate closed-loop stability for position feedback.

The Preisach hysteresis model is adopted for its generality. While the model was originally developed to represent magnetic hysteresis, experimental evidence indicates that it also describes other hysteretic dynamics, such as phase transition problems [11, 14, e.g.]. As a result, the main results are useful for a broad class of systems.

Section 2 of the paper discusses dissipative systems, and dissipation in hysteretic systems in particular. We use the concept of cyclo-dissipation to postulate a new supply rate for hysteretic systems. In Section 3, we give a brief description of the Preisach model of hysteresis. Section 4 demonstrates dissipativity of the Preisach model with respect to the supply rates $u\dot{y}$ and $\dot{u}y$. The two are not equivalent, and the connection is non-trivial. In Section 5, the dissipativity result is applied to define a class of stabilizing controllers for position control of an SMA actuator, using position feedback. Section 6 presents experimental results showing robust position control of an SMA actuator.

## 2 Dissipative Systems

In this section, we recall definitions of passivity and dissipativity. We then give a formal definition of a dissipative system that is more general than earlier definitions. We define cyclo-dissipativity and show its connection to dissipative systems. Cyclo-dissipativity is used to motivate appropriate supply rates for hysteretic systems.

We first present a formal definition of a dynamical system. If $u(t)$ is a function defined on the real line,

$$u_T(t) := \begin{cases} u(t) & t < T, \\ 0 & t \geq T. \end{cases}$$

**Definition 2.1** [25] We define a dynamical system $\Sigma$ through sets $U$, $U$, and $X$ and a map $\phi: \mathbb{R}^2 \times X \times U \mapsto X$. The following axioms are satisfied:

1. The input space, $U$, consists of a class of $U$-valued functions on the real line, for which $u_T(t) \in L_2(t_0, \infty; U)$ for all $t_0 \in \mathbb{R}, T > t_0$. 

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2. Define \( \mathbb{R}_2^+ := \{(t_2, t_1) \in \mathbb{R} \times \mathbb{R}; t_2 \geq t_1 \} \). The state transition function \( \phi : \mathbb{R}_2^+ \times X \times U \mapsto X \) and defines the state \( x \in X \) through the relation \( x(t_2) = \phi(t_2, t_1, x(t_1), u) \). This function satisfies the usual axioms for autonomous dynamical systems: For all \( t_0, t_1 \in \mathbb{R}, x_0 \in X \) and \( u, u_1, u_2 \in U \),

(a) Consistency of initial condition: \( \phi(t, t, x_0, u) = x_0 \),

(b) Semigroup property: \( \phi(t_2, t_1, \phi(t_1, t_0, x_0, u), u) = \phi(t_2, t_0, x_0, u) \),

(c) Causality: \( u_1(t) = u_2(t) \) for \( t_0 < t < t_1 \) implies \( \phi(t, t_0, x_0, u_1) = \phi(t, t_0, x_0, u_2) \) for \( t_0 < t < t_1 \),

(d) Time Invariance: \( \phi(t_1 + T, t_2 + T, x_0, u_2) = \phi(t_1, t_2, x_0, u_1) \) for all \( T \geq 0, t_2 \geq t_1, u_1 \in U, u_2(t) = u_1(t + T) \).

For a given dynamical system, we will say that \((x, u)\) is a path on \([t_0, t_1]\) if there is \( x_0 \in X \) such that \( x(t) = \phi(t, t_0, x_0, u) \), \( t_0 \leq t \leq t_1 \). The set \( X \) is called the state space, and \( U \) the input space. An important feature of a dynamical system representation is that the state must account for all memory in the system. For most systems, the state cannot be measured. We define an measured output \( y(t) \) in a set \( Y \) through the operator \( r : X \times U \rightarrow Y \):

\[
y(t) = r(x(t), u(t)).
\]

Note the output at time \( t \) depends only on the values of the state \( x \) and input \( u \) at time \( t \).

**Passivity**

Passivity theory is a set of energy-based definitions and theorems which can be useful in analyzing the stability of interconnected systems. Consider the non-linear time-varying scalar function

\[
y = f(t, u).
\]

This system is said to be passive if there exist non-negative constants \( \delta \) and \( \varepsilon \) such that

\[
\int_0^T u(t)y(t)dt \geq \varepsilon \|u\|_2 + \delta \|y\|_2
\]

for all inputs \( u \) defined on \([0, T]\). System (2.1) is said to be strictly input passive if \( \varepsilon > 0 \) and strictly output passive if \( \delta > 0 \).
A nice physical interpretation of equation (2.2) exists when the units of the product $u \, y$ are power. In this case, $u \, y$ is the energy supply rate, and the integral in (2.2) is the total energy supplied over $[0, T]$. Equation (2.2) says that the energy added to a passive system over $[0, T]$ must no greater than the energy supplied. In other words, a passive system cannot generate energy internally. System passivity is a very useful concept in feedback systems, as the passivity characteristics of interconnected systems can be used to show stability of the closed loop system.

Most dynamical systems, such as a capacitor, have the ability to store energy for later recovery. Suppose that such a system were allowed to have stored energy at time $t = 0$, which was then recovered over $[0, T]$. This would result in a negative net energy over $[0, T]$, and a violation of equation (2.2). One way to modify the definitions above is to qualify the definition (and any results) with the assumption that the system starts in a state of minimum stored energy: a “relaxed” state. This assumption is generally made implicitly. For instance, for many systems the zero state is a point of minimum energy, and it is typically assumed that all systems start with zero initial state.

An extension of this concept to allow for energy storage is the basis of dissipativity theory.

Dissipativity

Definition 2.2 [25] A dynamical system is dissipative if there exists a non-negative function $V : X \mapsto \mathbb{R}^+$ and a function $w : U \times Y \mapsto \mathbb{R}$ such that for all $t_1 \geq t_0$,

$$V(x_0) + \int_{t_0}^{t_1} w(u, y)dt \geq V(x_1)$$

(2.3)

where $u \in U$, $x_1 = \phi(t_1, t_0, x_0, u)$ and $y = r(x, u)$.

The function $V$ is generally called the storage function. If $V$ is identified with energy stored in the system, and $w$ identified with the power supplied; then dissipative systems are identified as those for which the stored energy is less than or equal to the initial energy plus the work done on the system. The supply rate and storage function are, in general, not unique. (In fact, the storage function is unique if and only if the system is conservative i.e. equality holds in the dissipation inequality [25].) Some possible abstract
storage functions are discussed in [25]. One common definition of supply rate is

\[ w(u, y) = \langle y, Qy \rangle + \langle y, Ru \rangle + \langle u, Su \rangle, \quad (2.4) \]

where \( Q, R, S \) are real matrices of appropriate dimensions. This is the basis of the definition of dissipativity in [19].

Given the supply rate in (2.4) and the trivial storage function \( V(x) = 0 \), passivity is the special case where \( Q = S = 0 \) and \( R = I \). Strict input passivity is obtained by defining \( S = -\varepsilon^2 I \) and \( R = I \).

The definitions of passivity and dissipativity considered thus far have an intuitive interpretation when the units of \( uy \) are power. However, the associated definitions and results hold, even when the “supply rate” and “storage function” have no actual physical meaning. Still, the determination of these functions is key to demonstrating dissipativity for a given system, and it is useful to be able to look to energy as a guide.

Many systems which are physically dissipative have more general energy supply rates than that in Defn. 2.2. In many applications the supply rate \( w \) is of the form \( \sigma(x, u)(\dot{x}) \) where \( \sigma(x, u) \) is a linear functional on the state-space. For example, in mechanical systems we are often concerned with controlling the position \( y \) of an object by applying an input force \( u \). The instantaneous power in this case is given by \( u \dot{y} \), rather than \( uy \). Similarly, in the classical example of linear elasticity, setting \( x \) to be strain and \( \sigma \) to be stress, the system is generally dissipative with \( w = \sigma \dot{x} \). In many cases, the measured quantity is not velocity \( \dot{x} \) and so the system is not dissipative in the classical sense. It is therefore useful to introduce more general definitions.

**Definition 2.3** Let \( w : X \times X \times U \to R \). Define for an arbitrary interval \([t_0, t_1] \) and state \( x_0 \in X \) the work function \( W \)

\[ W_{[t_0, t_1]}(x_0, u) = \int_{t_0}^{t_1} w(x_0, x(t), u(t))dt \]

where \( x(t) = \phi(t, t_0, x_0, u) \). The domain of \( W \) is all \((x_0, u) \in X \times U \) such that \( w(x_0, x(\cdot), u(\cdot)) \) is well-defined and in \( L_1(t_0, t_1) \). We say that \( u \) is admissible on \([t_0, t_1] \) for a given \( x_0 \) if \((x_0, u) \) is in the domain of the work function.

Note that the output of a system is not defined or used here. The following definition of dissipativity is identical to Definition 2.2 except that we allow
work functions other than
\[ \int_{t_0}^{t_1} w(u, y) dt. \]

**Definition 2.4** A dynamical system \( \Sigma \) is dissipative with supply rate \( w \) if there exists a non-negative functional \( V : X \mapsto \mathbb{R}^+ \), such that for all \( t_0, t_1 \in \mathbb{R} \), \( x_0 \in X \) and all admissible \( u \), the dissipation inequality
\[ V(x_0) + \int_{t_0}^{t_1} w(x_0, x(\tau), u(\tau)) d\tau \geq V(x_1) \quad (2.5) \]
is satisfied, where \( x_1 = \phi(t_1, t_0, x_0, u) \).

The following example is a system that clearly dissipates energy. However, it is not passive, nor dissipative in the usual sense. It is dissipative in the more general sense defined above.

**Example 2.1** [17, Theorem 4.1] Consider the second-order system
\[ M \ddot{z}(t) + D \dot{z}(t) + Kz(t) = Fu(t) \]
with position sensors
\[ y(t) = C_d z(t) \]
where we assume there exists \( Q \) such that \( C_d'Q = F \). Let \( x = (z, \dot{z}) \) be the state and assume that the state-space is reachable from 0. Choose any \( c \) so \( c^2 M < D \) and \( p \) so that \( p^2 C_d' C_d < c^2 K \). The system is dissipative with supply rate
\[ -p^2 \langle y, y \rangle + c^2 \langle y, Qu \rangle + \langle \dot{y}, Qu \rangle. \]

If two systems which both dissipate energy are connected in feedback, it does not necessarily follow that the closed-loop system is stable. This is illustrated by the following simple example.

**Example 2.2**
\[ \ddot{z}_1(t) + 12\dot{z}_1(t) + 100z_1(t) = u_1(t) \quad \ddot{z}_2(t) + .001\dot{z}_2(t) + .1z_2(t) = u_2(t) \]
\[ u_1 = r_1 - z_2 \quad u_2 = r_2 + z_1 \]
Each system can be considered as a model of damped linear spring, and each system dissipates energy. However, the interconnected system is unstable. The connection of two dissipative systems affects the stability of the closed loop.

Dissipativity theory is important in controller design because dissipation of subsystems can be used to show stability of the closed loop. There are two main approaches: (1) show that a closed loop storage function composed of storage functions for the subsystems is actually a Lyapunov function e.g. [26] or (2) combine the supply rates for the subsystems and show input/output stability for the closed loop e.g. [19]. The passivity theorem, small gain theorem and other sector-based stability results are all special cases of the second approach. Both techniques are demonstrated in [17].

**Cyclo-dissipativity**

As mentioned above, the choice of storage function can be guided by the energy within the system, or one of the more abstract choices may be made. Choice of supply rate can be more difficult. The following concept of cyclo-dissipativity has been found to be useful in that regard.

Many physical systems have the property that positive net work must be done on the system in order to bring it through a cycle from a given state back to the same state. This motivates the following definition, similar to [18, Defn. 8].

**Definition 2.5** A dynamical system $\Sigma$ is cyclo-dissipative with supply rate $w$ if

$$\int_{t_0}^{t_1} w(x_0, x, u) d\tau \geq 0$$  \hspace{1cm} (2.6)

whenever $x(t_1) = x(t_0) = x_0$.

Notice that every dissipative system is cyclo-dissipative; the converse is false. However, cyclo-dissipative systems do have several important connections with dissipative systems. As in [26], we will say that a system is \textit{partially dissipative} if there is a function $V : X \mapsto \mathbb{R}$, not necessarily non-negative, such that the dissipation inequality (2.5) is satisfied.

Many cyclo-dissipative systems are at least partially dissipative [26, Theorem IV.7]. Partial dissipativity can be used in closed loop stability/instability theorems in a manner identical to that for connections of dissipative systems. However, one of the most useful aspects of the notion of cyclo-dissipativity
Figure 1: **Hysteresis Loop**

is that it provides a guide to an appropriate supply rate for a system. This approach is very useful in the context of hysteretic systems.

Many hysteretic systems have a dynamic behaviour which is characterised by nested loops and branching in the input-output plane. Figure 1 illustrates one such loop.

Suppose that for any path on $[t_0, t_2]$ where $(u(t_0), y(t_0)) = (u(t_2), y(t_2))$ the loop $L$ is counter-clockwise, as shown in Figure 1. Consider first, without loss of generality, the case where $y$ increases from $y_-$ at time $t_0$ to $y_+$ at some intermediate time $t_1$, and then decreases to $y_-$ again at time $t_2$. Following a similar analysis in terms of energy in [4], denote the value of the input on the ascending loop by $l_a(y)$ and on the descending loop by $l_d(y)$. Since the loop is counter-clockwise, $l_a(y) > l_d(y)$, $y_- < y < y_+$. The area enclosed by any loop $L$ is

$$0 \leq \int_{y_-}^{y_+} l_a(y) - l_d(y) \, dy$$  \hspace{1cm} (2.7)
= \int_{t_0}^{t_1} u(t)\dot{y}dt

We obtain an identical inequality if \( y \) is first decreased to \( y_- \) and then increased to \( y_+ \). Comparing equations (2.6) and (2.7), the system appears to be cyclo-dissipative with supply rate \( u(t)\dot{y}(t) \).

Now, suppose that instead the system output is defined as

\[ e = y_+ - y. \]

A simple sketch shows that the loops are now clockwise. Denote the value of \( y \) as \( u \) increases from \( u_- \) to \( u_+ \) by \( l_u(u) \), and by \( l_d(u) \) as \( u \) decreases from \( u_+ \) to \( u_- \). We obtain, in a manner similar to that above, the inequality

\[ 0 \leq \int_{\mathcal{L}} \dot{u}(t)e(t)dt \]

over any loop \( \mathcal{L} \).

We have thus identified two potential supply rates for hysteretic systems: \( u\dot{y} \) and \( uce \). In order to show dissipativity with respect to these supply rates, we need to know more about the internal dynamics of the system. In other words, we need a model. We consider here the Preisach model because of its generality and ability to reproduce minor loops.

3 Preisach Model of Hysteresis

In this work, we use the Preisach model, which is appealing for its general structure. It was first developed to describe the hysteresis in magnetic materials. However, several experimental results suggest that it describes the behaviour in many other hysteretic materials, for instance SMA's e.g. [11, 14]. The chief assumption is that the hysteresis is rate-independent. Rate independence means that the curves \((u, y) \in \mathbb{R}^2\) are invariant for changes in the input rate, such as changes in frequency.

A very brief description of the Preisach model is given below. A more complete description can be found in [12]. A detailed study of Preisach models can be found in [4, 16]. The basic building block of the Preisach model is the hysteresis relay \( \gamma \). A relay is characterized by its half-width \( r > 0 \) and the input offset \( s \), and is denoted \( \gamma_{r,s} \). The behaviour of the relay is described schematically in Figure 2. The structure of the model is
illustrated in Figure 3. The output is computed as the weighted sum of relay outputs; the value \( \mu(r, s) \) represents the weighting of the relay \( \gamma_{r,s} \). The relay output, and hence the Preisach model, is only defined for continuous inputs \( u \). As this input varies with time, each individual relay adjusts its output according to the current input value, and the weighted sum of all relay outputs provides the overall system output

\[
y(t) = \int_{0}^{\infty} \int_{-\infty}^{\infty} \mu(r, s) \gamma_{r,s}[u](t) dsdr. \tag{3.1}
\]

The half-plane \( \mathbb{R}_+ \times \mathbb{R} \) is often referred to as the \textit{Preisach plane}, \( \mathcal{P} \). The collection of weights \( \mu(r, s) \) forms the \textit{Preisach weighting function} \( \mu : \mathcal{P} \mapsto \mathbb{R} \). This weighting function is experimentally determined for a given system; see
for two different approaches to the identification problem. In any physical setup, there are limitations which can be interpreted as a restriction on the support of \( \mu \). For instance, control input saturation, say at \( \hat{u} \), means that some relays in \( \mathcal{P} \) can never be exercised and cannot contribute to a change in output. This effectively restricts the domain of non-zero \( \mu \) to a triangle \( \mathcal{P}_r = \{(r,s) \in \mathcal{P} | |s| \leq \hat{u} - r\} \). The fact that relays outside \( \mathcal{P}_r \) do not contribute to output is modelled by setting \( \mu = 0 \) outside \( \mathcal{P}_r \).

For the remainder of this work, we will assume that the weighting function \( \mu \) belongs to the set \( \mathcal{M}_p \), defined as follows.

**Definition 3.1** \( \mathcal{M}_p \) is the set of all weighting functions \( \mu : \mathcal{P} \rightarrow \mathbb{R} \) which are bounded, piece-wise continuous, and non-negative.

The first two properties are true for physical systems having continuous, bounded outputs. The non-negativity requirement will limit the class of Preisach models for which dissipativity can be shown, but this limit is not overly restrictive. For example, models satisfying \( \mu \in \mathcal{M}_p \) have been reported for SMA actuators [11] and for piezoceramics [14].

In particular, it is easy to show that \( \mu \in \mathcal{M}_p \) means the maximum output value \( \hat{y} \) is finite and can be written

\[
\hat{y} = \int_{0}^{\infty} \int_{-\infty}^{\infty} \mu(r,s)dsdr,
\]

the value of output when all relays are “turned on.”

The Preisach plane can be used to track individual relay states by observing the evolution of the *Preisach plane boundary*, \( \psi \), in \( \mathcal{P}_r \). Figure 4 illustrates the Preisach plane concepts which follow.

First, the relays are divided into two time-varying sets \( \mathcal{P}_- \) and \( \mathcal{P}_+ \) defined as follows:

\[
\mathcal{P}_\pm(t) = \{(r,s) \in \mathcal{P}_r | \text{output of } \gamma_{r,s} \text{ at } t \text{ is } \pm 1\}.
\]

Each set is connected; \( \psi \) is the line separating \( \mathcal{P}_+ \) from \( \mathcal{P}_- \), with \( \mathcal{P}_+ \) below \( \mathcal{P}_- \). When an arbitrary input is applied, monotonically increasing input segments generate boundary segments of slope -1, while monotonically decreasing input segments generate boundary segments of slope +1. Input reversals cause corners in the boundary. The boundary is piecewise linear and Lipschitz continuous with Lipschitz constant 1.
The boundary $\psi$ always intersects the axis $r = 0$ at the current input value. With $\psi$ written as a map $\mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$, then $\psi(t, 0) = u(t)$. If the boundary at time $t$ is $\psi(t, r)$, applying an input for which $u(t) \neq \psi(t, 0)$ amounts to applying an input with a discontinuity at $t$. Such an input would not be admissible.

Defining the constant
\[ y_o = \int_0^\infty \int_{-\infty}^0 \mu(r, s) ds dr - \int_0^\infty \int_0^\infty \mu(r, s) ds dr, \]
then since $P_+$ is below $\psi$ and $P_-$ above, the output of a classical Preisach operator (3.1) with boundary $\psi$ can be rewritten
\begin{equation}
\label{eq:3.4}
y(t) = \int_0^\infty 2 \int_0^{\psi(t,r)} \mu(r, s) ds dr + y_o.
\end{equation}

The history of past input reversals—and hence of hysteresis branching behaviour—is stored in the corners of the boundary[10]. In other words, the Preisach plane boundary is the memory of the Preisach model. This suggests that it is an appropriate choice of state.

**Theorem 3.1** [10] The Preisach model described above forms a dynamical system with input space
\[ U = \{ u \in C^0(-\infty, \infty) \mid \|u\|_\infty \leq \hat{u} \text{ and } \lim_{t \to -\infty} u(t) = 0 \} \]
and state-space $X$ the subset of Lipschitz continuous functions $\psi \in C[0, \hat{u}]$ with $\psi(\hat{u}) = 0$. 

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Henceforth, a dynamical system having state space representation described above and with output behaviour described by (3.1) will be called a “Preisach system”.

4 Dissipation in the Preisach Model

In Section 2, we used the looping behaviour of hysteretic materials to define two potential supply rates. Here we show that Preisach systems are dissipative with respect to each of these supply rates. The proofs of this require defining a storage function for each supply rate, and then showing that inequality (2.5) holds. The first result of this section has been demonstrated previously, and is recalled here for completeness.

**Theorem 4.1** [12] Hysteretic systems admitting a Preisach model for which \( \mu \in \mathcal{M}_p \) are dissipative with respect to the supply rate \( u \dot{y} \).

One storage function satisfying the dissipation inequality (2.5) with this supply rate can be found in [12]. In that work, the storage function is derived from an analysis of energy loss in individual relays. Another, purely mathematical storage function for the same supply rate can be found in [4, Prop. 2.5.4], in the context of a general analysis of Preisach operators.

Graphical analysis in section 2 suggested that if \( e \) is taken as the output, we also have dissipation with respect to the supply rate \( \dot{ue} \). In order to prove that this is true, we first define a storage function candidate \( V \), and then show that the supply rate \( \dot{ue} \) and \( V \) satisfy the dissipation inequality (2.5).

**Lemma 4.1** Define

\[
V(\psi) = \int_{0}^{\infty} 2Q(r)dr, \quad (4.1)
\]

where

\[
Q(r) = \int_{-\infty}^{\psi} \int_{s}^{\infty} \mu(r, \xi) d\xi ds
\]

\[
+ \int_{\psi}^{\infty} \mu(r, s) ds [\psi(t, 0) - \psi(t, r) + r]
\]

\[
+r \int_{-\infty}^{\psi} \mu(r, s) ds.
\]
The function $V : X \mapsto \mathbb{R}^+$, where $X$ is the state space defined in Theorem 3.1, is finite and non-negative and is therefore a valid storage function candidate.

**Proof.** Since $\mu \in \mathcal{M}_p$, $\mu(r, s) = 0$ for all $|s| > \tilde{u}$ or $r > \tilde{u}$. Therefore, $Q(r)$ is finite for all $r \in [0, \tilde{u}]$ and zero for all $r > \tilde{u}$. As a result, $V$ is finite for all states $\psi$.

To show that $V$ is non-negative we will show that $Q$ is non-negative. Since $\mu \geq 0$ and $r \geq 0$, the only term in $Q(r, s)$ that is not obviously non-negative is the second term:

$$\int_{\psi}^{\infty} \mu(r, s) ds [\psi(t, 0) - \psi(t, r) + r].$$

However, for all $t$, $\psi$ is a Lipschitz function of $r$, so $|\psi(t, 0) - \psi(t, r)| \leq r$ and $Q$ (and therefore $V$) is non-negative.

**Lemma 4.2** For each $t \geq 0$,

$$\dot{\psi}(t, r) (u(t) - \psi(t, r)) \geq 0$$

for all $r$ for which $\dot{\psi}(t, r)$ is defined.

**Proof.** This result follows from the wiping-out property of the Preisach model [16]. Choose any time $t$. For every $r > 0$,

$$|u(t) - \psi(t, r)| \leq r.$$

If $|u(t) - \psi(t, r)| < r$, then by the wiping-out property $\dot{\psi}(t, r) = 0$. If $u(t) - \psi(t, r) = r$, then we necessarily have that $u$ is increasing and so $\dot{u}(t) = \dot{\psi}(t, r) > 0$ if this derivative is defined. Similarly, if $u(t) - \psi(t, r) = -r$, then $\dot{u}(t) = \dot{\psi}(t, r) < 0$.

**Theorem 4.2** Hysteretic systems admitting a Preisach model for which $\mu \in \mathcal{M}_p$ are dissipative with respect to the supply rate $\dot{u}e$.

**Proof.** In order to show that the dissipation inequality (2.5) holds, we will show that

$$\dot{u}(t)e(t) \geq \frac{d}{dt} V(\psi(t))$$

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at all times \( t \) where the storage function \( V \) is defined in (4.1).

Combining equations (3.2) and (3.4) gives
\[
e(t) = \dot{y} - y(t) = \int_0^\infty 2 \int_{\psi(t,r)}^{\infty} \mu(r, s) ds dr.
\]  

(4.2)

We rewrite
\[
V(\psi) = \int_0^\infty 2 \left[ \int_0^\psi \int_s^{\infty} \mu(r, \xi) d\xi ds \\
+ \int_0^\psi \mu(r, s) ds [\psi(t, 0) - \psi(t, r)] \right] dr + c
\]

where the constant
\[
c = \int_0^\infty 2 \left[ \int_{-\infty}^0 \int_s^{\infty} \mu(r, \xi) d\xi ds + r \int_{-\infty}^{\infty} \mu(r, s) ds \right] dr.
\]

We define
\[
\varepsilon(t, r) = \psi(t, 0) - \psi(t, r) = u(t) - \psi(t, r).
\]

Then,
\[
\dot{u}(t)e(t) - \frac{d}{dt} V(\psi) = \int_0^\infty 2 \left[ \dot{u}(t) \int_{\psi(t,r)}^{\infty} \mu(r, s) ds - \dot{\psi}(t, r) \int_{\psi(t,r)}^{\infty} \mu(r, \xi) d\xi \\
+ \mu(r, \psi) \dot{\psi}(t, r) \varepsilon(t, r) - \dot{\varepsilon}(t, r) \int_{\psi(t,r)}^{\infty} \mu(r, s) ds \right] dr
\]
\[
= \int_0^\infty 2 \left[ \mu(r, \psi) \dot{\psi}(t, r) \varepsilon(t, r) \right] dr.
\]

Since \( \mu \) is non-negative by assumption and from Lemma 4.2, \( \dot{\psi}(t, r) \varepsilon(t, r) \geq 0 \) for all \( r \),
\[
\dot{u}(t)e(t) - \frac{d}{dt} V(\psi) \geq 0
\]
and inequality (2.5) holds.  

Dissipativity of a dynamical system with respect to a particular supply rate may be used to design controllers for the system. Theorem 4.1 is useful in systems with output-derivative feedback. Such strategies have been applied in the past for beam vibration control using piezoceramic actuators [8, e.g.]. However, for many applications, the output position is the measured variable and that which we desire to control. In addition, it is sometimes easier to control $\dot{u}$ rather than $u$ itself; for example, it is easier to control thermal rate (proportional to power input) than it is to control temperature. In the next section we use the dissipativity result of Theorem 4.2 to obtain robust position control of an SMA actuator.

5 SMA Actuators

In SMA, hysteresis occurs as a result of temperature- and stress-induced phase transitions. When a shape memory alloy undergoes a temperature-induced phase transformation, its deformation characteristics are altered. This change is significant enough that SMA wire can be used as an actuator by cycling the temperature through the transformation range of the alloy. In the low-temperature phase, a mechanically loaded wire will stretch significantly; when it is heated to the high-temperature phase, it will contract, exerting a force on the load and recovering up to 6% strain with no permanent deformation. SMA actuators have been employed successfully in robotics research applications e.g. [13], and are beginning to be used in more commercial applications such as valves and dampers e.g. [27].

In [11] it was demonstrated that a Preisach model with $\mu \in \mathcal{M}_p$ yields a good fit to experimental data. Thus, we may conclude that SMA actuators are dissipative with respect to both supply rates discussed in the previous section. In [12] dissipation with respect to rate feedback was shown. However, it is difficult to control temperature and also to measure velocity of the wire. Furthermore, we are interested in controlling wire position, not velocity.

Theorem 4.2 could be used directly to obtain stability of a control system where rate of temperature change in the wire is controlled and the measurement is position. However, this ignores heat losses in the wire and also assumes that we control power. In practice, the closed loop control system is more accurately modelled as shown in Figure 5. A reference position $r_1$ is fed into the closed-loop system, and a controller $H$ reacts to the error between measured position $y_3$ and the reference. The controller attempts to
control the contraction of the SMA wire, and thus the actuator position, by modifying the temperature. This is done by controlling the current in the wire, \( i \). Noise on the control signal is modelled by the exogenous input \( r_2 \).

The two centre blocks describe the relationship between the control variable, current \( i \), and heat flow, \( \dot{T} \). The first of these two blocks models the conversion from current to power, and includes overcurrent protection at \( i_{\text{max}} \) as well as negative current cutoff to model the lack of active wire cooling. The model for this block is

\[
P(t) = \begin{cases} \frac{R i_{\text{max}}^2}{R} & i > i_{\text{max}} \\ \frac{R i^2(t)}{2} & 0 < i \leq i_{\text{max}} \\ 0 & i \leq 0 \end{cases}
\]  

(5.1)

The second block in the mapping \( i \mapsto \dot{T} \) is a first-order model for the relationship between heating and power. The temperature of the wire above ambient, \( T \), is related to the power by

\[
c_1 \dot{T}(t) = -c_2 T(t) + P(t)
\]

\[
y_2 = -\frac{c_2}{c_1} T(t) + \frac{1}{c_1} P(t).
\]  

(5.2)

The constants \( c_1 \) and \( c_2 \) are lumped physical parameters (\( c_i > 0 \)).

We now apply the dissipativity result of Theorem 4.2 to derive a robust stability result for the SMA actuator position control system shown in Figure 5.

**Lemma 5.1** The heating dynamics with input current \( u_2 = i \) and output \( y_2 = \dot{T} \) has finite gain and so is dissipative with supply rate

\[
w_2(y_2, u_2) = -\langle y_2, y_2 \rangle + k \langle u_2, u_2 \rangle
\]  

(5.3)

where \( k \) is the overall \( L_2 \)-gain.
Proof. First consider the model of current $i$ to power $P$ given in (5.1). As a result of the saturation, the model has finite gain $Ri_{\text{max}}$. The heating equation (5.2) with input $P$, output $y_2$ and state $T$ also has finite gain. The series connection from $u_2 = i$ to $y_2 = \dot{T}$ therefore has finite gain, and the heating dynamics are dissipative with supply rate (5.3).

The model (5.2) ignores the effect of latent heats of phase transformation. These latent heats affect heating in the following way: heating is slowed during the phase transformation from martensite to austenite, and cooling is slowed during the inverse transformation[3]. Also, equation (5.1) assumes a constant electrical resistance when in fact, the resistance of an SMA varies hysteretically with temperature. However, the stability proof which follows depends only on the fact that each of these equations has finite gain, and so including these effects would not have any effect on the stability result.

A large class of controllers will now be shown to stabilize the system. As a corollary, we will show that practical PID controllers can be used for position control of this system.

Theorem 5.2 The current-controlled SMA shown in Figure 5 is stable if

1. the heating dynamics with input current and output rate of temperature change have finite gain $k_2$,

   $$w_2(y_2, u_2) = -\langle y_2, y_2 \rangle + k_2 \langle u_2, u_2 \rangle$$

2. the controller is externally stable, strictly passive:

   $$\langle y_1, u_1 \rangle \geq \delta_1 \langle u_1, u_1 \rangle$$

   with $L_2$-gain $k_1$ of the controller satisfying $k_1 < \delta_1^2$.

Proof. The system can be considered to consist of three blocks, as shown in Figure 5: the controller, the heating dynamics, and the SMA wire hysteresis. The dissipativity of each of these blocks will be determined, and the result combined to show overall closed-loop stability. Refer to the block diagram of Figure 5 for notation.

Controller: The controller $H$ is assumed to be strictly passive with finite gain. This means that it is dissipative with supply rate

$$w_1(y_1, u_1) = -\alpha_1 \langle y_1, y_1 \rangle + (k_1 \alpha_1 - \delta_1) \langle u_1, u_1 \rangle + \langle y_1, u_1 \rangle$$
where $k_1$ is the gain and $\alpha_1 > 0$ is arbitrary.

**Heating:** From Lemma 5.1, the heating dynamics are dissipative with supply rate

$$w_2(y_2, u_2) = -\alpha_2 \langle y_2, y_2 \rangle + k_2 \alpha_2 \langle u_2, u_2 \rangle$$

where $k_2$ is the gain and $\alpha_2 > 0$ is arbitrary.

**SMA Wire:** From Theorem 4.2, the SMA wire is dissipative with supply rate

$$w_3(y_3, u_3) = \alpha_3 \langle y_3, u_3 \rangle$$

where $u_3 = \dot{T}$ and $\alpha_3 > 0$ is arbitrary.

The individual dissipativity characteristics of the three blocks can now be combined to show closed-loop stability. Defining

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad r = \begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix},$$

then the interconnection matrix

$$H = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$

satisfies (c.f., Figure 5)

$$u = r - Hy.$$

(5.4)

Define the diagonal matrices

$$Q = \text{diag}(-\alpha_1, -\alpha_2, 0)$$

$$R = \text{diag}(k_1 \alpha_1 - \delta_1, k_2 \alpha_2, 0)$$

$$S = \text{diag}(1, 0, \alpha_3).$$

The dissipativity of each block implies that

$$\langle y, Qy \rangle + \langle u, Ru \rangle + \langle y, Su \rangle \geq 0.$$  

Using (5.4) to eliminate $u$ and rearranging,

$$\langle y, \hat{Q}y \rangle - \langle y, \hat{S}r \rangle \leq \langle r, Rr \rangle$$
where
\[ \hat{Q} = -Q - H'RH + \frac{1}{2} [SH + H'S'], \quad \hat{S} = S - 2H'R. \]

Now, if \( \hat{Q} \) is positive definite, then it will follow as in [19, Thm. 1] that for some \( k_4 > 0 \),
\[ \|y\| \leq k_4\|z\|. \]
The matrix \( \hat{Q} \) will be positive definite if all three principal minors have positive determinants e.g. [22, Thm 6B].

We will now show that if the free parameters \( \alpha_1, \alpha_2, \alpha_3 \) are chosen carefully, then \( \hat{Q} \) is positive definite as required. Define \( \alpha_1, \alpha_2 \) in terms of \( \alpha_3 \) and a new parameter \( \epsilon \):
\[ \alpha_1 = 2\alpha_3 + \epsilon, \quad \alpha_2 = 2\alpha_3/k_2. \]

The 3 determinants are
\[ D_1 = \epsilon, \quad D_2 = \frac{2\alpha_3 \epsilon}{k_2}, \quad D_3 = \frac{\alpha_3 F}{4k_2} \]
where
\[ F = (-8k_1)\epsilon^2 + (8\delta_1 - \alpha_3 k_2 - 16k_1 \alpha_3)\epsilon - 2. \]
The determinants \( D_1 \) and \( D_2 \) will be positive if and only if \( \epsilon > 0 \) and \( \alpha_3 > 0 \). We will choose \( \epsilon \) and \( \alpha_3 \) so that the third determinant \( D_3 \) is also positive. Choose \( \epsilon \) to maximize \( F \):
\[ \epsilon = \frac{8\delta_1 - \alpha_3(k_2 + 16k_1)}{16k_1}. \quad (5.5) \]
Provided that \( \alpha_3 > 0 \) is sufficiently small, \( \epsilon > 0 \) as required. Substituting this value of \( \epsilon \) into \( F \)
\[ F = \frac{(k_2 + 16k_1)^2}{32k_1} \alpha_2^2 - \frac{\delta_1(k_2 + 16k_1)}{2k_1} \alpha_3 + 2 \left( \frac{\delta_1^2}{k_1} - 1 \right). \]
The function \( F \) is a quadratic in \( \alpha_3 \) and describes a parabola opening up, with \( F(0) = 2 \left( \frac{\delta_1^2}{k_1} - 1 \right) \). Since \( k_1 < \delta_1^2 \), this quantity is positive, so \( \alpha_3 > 0 \)
can be chosen sufficiently small that both $F(\alpha_3) > 0$ and the maximizing value of $\epsilon$ (5.5) are positive. With these values of the parameters, all three determinants are positive. It now follows that $\hat{Q}$ is positive definite, and the closed loop is externally stable.

While Theorem 5.2 is motivated by the configuration in Figure 5, the stability result can be generalized. The only property required of the heating dynamics was finite gain. As a result, Theorem 5.2 applies to any system with dynamics coupled in series with a hysteresis, so long as

- the coupled dynamics has finite gain; and
- the hysteresis can be represented by a Preisach model for which the weighting function $\mu$ is bounded, piece-wise continuous, and non-negative.

The result allows one to define a class of stabilizing controllers for these non-linear systems, which includes several standard linear controllers, as shown below.

**Corollary 5.3** If the PID controller with transfer function

$$H(s) = K_P + \frac{K_I}{s + \epsilon_I} + \frac{K_D s}{\epsilon_D s + 1}$$

where $\epsilon, K_P, K_I, K_D > 0$ satisfies $K_P^2 > k_1$ where $k_1$ is the $L_2$-gain of the controller, then it stabilizes a SMA actuator.

**Proof:** It is straightforward to show that the controller has passivity constant $\delta_1 = K_P$. The result then follows from the theorem.

It is difficult to write down a formula for the gain $k_1$ of a general PID controller. However, note that for a pure PI controller, the gain is $K_P + K_I/\epsilon_I$, while for a pure PD controller this quantity is $K_P + K_D/\epsilon_D$. The condition thus represents a constraint on the lower bound of the proportional action $K_P$. PID controller parameters can be chosen to optimize performance considerations, provided that $K_P^2 > k_1$ so that the closed-loop is stable.

### 6 Experimental Results

This section describes the experimental apparatus used to verify the controller design results. Closed-loop stability results for non-linear systems are
often quite conservative, with the result that controllers which meet the stability requirements result in poor performance. Experimental data is shown which demonstrates that the class of stabilizing controllers defined in this work contains useful controllers.

The experimental verification is carried out on a one-wire SMA actuator under constant-load conditions. Figure 6 shows a diagram of the experimental position control setup.

The actuator consists of a 79cm (loaded, martensite) length of 0.25mm diameter NiTi “Flexinol” wire from Dynalloy Inc., biased by a load of 800g. This represents a constant stress on the wire of roughly 163 MPa. The austenite transformation temperature of the wire is given as 90°C; the alloy is martensite at room temperature. The wire is routed over a 3cm diameter pulley, with a contact arc length of approximately 90°. The shaft of the pulley drives a 2000 count/revolution optical encoder, for a linear resolution of approximately 47µm. The encoder output is converted to position data, the control variable. To heat the wire, control current is driven through the SMA by a voltage-controlled current amplifier. Wire cooling is not actively
controlled. Control calculations are performed in real-time on a 50MHz i386 PC with an i387 numeric coprocessor, at a sampling rate of 100Hz.

The control structure is similar to that shown in Figure 5. The wire is not actively cooled, so a negative control output is cut off at zero. In addition, the wire current is saturated at 1 amp in order to protect the wire from overheating. These non-linearities were included in the heating model used in the stability analysis of the previous section.

The following figures show experimental responses using various controllers of the form

\[ C(s) = K_p + \frac{K_i}{s + \varepsilon} + \frac{K_ds}{\varepsilon s + 1}. \]

This is an approximated PID controller; in all cases, \( \varepsilon = 0.01 \). These controllers were discretized and implemented as difference equations on the control computer.

Figure 7 shows the step response of the actuator to various input step sizes. The PI controller used was tuned manually for a reference step of 20 mm. Figure 8 shows the expected reduction in the overshoot after the addition of a derivative term in the controller.

Figure 9 shows the tracking response of the system to a reference which is a combination of sinusoids, and exercises the minor loop branching behaviour of the hysteresis.

The controllers used for these three previous experimental responses satisfy the stability condition of Corollary 5.3, \( K_p^2 > K_p + K_i/\varepsilon \). Outside the class of stabilizing controllers predicted by Corollary 5.3, it is possible to obtain marginal stability (see Figure 10). This behaviour is unexpected, and demonstrates the importance of complete modelling and proper stability work.

As with the majority of non-linear stability results, Theorem 5.2 is a sufficient condition. It is possible to see an improvement in performance using controllers which are outside the class of stabilizing controllers predicted by Corollary 5.3 (see Figure 11).

7 Conclusions

A new form of dissipativity, useful for the design of output-feedback controllers, was shown for Preisach systems with \( \mu \in \mathcal{M}_p \). Using this new dissipativity result, energy-based techniques were applied to demonstrate position
feedback stability for a one-wire SMA actuator under constant load. Heating dynamics and saturation non-linearities were included in the modelling and stability analysis, in addition to the hysteresis non-linearity inherent in the SMA wire.

A common concern is the conservative nature of some non-linear stability results. Experiments demonstrate here that useful controllers are included in the class of stabilizing controllers. In addition, it was demonstrated experimentally that marginal stability could be observed using a simple PI controller, and the class of stabilizing controllers excludes the controller producing this behaviour.
Figure 8: PID-controlled step response ($K_p = 2.6, K_i = K_d = 0.04$)

Figure 9: P-controlled tracking response ($K_p = 1.1$)
Figure 10: PI-controlled marginally stable step response ($K_p = 0.1$, $K_i = 1$, $K_d = 0$)

Figure 11: Improved-performance PI-controlled step response ($K_p = 0.5$, $K_i = 0.06$, $K_d = 0$)
References


